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Convergent polynomial expansion and scaling in diffraction scattering. III. Conformal mapping without spurious cut and scaling for elastic diffraction scattering processes

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In this and a following paper we generalize the method of approach by the convergent polynomial expansion (CPE) for both elastic diffractive and inelastic nondiffractive hadron-hadron collision processes at high energies. The presence of spurious cuts in some of the conformal mappings used earlier is pointed out. A conformal mapping of the unsymmetrically cut $\cos\theta$ plane, which does not develop any spurious cut or require any knowledge of zeros, is combined with that of the s plane to construct a variable $\chi(s,t)$ which has the potentialities to reproduce some known scaling variables and Regge behavior and to provide information about asymptotic behavior of slope parameters of the type $\sim(\ln s)^n$, with $n=0,1,2$. Away from the diffraction peak the variable becomes $b(s)(\ln t)^2$. Because of the absence of spurious cuts in the mapped plane, the variable has the potentialities to provide information on the possible existence of entire functions for the differential-cross-section ratio $f(s,t)$ at asymptotic energies. However, the rate of convergence and the nature of the polynomials in the proposed CPE are not uniquely fixed at finite energies. Only at asymptotic energies the polynomials are uniquely the Laguerre polynomials and the CPE goes over to the optimized polynomial expansion. The possible existence of a scaling function for the differential-cross-section ratio at asymptotic energies as a series in Laguerre polynomials in the variable χ is pointed out. The first term in the expansion in the CPE gives a good description of the energy dependence of the forward slopes for different processes without needing any effective shape of spectral function. From the asymptotic behaviors of slope parameters obtained from data analysis we find that qualitatively the forward slopes for $\pi^{\pm}p$ and $K^{\pm}p$ scattering grow at the some rate, like $\sim\ln s$, as $s\to\infty$. Quantitatively there is also a positive indication that asymptotically the forward slopes for $\pi^{\pm}p$ scattering may be equal. Available data on the cross-section ratio for any elastic diffraction scattering process at high energies with $P_{\text{lab}}\geq 50$ GeV/c and all values of $|t|$ exhibit scaling in a remarkable manner and lie on a scaling curve. However, for lower values of P_{lab}, experimental data with larger values of $|t|$ deviate from the scaling curve. Because of the simple structure of the variable χ, the possibility of predicting $f(s,t)$ as a function of $|t|$ and for higher values of s is pointed out. Our data analysis reveals that the differential-cross-section ratios for pp, $\pi^{\pm}p$, and $K^{\pm}p$ scattering are entire functions of the corresponding χ's for $s\to\infty$.

I. INTRODUCTION

Recently some scaling variables have been proposed for the differential-cross-section ratio of elastic hadron-hadron collision processes at high energies. Brief reviews of the works on scaling have been reported in Refs. 1 (paper I) and 2 (paper II). In no other variable has scaling been shown to be exhibited by the experimental data in such a remarkable fashion both for small and large momentum transfers as in the variables proposed in I and II, constructed using conformal mapping and the convergent polynomial expansion (CPE). Significance of scaling in the context of the CPE has been discussed in I and II. Usually the optimized polynomial expansion (OPE) for scattering amplitudes[3] involves unknown parameters which depend upon energy. While analyzing the differential-cross-section data by means of OPE, a set of parameters at every energy has to be determined with the help of a computer. Although this program has been found to yield meaningful results,[4,5] it is definitely cumbersome and may even be untractable for data analysis at high energies. However, if scaling can be shown to be exhibited in OPE in a suitably chosen conformally mapped variable, the scaling function and hence fits to the data at all high energies can be known once the parameters are determined by fitting the experimental data at any single energy in the scaling region. Thus, realization of scaling in the context of OPE is very important from the point of view of economic use of computer time in fitting the high-energy data. An important consequence of scaling which missed our earlier observations,[1,2] but which has been discussed in this paper, is that even without fitting the scaling curve, the cross-section-ratio data for higher energies and as a function of $|t|$ can be predicted from the knowledge of the scaling curve and the scaling variable, particularly in the cases where the latter is a simple function of t.

In I the conformal mapping of the $x=\cos\theta$ plane developed earlier[6,7] has been combined with the conformal mapping of the s plane for constructing the scaling variable for pp scattering for which no spurious cuts are introduced by the mapping functions. However, when the method adopted in I is applied to processes possessing unsymmetrically cut x planes of analyticity using conformal mapping of Ref. 7, spurious cuts are introduced in the mapped plane. Although the presence of such cuts has been discussed in I and II, a fully correct picture about their locations has not been supplied. In II a conformal mapping of the x plane developed in Ref. 8, which guarantees a convergent expansion in Laguerre polynomials for all energies, has been used to construct a scaling variable for pp, $\bar{p}p$, $K^{\pm}p$, and $\pi^{\pm}p$ scattering. Apart from explaining shrinkage-antishrinkage of forward peaks at all energies,[8] early onset of scaling is exhibited by the cross-section-ratio data in this variable.[2] There are other beautiful features of this variable which have been sufficiently well discussed in II. But two of the several limitations[2] of this approach, which are important for the present paper, are the following: First, for the construction of the conformal mapping which guarantees CPE in terms of Laguerre polynomials for all energies, existence of at least one real zero on the physical region of the x plane has to be assumed. While fitting the slope-parameter-data, equations to real zero trajectories have to be taken either from experimental information or model predictions or else they have to be assumed. Undoubtedly such a prescription has yielded a universal description of the slope-parameter data for all energies[8] and the real zero trajectories, which get theoretical support from the predictions of the Veneziano model,[9] might be really existing for several processes,[9] but recent computation of zero trajectories[10-13] indicates that they are, in general, complex at least for certain processes. The main problem is that, except for π^-p scattering,[12,13] the presence of real zero trajectories has not been established yet for other processes from data analysis.[10-12] Second, the variable used in Ref. 8 and II develops spurious cuts in the mapped plane, which may be an objectionable feature from the point of view of the correct analytic representation and the CPE. As clarified in Sec. II of this paper, since the spurious cut is present in the mapped plane for all energies and for all the elastic scattering processes considered in II, the possibility that $f(s,t)$ may become an entire function for $s\to\infty$ in the scaling variable χ suggested in II is ruled out for all those processes.[2] A third limitation arises when an attempt is made to predict the cross-section-ratio data for higher energies in the scaling region as a function of t, because of the complicated t dependence of the mapped variable.[2,8] Further, if a method of representation of scaling is developed for processes with unsymmetrically cut x planes of analyticity, but without requiring any knowledge of zeros, it will be easier to generalize it to include scaling in inelastic process[14,15] such as $\pi^-p\to\pi^0n$ and $\pi^-p\to\eta n$.

Since the OPE has been found to be extremely useful in numerous applications, both in particle and nuclear physics,[5] where emphasis is given on convergence and correct analytic representation, it is important to know the location of the spurious cuts introduced by the mapping functions used by others. In this paper we point out that spurious cuts are present in the conformal mappings used by some authors[16-18] who did not mention the presence of such cuts in their works. For the sake of completeness we also supply further clarification on the spurious cut present in the conformal mapping used by Deo and Parida,[7] and provide a brief summary on the spurious cut in the conformal mappings used recently.[2,8,13]

Energy dependence of the slope-parameter data and scaling of the cross-section-ratio data for pp scattering have been described effectively in the context of the CPE in I without introducing any spurious singularities. In view of the presence of the spurious cuts in the variables used in II and Ref. 8, doubts may be raised whether an unquestionable representation of the scaling of the data in the context of CPE exists for processes possessing unsymmetrically cut x planes of analyticity. In this paper we show that it is possible to describe the energy dependence of the slope-parameter data and represent scaling of the corss-section-ratio data for such processes at high energies by means of a simple conformal mapping which does not introduce any spurious cut in the mapped plane or require any knowledge of zeros of the amplitude. Whereas it was necessary to fit the slope-parameter data at both forward and nonforward angles in order to know the unknown parameters in the scaling variable in II, the parameters in the present variable, whose number is one less than that in II, are determined by fitting the slope-parameter data at forward angles only. Since the construction of the variable proposed here does not require any knowledge of zeros, it is easy to generalize such an approach to inelastic processes for which the variable also succeeds in a remarkable manner.[19]

The formula developed for the slope parameter has the potentialities, as in I and II, to yield asymptotic behaviors of the type $\sim(\ln s)^n$, with $n=0,1,2$. In fitting the slope-parameter data at high energies, no effective shapes of spectral functions are needed. Data analysis indicates that, qualitatively, the slope parameters for $K^{\pm}p$ and $\pi^{\pm}p$ scattering grow asymptotically at the same rate as ~ln$s$. Quantitatively also, as an evidence in support of model-independent predictions, we find that there is a positive indication that the slopes for $\pi^{\pm}p$ scattering may grow at the same rate. Contrary to the analysis of II, where $f(s,t)$ is not an entire function of χ for $s\rightarrow\infty$ because of the presence of the spurious cut in the mapped plane, we conclude here from the information on the asymptotic behavior of slope parameter that the differential-cross-section ratio becomes an entire function in the scaling variable $\chi(s,t)$ for $s\rightarrow\infty$, at least for pp, $K^{\pm}p$, and $\pi^{\pm}p$ scattering. Unlike the situation in II, here the nature of the polynomials is not uniquely fixed for all energies but, as in the results of I, the length of the physical region and the nature of the polynomials vary with energy for finite energies. Only at asymptotic energies the polynomials are uniquely the Laguerre polynomials and the CPE goes over to the OPE. Scaling of the data on $f(s,t)$ in the variable χ is similar to that in II for the cases of K^-p and π^-p scattering. For the cases of $\bar{p}p$ and π^+p scattering, scaling is almost similar to that in II, except for the fact that there are a few more deviations from the scaling curve for lower values of P_{lab}. There are significant deviations of the data from the scaling curve in K^+p scattering, as compared to that in II, for lower values of P_{lab} and larger values of $|t|$. However, all the available data for every process with $P_{\text{lab}}\geqslant 50$ GeV/c scale in the variable χ in a remarkable manner. If the present variable and CPE are applied for pp scattering, the same results as in I would be obtained for high energies. Several limitations of this approach are pointed out. Applications of this method for inelastic processes, the success of which has been demonstrated in a subsequent paper,[19] and predicting $f(s,t)$ for higher values of s have been pointed out.

This paper is organized in the following manner: In Sec. II we report a brief review of the spurious cuts present in different conformal mappings. In Sec. III we propose conformal mappings of the s and $\cos\theta$ planes and discuss scaling of the differential-cross-section ratio by means of the CPE. Section IV is devoted to the analysis of the experimental data on forward-slope parameters at high energies, obtaining information on the asymptotic behavior of slope parameters, realization of the possibility that $f(s,t)$ may be an entire function of χ for some processes, and demonstration of scaling of the cross-section-ratio data. In Sec. V we summarize the results of this paper and discuss several limitations and applications of this approach. In Sec. VI we state our conclusions briefly.

II. SPURIOUS CUTS IN CONFORMAL MAPPING METHODS

The OPE for scattering amplitudes and form factors[3] has found its successful and extensive applications in many areas of particle physics. From the time of inception[3] of optimal convergence by conformal mapping, many mapping functions have been proposed and approximate forms of amplitudes constructed, but not all of them go without flaws. In this section we point out that spurious cuts are present in the conformal mapping used by Dumbrais and Chernev,[16] Dumbrais, Chernev, and Zlatanov,[17] and Deo and Mahapatra.[18] Although these cuts clearly violate analyticity properties, no mention of these has been made by the authors.[16-18] For the sake of completeness we also provide further clarification on the nature of the spurious cut present in the mapping of Deo and Parida[7] and a brief summary of the spurious cut in the mapping function used recently by Parida[2,8] and others.[13]

According to Mandelstam analyticity a schematic picture of the general analytic structure of scattering amplitude for a process $a+b\rightarrow a+b$ in the $x=\cos\theta$ plane has been shown in Fig. 1(a). Here x_+ $(-x_-)$ denotes the start of the right- (left)-hand cut and x_p^+ $(-x_p^-)$ denotes the image of the t-(u-) channel pole. Generally, the cuts are unsymmetrically placed with respect to the origin

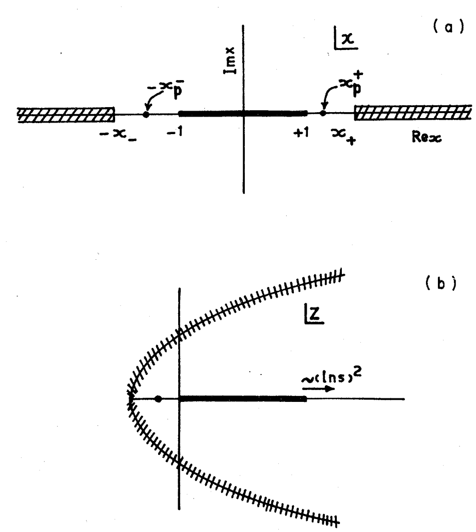

FIG. 1. A schematic picture of the analytic structure of the amplitude in the $x=\cos\theta$ plane. The solid circles represent poles. (b) Conformal mapping of the x plane onto the Z plane as proposed in Ref. 6 for pp scattering. The solid circle represents images of the poles.

except for a few processes. In the case of pp scattering there is $t \leftrightarrow u$ symmetry which imposes $x_+ = x_-$ and $x_p^+ = x_p^-$. The first attempt towards understanding the already existing phenomenological fits to differential-cross-section data was proposed by Deo and Parida[6] by means of a parabolic conformal transformation onto the Z plane,

$$\omega = \frac{1-x^2}{x_+^2 - 1}, \tag{2.1}$$

$$Z = (\sinh^{-1}\sqrt{\omega})^2. \tag{2.2}$$

This conformal transformation has been shown in Fig. 1(b) which does not introduce any spurious cut in the mapped plane. Neglecting the contribution of the pseudoscalar pion poles to the imaginary part, which dominates the scattering data near forward angles, a convergent expansion of the type

$$\frac{d\sigma}{dt} = e^{-\alpha Z} \sum_n C_n(s) P_n(2\alpha Z) \tag{2.3}$$

provides a good analytic approximation for the cross section at least in the diffraction region. It has been shown in I that the series (2.3) converges for all energies, but only at asymptotic energies is the convergence maximum and the polynomials $\{P_n(Z)\}$ are uniquely the Laguerre polynomials $\{L_n(Z)\}$.

Dumbrais and Chernev[16] have argued that the convergence of (2.3) is limited by the image of the pion poles in the interior of the parabola in the Z plane. They claim to have proposed improved convergence of Laguerre polynomial expansion in terms of a mapped variable that maps the poles of pp scattering onto the apex of the parabola by the following steps:

$$U = \frac{1-x^2}{x_p^2 - 1}, \tag{2.4}$$

$$Z_a = (\sinh^{-1}\sqrt{U})^2, \tag{2.5}$$

where $x_p^+ = x_p^- = x_p = 1 + m_\pi^2/2p^2$ for pp scattering. The authors then use the polynomial expansion

$$\frac{d\sigma}{dt} = \sum_{n=0}^{N} \mathfrak{a}_n(s) L_n(Z_a) \tag{2.6}$$

to fit the data at different fixed energies and extract information on the slope parameter.[16] The values of the forward-slope parameter obtained by these authors are found to be very different from those obtained by conventional methods of parametrization. But it is very clear that the transformations (2.4) and (2.5) introduce spurious branch points at $x = \pm x_p$, giving rise to the spurious left- and right-hand cuts in the regions $-\infty \leq x \leq -x_p$ and $x_p \leq x \leq \infty$, respectively. In fact, by the conformal transformations (2.4) and (2.5), the

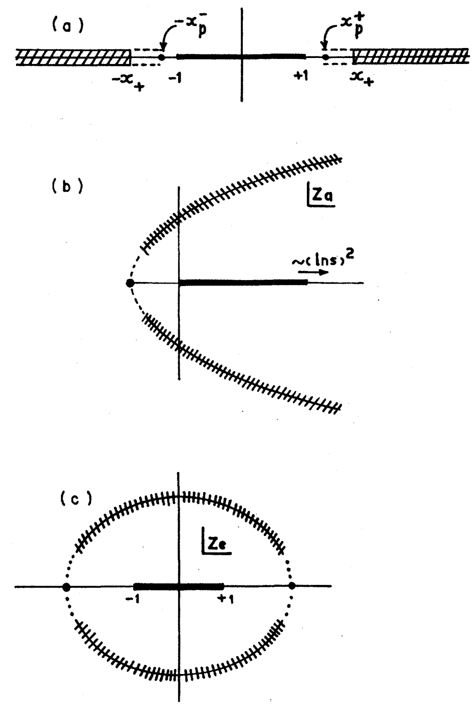

FIG. 2. Spurious cuts in the x plane, shown by the dotted contours, as introduced by the conformal mappings adopted by Dumbrais and Chernev (Ref. 16) and Dumbrais, Chernev, and Zlatanov (Ref. 17). (b) Parabolic conformal mapping of the x plane onto the Z_a plane used in Ref. 16. The dotted lines represent the images of the spurious cuts. (c) Elliptic conformal mapping of the x plane onto the Z_e plane used in Ref. 17. The dotted lines represent the images of the spurious cuts.

singularities are logarithmic branch points at $x = \pm x_p$ instead of being simple poles. Perhaps the authors have missed this vital point regarding correct Mandelstam analyticity. The spurious cuts introduced in the x plane in this process have been shown by the dotted curves in Fig. 2(a). Figure 2(b) shows the conformal mapping onto the Z_a plane, the dotted lines being the image of the spurious cuts. Thus the conformal mapping adopted by Dumbrais and Chernev[16] directly violates the basic principle of Mandelstam analyticity.

Using elliptic conformal mapping and expansion in terms of Tschebysheff polynomials, Dumbrais, Chernev, and Zlatanov[17] claim to have proposed a new analytic parametrization of the differential cross section in the diffraction region. The authors extract new values of the ratio of the real to the imaginary part and the forward-slope parameter for pp, $\bar{p}p$, pd, $\pi^\pm p$, and $K^\pm p$ scattering, which, in most cases, are found to be completely different from those obtained by the conventional method of parametrization. The argument adopted by the authors[17] is the same as that of Ref. 16, which is that the presence of the pole singularities in the interior of the figure of convergence limits the rate of convergence of polynomial expansion. In their conformal transformations the pole singularities are mapped onto the turning points, and other parts of the real x axis, for which $\text{Re}x > x_p^+$ and $\text{Re}x < -x_p^-$, are mapped onto the boundary of a unifocal ellipse. The desired conformal mapping is achieved through the following steps:

$$y = \frac{2x - x_1 - x_2}{x_2 - x_1}, \tag{2.7}$$

$$w = \frac{y - y_0}{1 - yy_0}, \tag{2.8}$$

$$u(w) = E_1(\sin^{-1}w, k) = \int_0^w \frac{dt}{[(1-t^2)(1-k^2t^2)]^{1/2}}, \tag{2.9}$$

$$v(w) = i\exp\left[-\frac{i\pi u(w)}{2E_2(k)}\right], \tag{2.10}$$

$$Z_e = \tfrac{1}{2}(v + v^{-1}) = \sin\left(\frac{\pi u}{2E_2}\right), \tag{2.11}$$

where E_1 and E_2 are incomplete and complete elliptic integrals of first kind, respectively, and

$$y_0 = \frac{y_- - y_+}{y_+y_- + Y_+Y_- - 1}, \tag{2.12}$$

$$W = \frac{1}{k} = \frac{y_+Y_- + y_-Y_+}{Y_+ + Y_-}, \tag{2.13}$$

with

$$Y_\pm = (y_\pm{}^2 - 1)^{1/2}. \tag{2.14}$$

Here x_1 and x_2 are the boundary points of the segment of the physical region in the x plane on which the data are known. In the mapping adopted here the x plane is mapped onto the y plane stretching the region $[x_1, x_2]$ onto the region $[-1, 1]$. The mapping from y to the w plane symmetrizes the positions of the nearest singularities whether they are branch points or poles; and the points $-x_p^-$ $(-x_-)$ and x_p^+ (x_+) are mapped onto the points $-W$ and W, respectively. The mapping onto the Z_e plane by (2.11) through (2.10) then gives the desired ellipse. Whereas Cutkosky and Deo[3] map the start of the cuts onto the turning points of the ellipse, these authors map the poles onto the same points. But while doing so, spurious branch points are introduced at the pole positions affecting the analyticity property. This can be easily checked from (2.9) where $u(w)$ has branch points[20] at $w = \pm 1$, $w = \pm(1/k) = \pm W$, and $w = \pm\infty$. By construction of the variable Z_e through the function $v(w)$ in (2.10), the branch points at $w = \pm 1$ disappear but others survive. The points $w = \pm W$ correspond to the images of the nearest singularities by construction whether they are poles or branch points. Thus the conformal transformation introduces two spurious cuts in the x plane in the regions $-\infty \leq \text{Re}x \leq -x_p^-$ and $x_p^+ \leq \text{Re}x \leq \infty$ as shown by the dotted curves in Fig. 2(a). The mapping onto the ellipse along with the images of the spurious cuts are shown in Fig. 2(c). In view of these, the formula proposed by the authors[17] does not conform to the correct analyticity property of the differential cross section for pp, $\bar{p}p$, pd, $\pi^\pm p$, and $K^\pm p$ scattering, where poles are nearest singularities. In view of this, the claim made by the authors[17] regarding correct analytic representation turns out to be false.

In a very explicit form and clearly violating analyticity properties, the spurious cuts are present in the conformal mappings adopted by Deo and Mahapatra[18] who apply OPE to propose a scheme of parametrization for the S- and P-wave $\pi\pi$ partial-wave amplitudes. While mapping of the left- (right-) hand inelastic cuts in the s plane shown in Fig. 3(a) onto a circle in the Z_- (Z_+) plane, the authors use

$$Z_- = (2 + \sqrt{3})(\sqrt{3+s} - \sqrt{3})/\sqrt{s}, \tag{2.15}$$

$$Z_+ = (2 + \sqrt{3})(2 + i\sqrt{s-4})/\sqrt{s}. \tag{2.16}$$

Clearly these mappings introduce square-root branch points in the s plane at $s = 0$ and $s = \infty$, giving rise to the spurious cut in the region $0 \leq \text{Re}s \leq \infty$ shown by the dotted line in Fig. 3(a). The images of the spurious cuts in the Z_- and Z_+

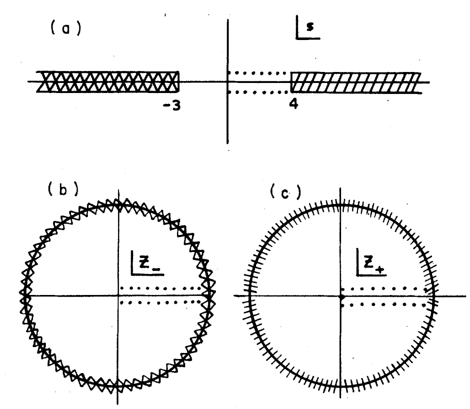

FIG. 3. (a) Analytic structure in the s plane used by Deo and Mahapatra (Ref. 18). The dotted contour is the spurious cut introduced by the conformal mapping used by the authors. (b) Conformal mapping of the right-hand cut onto the Z_- plane used in Ref. 18. The dotted line represents the image of the spurious cut. (c) Conformal mapping of the left-hand cut onto the Z_+ plane used in Ref. 18. The dotted line represents the image of the spurious cut.

planes are shown in Figs. 3(b) and 3(c). In view of this, the results[18] derived from extensive data analysis lose their weight.

While generalizing the conformal transformation (2.1) for unsymmetrically cut planes of analyticity, Deo and Parida[7] used the following mapping functions:

$$y = 2x + x_- - x_+ , \quad (2.17)$$

$$w_u = \frac{y_{\max}^2 - y^2}{y_{\text{in}}^2 - y_{\max}^2} , \quad (2.18)$$

$$Z_u = (\sinh^{-1}\sqrt{w_u})^2 , \quad (2.19)$$

where

$$y_{\max} = 2 + x_- - x_+ , \quad (2.20)$$

$$y_{\text{in}} = x_- + x_+ . \quad (2.21)$$

Then a Laguerre-polynomial expansion of the type (2.3) in the parabolic variable Z_u has been used to describe the energy dependence of the slope-parameter data for pp, $\bar{p}p$, and $K^\pm p$ scattering. The variable Z_u which reduces to Z given in (2.2), when $x_+ = x_-$, causes no problem for pp scattering. But for a general unsymmetrically cut x plane of analyticity the transformation y^2 in (2.18) introduces a spurious cut in the y^2 plane in the region $0 \leq \text{Re}y^2 \leq \infty$ by folding a part of the y plane on top of the other part. In the Z_u plane, the spurious branch point at $y^2 = 0$ is mapped onto the point

$$Z_u^1 = (\sinh^{-1}\sqrt{w_{\max}})^2 , \quad (2.22)$$

where

$$w_{\max} = y_{\max}^2/(y_{\max}^2 - y_{\text{in}}^2) . \quad (2.23)$$

The branch point at $y^2 = \infty$ is mapped onto two points, $Z_u^2 = \infty + i\infty$ and $Z_u^3 = \infty - i\infty$. These three branch points give rise to two cuts starting from the common point Z_u^1 in the Z_u plane as shown by wavy lines in Fig. 4. Obviously, because of their presence inside the figure of convergence these spurious cuts affect the convergence of polynomial expansion. As can be easily seen from Eqs. (2.22) and (2.23), Z_u^1 lies at one of the end points of the image of the physical region which moves on the $\text{Re}Z_u$ axis with energy. This can be verified by rewriting Eq. (2.23) as

$$w_{\max} = \frac{q^2}{t_R} \frac{[1 + (t_L - t_R - \Delta/s)/4q^2]^2}{[1 + (t_L - \Delta/s)/4q^2]} . \quad (2.24)$$

For very high energies $w_{\max} \sim s$ and Z_u^1 moves onto infinity like $\sim(\ln s)^2$, along with the end point of the image of the physical region. Therefore, although the spurious cuts affect the convergence of polynomial expansion at finite energies, they cause no problem at asymptotic energies. Looked at as an analytic function of x, $Z_u(x)$ has no other singularities in the x plane except the dynamical cuts. Therefore expansion in $Z_u(x)$ does not violate analyticity.

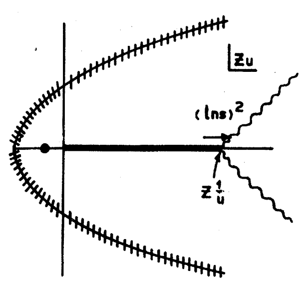

FIG. 4. Conformal mapping of the x plane onto the Z_u plane used in Ref. 7. The wavy lines represent the spurious cuts in the mapped plane.

Recently,[8] shrinkage-antishrinkage of forward peaks has been related to the experimentally observed or assumed real zero trajectories in pp, $\bar{p}p$, $K^\pm p$, and $\pi^\pm p$ scattering. A convergent polynomial expansion has been developed for all energies[8] by means of a conformal mapping constructed using the following steps:

$$g(x) = \left(\frac{c + x}{x + x_0}\right) , \quad (2.25)$$

$$w_0 = \frac{x_+ - x}{x_- + x} \frac{x_- + 1}{x_+ - 1} , \quad (2.26)$$

$$Z_0 = (\cosh^{-1}\sqrt{w_0})^2 , \quad (2.27)$$

$$Z_c = g^2(x) Z_0 . \quad (2.28)$$

Here c is a real constant corresponding to the position of the strong behavior introduced by the mapping, and $x = -x_0$ corresponds to the position of the real zero on the physical region. It can be verified that[8] the forward-slope-parameter data alone cannot determine the value of c. The value of c has been chosen to be unity in Ref. 8 for data analysis and also in Ref. 13 where oscillation of the slope parameter has been well explained using this variable and zero trajectories for $\pi^- p$ scattering. In II, while demonstrating early onset of scaling by means of this variable, the value of c has been determined using the slope-parameter data for $t = 0$ and $|t| = 0.2$ GeV^2 for different processes. It may be noted that the occurrence of the function $g^2(x)$ in the final transformation in (2.28) introduces a spurious cut in the $g^2(x)$ plane in the region $0 \leq \text{Re}g^2(x) \leq \infty$ by folding a part of the $g(x)$ plane on top of the other part. The image of this spurious cut lies in the region $0 \leq \text{Re}Z_c \leq \infty$ in the Z_c plane, as shown by the dotted contours in Fig. 5, covering the image of the entire physical region in the Z_c plane. Looked at as an

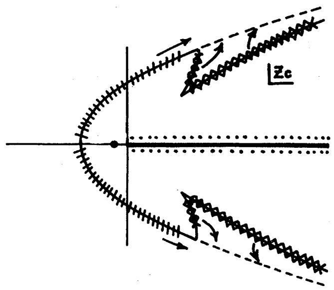

FIG. 5. Conformal mapping of the x plane onto the Z_c plane used in Refs. 2, 8, and 13. The dotted contour is the spurious cut. The slashed (crossed) line indicates the image of the left- (right-) hand cut.

analytic function of x, the mapping function $Z_c(x)$ possesses no other singularities except for the dynamical branch points allowed by Mandelstam analyticity. Therefore the CPE in terms of this variable does not violate analyticity properties. It has been pointed out that this spurious cut would not affect the convergence of polynomial expansion either, since the cut is confined to the physical region of the Z_c plane about which the polynomial expansion is being made.[2,8,13] Since the theoretical basis of such an argument is not yet clear, the presence of the spurious cut would not make the proposed CPE go without any criticism. Further, as pointed out in Sec. I, the presence of this spurious cut in the χ plane, as defined in II, may spoil the prospects of obtaining information on the possible existence of entire functions from data analysis.

In spite of the presence of the spurious cut in the variables used in Refs. 2, 8, and 13, these works justify their importance in view of their several novel features such as the description of shrinkage-antishrinkage,[8] oscillation[13] of the slope parameter, and scaling of the differential-cross-section ratio. But since OPE[3] and CPE[1,2,8] emphasizes correct analytic representation, and accelerated convergence by conformal mapping and polynomial expansion, it will be extremely useful if a CPE can be devised to represent scaling without introducing any spurious cut. There exists a CPE in I which has been successfully used to describe energy dependence of the slope-parameter data and scaling of the differential-cross-section-ratio data remarkably well for pp scattering. So far no CPE has been developed by means of a parabolic variable for any process possessing unsymmetrically cut x plane of analyticity which does not introduce any spurious cut. Further, the variable of II also requires the knowledge of at least one real zero trajectory for its construction, but at present, both the theoretical and experimental information on the existence of such zero trajectories is very meager, which makes such a construction difficult. In Sec. III we discuss the construction of a simpler conformal mapping which neither introduces any spurious cut in the mapped plane nor requires any knowledge of zero trajectories.

III. CONFORMAL MAPPING WITHOUT SPURIOUS CUT AND SCALING BY CONVERGENT EXPANSION

The method of construction of CPE by using both the s- and the $\cos\theta$-plane analyticity has been discussed in sufficient detail in I and II. In this section we will emphasize some aspects of conformal mapping of the x plane without introducing spurious cuts and scaling by convergent expansion.

A. Conformal mapping of the s and $\cos\theta$ planes

For fixed energies above threshold the right-hand cut is closer to the forward direction in the x plane, whereas the left-hand cut is farther away. The right-hand cut is considered so important that while exploiting analyticity of the x plane by conformal mapping the presence of the left-hand cut has been ignored by Lovelace.[21] The conformal mapping of the x plane onto the Z_0 plane defined by (2.26) and (2.27) has been shown in Fig. 6(a). By this mapping the entire x plane, excluding the cuts, is mapped onto the interior of the

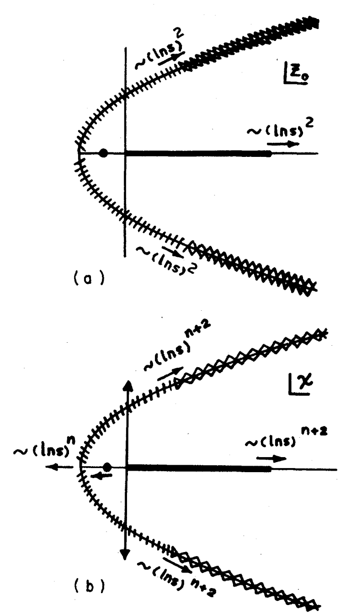

FIG. 6. (a) Conformal mapping of the x plane onto the Z_0 plane which does not introduce any spurious cut. (b) Conformal mapping of the x plane onto the χ plane for physical values of energy and for $(\ln s)^n$ type of asymptotic behavior of the slope parameter. The slashed (crossed) line indicates the image of the left- (right-) hand cut.

parabola, and the image of the start of the right-hand cut is placed at the apex of the parabola lying closest to the image of the forward direction which, in this case, is the origin and focus of the parabola in the Z_0 plane. The whole of the right-hand cut is mapped onto the forward portion of the parabola surrounding the focus like a crescent. The rest of the boundary of the parabola is formed out of the images of the left-hand cut. The points $x=\infty\pm i\epsilon$ on the right-hand cut are mapped onto the points Z_0^+ and Z_0^-, where

$$\mathrm{Re}\,Z_0^+=\mathrm{Re}\,Z_0^-=\frac{-\pi^2}{4}+(\cosh^{-1}\sqrt{w_\infty})^2\,, \tag{3.1}$$

$$\mathrm{Im}\,Z_0^+=\pi\cosh^{-1}\sqrt{w_\infty}\,, \tag{3.2}$$

$$\mathrm{Im}\,Z_0^-=-\pi\cosh^{-1}\sqrt{w_\infty}\,, \tag{3.3}$$

$$w_\infty=\left(4q^2+t_L-\frac{\Delta}{s}\right)\Big/\,t_R\,. \tag{3.4}$$

Because of the closer vicinity of the image of the right-hand cut to that of the forward direction in the Z_0 plane, it is possible to maintain the relative weights between the two cuts. For $s\to\infty$, $\mathrm{Re}\,Z_0^+$ and $\mathrm{Re}\,Z_0^-$ approach infinity as $\sim(\ln s)^2$, $\mathrm{Im}\,Z_0^+$ and $\mathrm{Im}\,Z_0^-$ approach infinity as $\sim\ln s$, and the image of the right-hand cut covers the entire parabola. At high energies the images of the distant parts of the right-hand cut are farther removed from the image of the forward direction, making them less important as compared to the nearest parts. Such a picture agrees with the general notion that scattering of hadrons at high energies is mainly peripheral and due to long-range forces. Looking to the analytic structure in the t plane for $s\to\infty$, the start of the left-hand cut is pushed to $-\infty$, the physical region is in the region $-\infty\leq\mathrm{Re}\,t\leq 0$, and there is only the right-hand cut in the region $t_R\leq\mathrm{Re}\,t\leq\infty$. Such an analytic structure becomes identical with that of the pion and the nucleon form factors, and also the deuteron form factor in the absence of the anomalous cut. The ideal conformal transformation for convergent expansion for form factors is[22,23]

$$w_f=\frac{-t}{t_R}\,, \tag{3.5}$$

$$Z_f=[\sinh^{-1}\sqrt{w_f}\,]^2\,. \tag{3.6}$$

A comparison between Z_f and Z_0 can be made if we rewrite w_0 as

$$w_0=\left(1-\frac{t}{t_R}\right)\frac{(4q^2+t_L-\Delta/s)}{(4q^2+t_L+t-\Delta/s)}\,. \tag{3.7}$$

For values of s and t such that

$$\left|\left(4q^2+t_L-\frac{\Delta}{s}\right)\right|\gg|t|\,, \tag{3.8}$$

$$w_0\simeq 1-\frac{t}{t_R}\,, \tag{3.9}$$

and

$$Z_0\simeq Z_f\,. \tag{3.10}$$

The condition (3.8) can be achieved for small momentum transfers even for intermediate energies and for larger $|t|$ for high energies. In this kinematical domain a convergent expansion in Z_0 would make the effects of the left-hand cut negligible as compared to the right-hand cut and the distant parts of the right-hand cut less important as compared to its nearer parts. The convergent expansion for the differential cross section can be written as

$$\frac{d\sigma}{dt}=e^{-\alpha Z_0}\sum_n a_n(s)P_n(2\alpha Z_0)\,. \tag{3.11}$$

At this stage it is necessary to clarify the nature of the polynomials $\{P_n(z)\}$ and the rate of convergence of the series (3.11). For any fixed but finite energy the physical region in the x plane is mapped onto a finite portion of the right half of the real axis in the Z_0 plane, although the image of the cuts forms the parabola with focus at the origin. The length of the image of the physical region increases with energy and only at asymptotic energies it spreads onto the entire right half of the $\mathrm{Re}\,Z_0$ axis as $\sim(\ln s)^2$. It is well known[24,25] that the length of the physical region decides the nature of a sequence of orthogonal polynomials and the correct physical region for Laguerre polynomials, in terms of which an expansion of the type (3.11) converges within the entire parabola in the Z_0 plane, is the whole of the right half of the $\mathrm{Re}\,Z_0$ axis. Since the length of the image of the physical region in the Z_0 plane varies with energy, the nature of the polynomials $\{P_n(z)\}$ also varies.[25] In general, the domain of convergence of a polynomial expansion depends upon the nature of the polynomials[24] which in turn are determined by the length of the physical region. Thus for different finite energies the polynomials and the corresponding domains of convergence in the Z_0 plane are not uniquely fixed and the domains of convergence may not coincide with the parabolic figure of convergence in the Z_0 plane. According to the theory of OPE,[3] maximum convergence of a polynomial expansion occurs only if the domain of convergence of the polynomial expansion coincides with the figure of convergence in the mapped plane whose interior is the image of the entire domain of analyticity of the x plane. Thus the rate of convergence of (3.11) is not maximum at finite energies, but varies with energy.

However, for asymptotic energies the polynomials are uniquely the Laguerre polynomials and the domain of convergence of (3.11) coincides with the parabola in the Z_0 plane, thus making the rate of convergence maximum. Although the rate of convergence of (3.11) may not be maximum for finite energies, it may be faster than that of the conventional Legendre polynomial expansion in x because of the following region: We are considering convergence of (3.11) in the mapped plane and the area enclosed by the domain of convergence at any finite energy is likely to contain the image of a larger portion of the x plane than the smaller area of the Lehmann ellipse within which the Legendre-polynomial expansion converges. For this reason we call (3.11) a convergent polynomial expansion (CPE). Only at asymptotic energies CPE goes over to OPE and $\{P_n(z)\} \to \{L_n(z)\}$. Besides achieving accelerated convergence by conformal mapping the cut plane into the interior of the parabola, convergence of (3.11) is further accelerated near forward angles because for $|t| \ll t_R$ and energies such that $|[4q^2 + t_L - (\Delta/s)]| \gg |t|$, $Z_0 \simeq -t/t_R$. In this kinematical region, taking only the first term in (3.11) yields the well-known form of the exponential fit to the forward peak.

To fix the polynomials $\{P_n(z)\}$ uniquely as Laguerre polynomials for all energies, the physical region of the x plane was mapped onto the entire right half of the real axis in a mapped plane with the assumption that a real zero exists on the physical region.[2,8] But, as has been discussed in Sec. II, the corresponding conformal mapping introduces spurious cut and also information on real zeros are not available for several processes. We therefore plan to test the success of (3.11) in representing the experimental data for processes possessing unsymmetrical cut planes of analyticity.

Before proceeding further, it is necessary to point out certain limitations of the analytic approximation given by (3.11). Since $t_R(s)$ and $t_L(s)$ represent boundaries corresponding to the absorptive part alone, the conformal mapping Z_0 does not contain the cut contributions of the real part and also the contribution due to the poles are not included in (3.11). However, for moderately large values of s there is very little difference between the domains of analyticity of the real and absorptive parts of the x or t plane. If we adopt the same conformal mapping for the real part also, a part of CPE on the right-hand side in (3.11) can be taken as the contribution due to the real part. Thus the limitation of (3.11) in not representing the correct Mandelstam analyticity is that it does not contain the pole contributions. There is a host of papers[26-31] which contain the assumption that the scattering near forward angles is due to the absorptive part alone. It has been shown[27,30] that the unitarity upper bound for the absorptive part of scattering amplitude derived by Singh and Roy[27] saturates the high-energy data near forward angles for NN and πN scattering.[31] Experimentally it is found that the real part has a negligible contribution near forward angles for all the elastic diffractive processes at high energies. Taking the view that the poles contribute to the real part alone, our representation is expected to be very good near forward angles. However, away from the forward angles the real part effects may be significant. Theoretically the pole contribution and its interference with the cut controbution may be significant for scattering at larger angles. The CPE developed cannot account for such contributions, if they are present. However, the simple picture of scaling as discussed in Sec. IV will be spoiled if pole terms are explicitly retained in (3.11).

Besides the "partial-wave" amplitudes a_n's being energy dependent, the parameter α occurring (3.11) also depends upon s. It is possible to take into account the energy dependence of a_n's by some conformal mapping using analyticity of the s plane. But while approximating $a_n(s)$ it may be borne in mind that the analyticity properties of these "partial-wave" amplitudes are different from those of the total amplitudes for unequal-mass scattering. However, as has been demonstrated earlier and also in this paper, energy dependence of $\alpha(s)$ alone is sufficient to account for the energy dependence of the forward slope-parameter data and scaling of the cross-section-ratio data. The need for exploiting the analyticity properties of the s plane along with those of the x plane and the method of construction of the CPE for $\alpha(s)$ have already been discussed in sufficient detail in I and II. The Taylor series expansion, taking into account the energy dependence of α, can be written as[1,2]

$$\alpha(s) = \begin{cases} c_0 + c_1\zeta\,, & (3.12) \\ d_0 + d_1\eta + d_2\eta^2\,, & (3.13) \end{cases}$$

where ζ (η) maps the left-hand cut[32] in the s plane in the region $-\infty \leq \mathrm{Re}\, s \leq s_1$ onto a parabola (strip) with focus at the origin. By these mappings the physical region of the s plane lying in the region $s_{\mathrm{th}} \leq \mathrm{Re}\, s \leq \infty$ is mapped onto the right-half axis in the ζ (η) plane. In (3.12) the transformation η (ζ) is defined in the following manner:

$$w_s = \frac{s - s_{\mathrm{th}}}{s_{\mathrm{th}} - s_1}\,, \tag{3.14}$$

$$\eta(s) = \sinh^{-1}\sqrt{w_s}\,, \tag{3.15}$$

$$\zeta = \eta^2\,. \tag{3.16}$$

With the prescription (3.13) the slope parameter has the potentiality to saturate the asymptotic behaviors of the type

$$\lim_{s\to\infty} b(s) \sim (\ln s)^n, \quad n=0,1,2\,, \tag{3.17}$$

where $n=0$ and 1 would correspond to constant and Regge types of asymptotic behaviors, respectively, and $n=2$ corresponds to saturation of the unitarity upper bound.[30]

B. Scaling by convergent polynomial expansion

Since (3.11) is a convergent expansion in Z_0, it is useful to define

$$\chi(s,t)=\alpha(s)Z_0(s,t) \tag{3.18}$$

and

$$f(s,t)=\frac{d\sigma}{dt}(s,t)\Big/\frac{d\sigma}{dt}(s,0)\,. \tag{3.19}$$

Then from (3.11) we get

$$f(s,t)=\bar{e}^{\chi}\sum_{n=0} e_n P_n(2\chi)\,, \tag{3.20}$$

where

$$e_n=\frac{a_n}{\sum_n a_n P_n(0)}\,. \tag{3.21}$$

The coefficients e_n's are, in general, energy-dependent parameters. For high energies $t_R \simeq 4m_\pi^2$ and

$$\chi(s,t)\sim b(s)Z_0 \tag{3.22}$$

for all angles within the forward hemisphere. For large energies and momentum transfers for which $|t|\gg 4m_\pi^2=0.078\ \text{GeV}^2$, which corresponds to the region in which a large majority of data points exist, (3.22) yields

$$\chi(s,t)\sim b(s)(\ln t)^2\,. \tag{3.23}$$

It may be noted that in this kinematical region the variable χ proposed here and also in earlier works[1,2] is completely different from all other scaling variables.[14,15,26-29,33,34] Only for high energies and for values of $|t|\ll t_R\simeq 0.078\ \text{GeV}^2$ the present variable reduces to the known scaling variable[28,29] $tb(s)$. Retaining terms up to first [second] in (3.13) yields the scaling variable $t\ln s$ $[t(\ln s)^2]$, corresponding to the saturation of Regge behavior (unitarity bound) in the same kinematical domain.

Since one of our objectives is to propose χ as a scaling variable through CPE, but without introducing any spurious cuts, it is necessary to examine the nature of convergence in the χ plane. For large physical values of s, the image of the physical region spreads the entire right half of the Reχ axis like $\sim(\ln s)^{n+2}$, where $n=0,1,2$, fixed from the asymptotic behavior of slope parameter. Then if the asymptotic behavior is either of the Regge type or saturates the unitarity bound, convergence of (3.20) approaches maximum and $\{P_n(2\chi)\}\to\{L_n(2\chi)\}$ faster in the energy scale than what would have been achieved considering only the Z_0 plane and the series (3.11). Since the apex of the parabola lies at

$$\text{Re}\chi=\frac{-\pi^2}{4}\alpha(s) \tag{3.24}$$

and the real and the imaginary parts of χ, corresponding to the images of points on the cuts, are proportional to $\alpha(s)$, the branch points are pushed on to infinity in the χ plane at asymptotic energies for those processes for which the slope parameter might yield $n=1$ or 2. As it has been mentioned in Sec. III, the images of pole singularities are mapped onto the interior of the Z_0 plane and if they are important (which is not the case at least for pp, $\bar{p}p$, $\pi^\pm p$, and $K^\pm p$ scattering), the convergence of (3.11) is limited by the images of the poles. But if $n=1$ or 2, the images of the poles are also pushed on to Re$\chi=-\infty$ like $\sim(\ln s)^n$ for asymptotic energies in the χ plane. Thus in the limit $s\to\infty$, all the singularities are removed on to infinity and the entire χ plane is available as the plane of analyticity. A schematic diagram of the conformal mapping in the χ plane for a given physical but large value of s has been shown in Fig. 6(b). Auberson, Kinoshita, and Martin[26] have proved, using results of axiomatic field theory (AFT), that for an amplitude violating Pomeranchuk theorem and saturating the unitarity bound, the amplitude ratio $A(s,t)/A(s,0)$, where $A(s,t)$ is the absorptive part, becomes an entire function of the scaling variable $\tau=t(\ln s)^2$ for $s\to\infty$. In the present case the function $f(s,t)$ becomes an entire function not only in the case when unitarity bound on the slope parameter is saturated ($n=2$), but also when the amplitude has Regge-type asymptotic behavior ($n=1$), but $f(s,t)$ scales only when e_n's are independent of s. From the analysis in Sec. IV it will be seen that, at least for the slope-parameter data for $K^\pm p$ and $\pi^\pm p$ scattering, $n=1$.

At finite energies the length of the physical region in the χ plane varies. Since the length of the physical region determines the nature of the orthogonal polynomials $\{P_n(2\chi)\}$, their nature also varies with energy.[25] At finite energies the domain of convergence of these polynomials may not be the whole interior of the parabola. But only at asymptotic energies the physical region is $0\leq\text{Re}\chi\leq\infty$, and the orthogonal polynomials determined by it are uniquely the Laguerre polynomials, whose domain of convergence is the parabola in the χ plane with focus at the origin. Since the whole of the cut plane of analyticity in the x plane has been mapped onto the interior of the parabola, the convergence of (3.20) is also maximum at asymptotic energies.[3] It may be possible that in this kinematical region the same number of coefficients e_n's having the same values may represent $f(s,t)$ by (3.20) for all energies. Thus, if (3.20) defines a scaling function at all it must be at asymptotic energies. At finite energies scaling cannot be achieved by (3.20), since the rate of convergence and also the nature of $\{P_n(2\chi)\}$ are not unique.

In the case of exact results[26-29] using principles derived from AFT, scaling in the variables $t(\ln s)^2$ or $tb(s)$ has been proved *a priori* in the limit $s\to\infty$ and $|t|$ lying within the diffraction peak region. Roy and Singh[27] have proved that upper bound on the ratio $A(s,t)/A(s,0)$ scales in the variable $t(\sigma_{\text{tot}}{}^2/\sigma_{\text{el}})$. In the present case energy independent of e_n's is hypothesized. Of course there are sufficient reasons, as described above, to induce such a hypothesis by CPE. Evidence for such a hypothesis will be strongly supported by the experimental data as described in Sec. IV. There are other variables such as $t\sigma_{\text{tot}}$ of Dias de Deus[33] and $(ut/s)[\sigma_{\text{tot}}(s)/\sigma_{\text{tot}}(s_0)]$ of Hansen and Krisch[34] in which scaling is also hypothesized.

IV. ENERGY DEPENDENCE OF SLOPE PARAMETER AND SCALING OF THE CROSS-SECTION-RATIO DATA

In Sec. III we have developed a CPE for diffraction scattering without introducing any spurious cut and without using any zero trajectory. In this section we examine the success of the CPE in representing energy dependence of forward-slope parameter and scaling of the cross-section-ratio data at high energies for elastic scattering processes possessing an unsymmetrical cut x plane of analyticity. It is not necessary to repeat data analysis for pp scattering using the present variable,[35] since such an analysis has already been done in I using a better variable which explicity preserves $t\leftrightarrow u$ symmetry for such a process, but does not introduce any spurious cut or require any knowledge of zeros. Even though such analyses, as reported here for $\bar{p}p$, $\pi^{\pm}p$, and $K^{\pm}p$ scattering are repeated for pp scattering, they would yield very nearly the same results[35] as in I.

From the formula obtained for the slope parameter we first determine the unknown parameters in χ by fitting the data for forward slopes at high energies. Fit to the slope-parameter data yields information on the asymptotic behavior. From the asymptotic behaviors we conclude whether, for a given process, $f(s,t)$ is an entire function of χ. For $\pi^{\pm}p$ scattering we get an indication that the asymptotic equality of particle-antiparticle slopes may be satisfied. From the knowledge of the unknown parameters in χ we plot the data on $f(s,t)$ against $\chi(s,t)$ in order to test the success of the scaling variable.

From (3.20) the formula for the slope $b(s)$ of the forward peak can be derived, i.e.,

$$b(s)=\frac{d}{dt}\ln f(s,t)\bigg|_{t=0}=-\frac{d\chi(s,t)}{dt}\bigg|_{t=0}$$

$$=\frac{\alpha(s)}{t_R}\left(1+\frac{1}{4q^2+t_L-\Delta/s}\right) \tag{4.1}$$

or

$$b(s)=\frac{d_0+d_1\eta+d_0\eta^2}{t_R}\left(1+\frac{t_R}{4q^2+t_L-\Delta/s}\right), \tag{4.2}$$

where we have retained only the first term in (3.20) and used (3.13) for $\alpha(s)$. For all the processes to be considered here we have used theoretical elastic boundaries of spectral functions, computed from box diagrams. Even for moderate energies $t_R\simeq 4m_\pi{}^2$. Thus for high energies such that

$$|(4q^2+t_L-\Delta/s)|\gg 1\text{ GeV}^2,$$

the formula (4.2) for the slope parameter becomes

$$b(s)\simeq(d_0+d_1\eta+d_0\eta^2)/4m_\pi{}^2. \tag{4.3}$$

By formula (4.2) shrinkage-antishrinkage and the oscillatory pattern of the data observed at low and intermediate energies cannot be fitted.[8] However, it is possible to account for the slope-parameter data for $\bar{p}p$, $K^{\pm}p$, and $\pi^{\pm}p$ scattering at high energies even without using any effective shape of spectral function. Since scaling is supposed to be valid at high energies we will confine our attention for data analysis at such energies.

(*a*) *$\bar{p}p$ scattering.* Available data[36] on the slope parameter for $\bar{p}p$ scattering for $s\geq 4$ GeV^2 and with $|t|\leq 0.11$ GeV^2 were fitted with formula (4.2) taking elastic boundaries of spectral functions.[2,7,8] Taking only the first term in (4.2) with only one parameter

$$d_0=1.045, \tag{4.4}$$

a χ^2/DOF $=8.217$ was obtained for 22 data points. The fit has been shown in Fig. 7. In this case the asymptotic behavior is of constant type and the formula cannot reproduce the rapid antishrinkage observed at low energies.[2] However, as has been

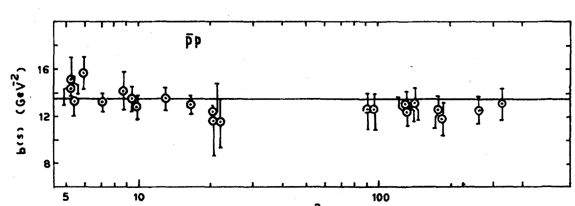

FIG. 7. Fit to the forward-slope-parameter data for $p\bar{p}$ scattering at high energies.

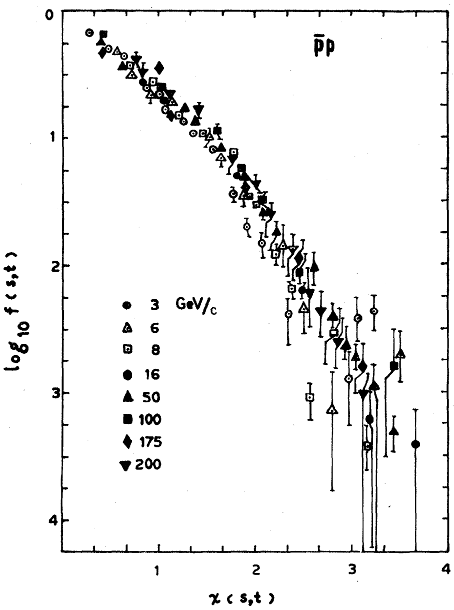

FIG. 8. Scaling of the available differential-cross-section-ratio data for $\bar{p}p$ scattering at high energies.

done in Ref. 8 and II, χ^2/DOF can be somewhat improved by taking all three terms in $\alpha(s)$ with d_0 and d_2 positive and d_1 negative. This would correspond to an asymptotic behavior with $n=2$. But the slope-parameter data at high energies are not sufficient to attach more relative importance to this type of fit. Thus it cannot be conclusively stated whether $f(s,t)$ is an entire function of χ for $s\to\infty$. With the knowledge of χ from (3.13), (3.18), and (4.4) we now plot the available data[36] on $f(s,t)$ against χ as shown in Fig. 8. In this case scaling appears to be very good. All the data starting from $P_{lab}=3$ GeV/c and for $|t|\leq|t|_{max}\simeq 0.5$ GeV^2 almost lie on the scaling curve. For higher energies data for larger values of $|t|$ also appraoch the scaling curve. In particular, all the available data with $P_{lab}\geq 16$ GeV/c fall on the scaling curve. Compared to the scaling in II it is found that the data for $P_{lab}<16$ GeV/c and away from the forward direction deviate a little more from the scaling curve, but for energies with $P_{lab}\geq 16$ GeV/c scaling of the data is exhibited in a remarkable manner very much similar to that in II.

(*b*) *π^+p scattering*. We have taken 47 points[36] on $b(s)$ with $s>5$ GeV^2 and tried to fit them using the first two terms in $\alpha(s)$ in (4.2). A very good average of the existing oscillatory pattern of the data at high energies is described by the fit with elastic boundaries of spectral functions and the following values of parameters

$$d_0=0.346\,,\qquad d_1=0.100\,, \tag{4.5}$$

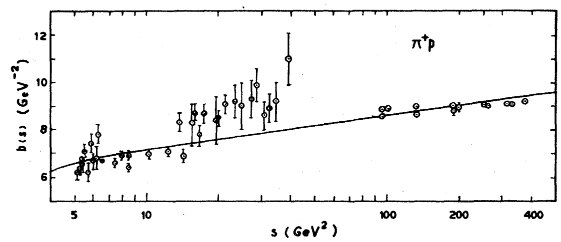

FIG. 9. Fit to the forward-slope-parameter data for π^+p scattering at high energies.

yielding a χ^2/DOF = 3.395. This fit has been shown in Fig. 9, which yields an asymptotic behavior of ~ln$s$ type corresponding to $n=1$, and definitely suggests that $f(s,t)$ is an entire function of χ at asymptotic energies. The fit does not improve if the third term in $\alpha(s)$ is included. With the knowledge of the unknown parameters as given above we now plot the data[36] on $f(s,t)$ against χ as shown in Fig. 10. All the data for $P_{lab}\geq 3$ GeV/c and with $|t|\leq|t|_{max}\simeq 1.2$ GeV^2 almost lie on the same scaling curve. Scaling in this case is also similar to that of II, except for the fact that there is a little more deviation of the data at $P_{lab}=3$, 5, and 6 GeV/c and away from forward angles. However, all the available data with $P_{lab}\geq 50$ GeV/c lie on the same scaling curve.

(*c*) *π^-p scattering*. We have taken 44 data points[36] on $b(s)$ with $s>4$ GeV^2 and fitted them with formula

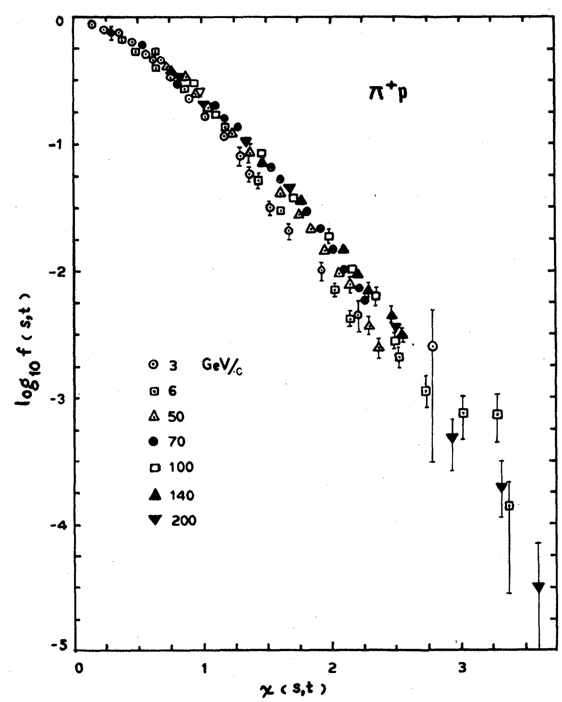

FIG. 10. Scaling of the available differential-cross-section-ratio data for π^+p scattering at high energies.

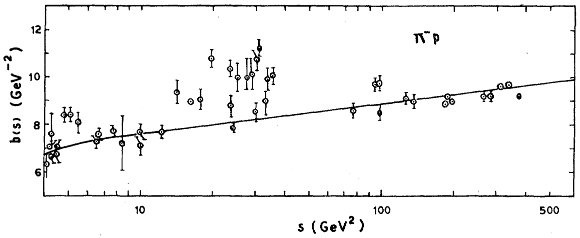

FIG. 11. Fit to the forward-slope-parameter data for π^-p scattering at high energies.

(4.2). A very good average of the existing oscillatory pattern of the data at high energies is described with elastic boundaries[2,8] of spectral functions taking the first two terms in $\alpha(s)$ with

$$d_0 = 0.43\,, \qquad d_1 = 0.081\,. \tag{4.6}$$

This fit yields a $\chi^2/\text{DOF} = 10.168$ and has been shown in Fig. 11. The asymptotic behavior is ~$\ln s$ type, corresponding to $n = 1$, which suggests that $f(s,t)$ is an entire function of χ for $s \to \infty$. The fit does not improve if the third term in $\alpha(s)$ is included. With the knowledge of the parameters occurring in χ we now plot the data[36] on $f(s,t)$ against χ as shown in Fig. 12. It is found that all the available data with $P_{\text{lab}} \geq 3$ GeV/c and $|t| \leq |t|_{\max} \simeq 1.2$ GeV^2 lie on the scaling curve. Scaling in this case is similar to that in II. All the available

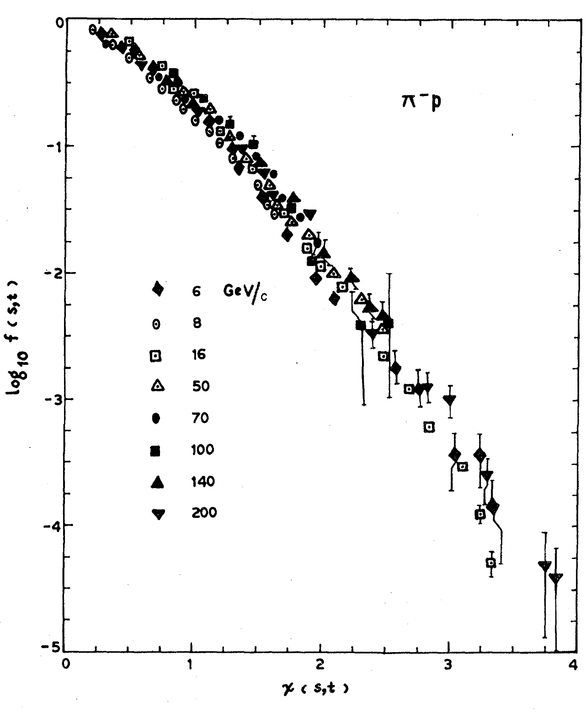

FIG. 12. Scaling of the available differential-cross-section-ratio data for π^-p scattering at high energies.

data for $P_{\text{lab}} \geq 50$ GeV/c lie on the same scaling curve.

(*d*) K^+p *scattering*. For this process we have taken 40 data points[36] on $b(s)$ with $s > 3$ GeV^2 and fitted them with formula (4.2), taking theoretical elastic boundaries of spectral functions.[2,7,8] The best fit has been obtained with the following values of the parameters:

$$d_0 = 0.051\,, \qquad d_1 = 0.180\,, \tag{4.7}$$

with $\chi^2/\text{DOF} = 2.1$. This fit has been shown in Fig. 13. The asymptotic behavior for this fit corresponds to $n = 1$, suggesting that $f(s,t)$ is an entire function of χ for $s \to \infty$. With the knowledge of the parameters in χ, we plot the experimental data[36] on $f(s,t)$ against χ as shown in Figs. 14(a) and 14(b). Figure 14(a) shows scaling of the cross-section-ratio data for smaller $|t|$ values and for all energies starting from $P_{\text{lab}} = 3$ GeV/c. It is found that all the available data above $P_{\text{lab}} = 3$ GeV/c and for $|t| \leq |t|_{\max} = 0.5$ GeV^2 almost lie on the same scaling curve. However, it is also clear that as $|t|$ increases above $|t|_{\max}$, data for lower values of P_{lab} deviate more and more from the scaling curve. Also, the value of $|t|_{\max}$ is nearly one fourth of that observed in the variable of II. This clearly shows that early onset of scaling as observed in II is not present in the present variable in K^+p scattering. Figure 14(b) shows scaling of the data for higher energies with $P_{\text{lab}} \geq 50$ GeV/c. All the available data[36] for this energy range appear to lie almost on the same scaling curve. Scaling in this energy region can be further improved if formula (4.2) is used to fit the slope-parameter data for $P_{\text{lab}} \geq 50$ GeV/c. The dotted line in Fig. 13 shows such a fit with $\chi^2/\text{DOF} = 3.316$ for 13 data points and for the following values of the pararameters:

$$d_0 = 0.39\,, \qquad d_1 = 0.068\,. \tag{4.8}$$

Scaling of the cross-section-ratio data correspoinding to these parameters is shown in Fig. 14(c).

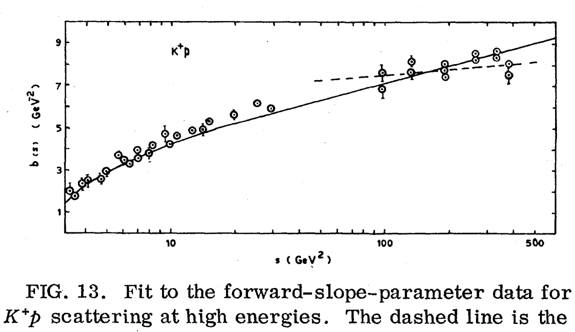

FIG. 13. Fit to the forward-slope-parameter data for K^+p scattering at high energies. The dashed line is the fit obtained using the data for $P_{\text{lab}} \geq 50$ GeV/c only.

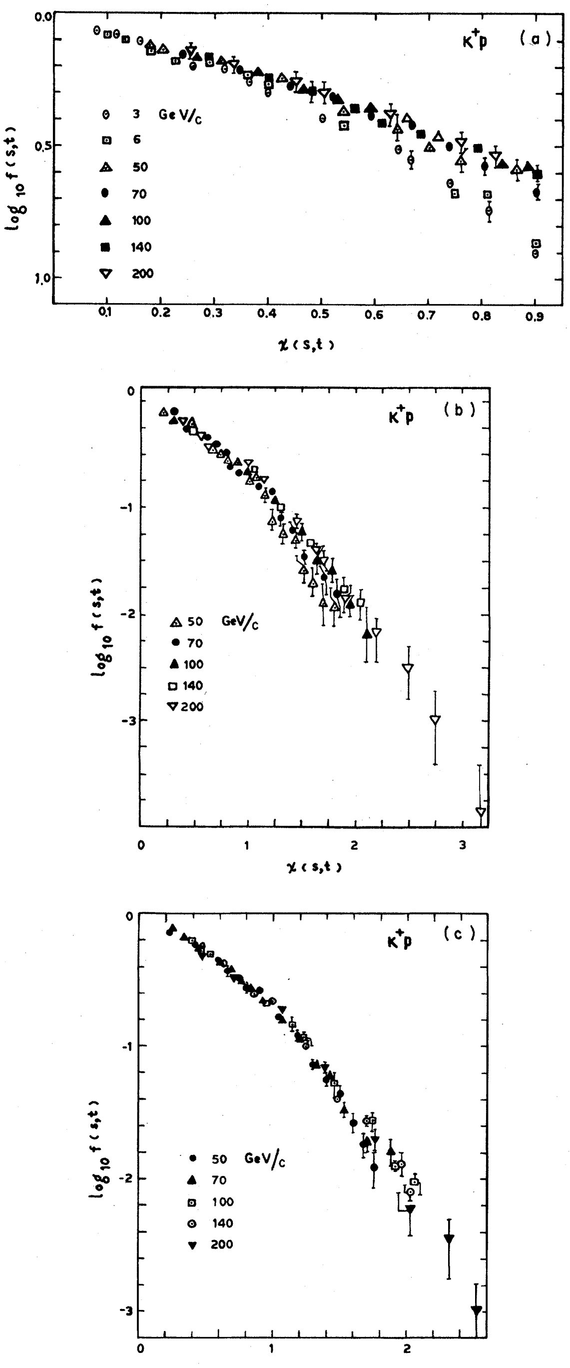

FIG. 14. (a) Scaling of the available data on differential-cross-section ratio for K^+p scattering for smaller $|t|$ values and with $P_{lab} \geqslant 3$ GeV/c. (b) Scaling of the available data on differential-cross-section ratio for K^+p scattering in the larger-$|t|$ region with $P_{lab} \geqslant 50$ GeV/c corresponding to the solid-line fit of the slope-parameter data of Fig. 13. (c) Scaling of the available data on differential-cross-section ratio for K^+p scattering in the larger-$|t|$ region with $P_{lab} \geqslant 50$ GeV/c corresponding to the dashed-line fit of Fig. 13.

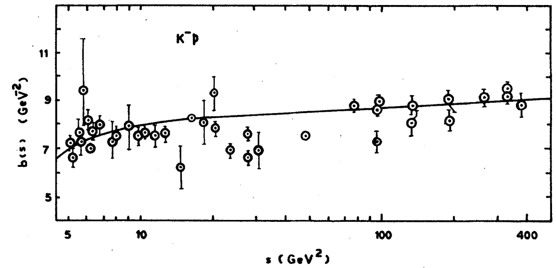

FIG. 15. Fit to the forward-slope-parameter data for K^-p scattering to high energies.

It is clear that scaling for large $|t|$ data has been improved as compared to that in Fig. 14(b).

(e) *K^-p scattering.* Using formula (4.2) we have fitted 38 data points[36] on $b(s)$ with $s \geqslant 5$ GeV^2 taking elastic boundaries of spectral functions[2,7,8] and the first two terms of $\alpha(s)$ with

$$d_0 = 0.593\,, \qquad d_1 = 0.032\,. \tag{4.9}$$

This fit, which has been shown in Fig. 15, yields a $\chi^2/\text{DOF} = 6.6$. The asymptotic behavior is like ~ln$s$ suggesting that $f(s,t)$ is an entire function of χ for $s \to \infty$. With the knowledge of the unknown parameters in χ we now plot the data[36] on $f(s,t)$ against χ as shown in Fig. 16. It is found that all the available data[36] for $P_{lab} \geqslant 3$ GeV/c and with $|t| \leqslant |t|_{max} = 1.0$ GeV^2 lie on the scaling curve. For

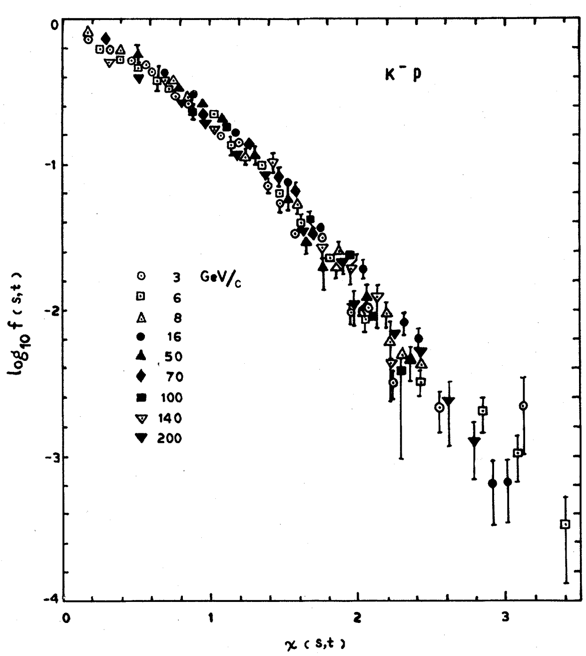

FIG. 16. Scaling of the available data on the differential-cross-section ratio for K^-p scattering at high energies.

TABLE I. Information on the asymptotic behavior of slope parameter and scaling of cross-section-ratio data for different diffraction scattering processes. Here $|t|_{max}$ denotes the maximum range of $|t|$ of the available data for $P_{lab} \geq 3$ GeV/c which lie on the scaling curve.

Process	Asymptotic behavior of slope parameter	Range of P_{lab} (GeV/c) investigated for scaling	$\|t\|_{max}$ (GeV^2)
$\bar{p}p$	Const	3–200	0.5
π^+p	ln$s$	3–200	1.2
π^-p	ln$s$	3–200	1.2
K^+p	ln$s$	3–200	0.5
K^-p	ln$s$	3–200	1.0

higher energies, data for larger values of $|t|$ approach the scaling curve. In particular, all the available data for $P_{lab} \geq 50$ GeV/c lie on the same curve. Unlike the case of K^+p scattering, scaling is exhibited by the data in a remarkable manner similar to that in II.

Some of the results obtained in this section have been summarized in Table I. Before concluding this section we note that using analyticity of the s plane along with that of the cosθ plane, reasonably good fits to the high-energy data on forward slope parameters for $\bar{p}p$, $\pi^{\pm}p$, and $K^{\pm}p$ scattering have been obtained without using any effective shapes of spectral functions[2,7,8] and any information on the zeros, but using only one parameter for $\bar{p}p$ scattering and two parameters for others. The asymptotic behavior for the slope parameter for $\bar{p}p$ scattering is constant type, although there is some weak indication[2] that it can be like ~(ln$s$)^2. Asymptotic behaviors for $\pi^{\pm}p$ and $K^{\pm}p$ scattering are like ~ln$s$, suggesting that for $s \to \infty$ cross-section ratios for these processes are entire functions of χ. Using (4.2) for pp scattering would yield exactly the same type of fit[35] obtained in I which reveals that the present data are consistent with asymptotic behaviors of the type ~ln$s$ or ~(ln$s$)^2 suggesting strongly that for pp scattering also the cross-section ratio is an entire function of χ for $s \to \infty$. Such a conclusion holds for the variable of I since it does not introduce any spurious cut in the mapped plane. Because of the presence of the spurious cuts in the χ plane, such conclusions cannot be drawn in the case of the variable of II. Although scaling in the present variable is almost the same as in II for $\bar{p}p$, $\pi^{\pm}p$, and K^-p scattering and exhibited in a remarkable manner by the data, deviations from scaling are observed for the data within $3 \leq P_{lab} < 50$ GeV/c and with $|t|$ away from forward direction for K^+p scattering. Even for $\bar{p}p$ and π^+p scattering there are little more deviations of the data from the scaling curve as compared to that in II at lower values of P_{lab}. If this method is repeated for pp scattering, scaling would be very much the same as in I but not like that in II, where early onset of scaling has been observed for the data with $P_{lab} \geq 3$ GeV/c. Such a discrepancy probably arises because the variable of II has been specifically designed to work even for lower energies. But in that case,[2] the price we pay is the presence of a spurious cut in the mapped plane for all the processes.[2]

But for higher energies with $P_{lab} \geq 50$ GeV/c, data at all available values of $|t|$ scale in a remarkable manner as observed in I and II. It can be argued that the present model has been developed for near forward angles and need not work in representing the data for larger angles. Although there is no convincing explanation yet as to why there is scaling for larger $|t|$ data, some heuristic plausibility arguments can be put forward.[1,2] For the real part, which may be important for larger $|t|$ data at high energies, the domain of analyticity is the same as that of the absorptive part if we ignore contributions due to poles. Hence, the same conformal mapping can be used and also a part of the CPE in (3.20) may be taken to represent the real part. Further, the short-range forces represented by the distant parts of the right-hand cut, which possibly influence scattering for larger $|t|$, have been indirectly taken into account in a crude manner by the conformal mapping, although the effects of very distant parts of the right-hand cut and the distant parts of the left-hand cut have been made less and less important. An alternative heuristic explanation may be that the real part effects and/or the influence due to short-range forces are negligible for high energies and away from forward direction.

V. SUMMARY, DISCUSSION, LIMITATIONS, AND APPLICATIONS

In this section we summarize the results obtained in this paper and discuss their relevance.

We point out other processes for which this approach can be effectively applied. Several limitations of this method are also discussed in this section.

A. Summary and discussion of results

In this paper we have pointed out the presence of spurious cuts introduced by the conformal mappings adopted by several authors[16–18] who proposed analytic approximations for scattering amplitude by OPE. In view of the importance and wide applications of OPE, such information on the spurious cuts is necessary so as not to mislead readers in the future. We have supplied correct pictures of the spurious cuts in the conformal mapping used by Deo and Parida[7] earlier and recently by Parida.[2,8] In view of the presence of spurious cuts in these works,[2,7,8] there exists no other unquestionable representation by means of CPE except the one proposed in this paper for scaling in diffraction scattering processes possessing an unsymmetrically cut x plane of analyticity. Further, in addition to developing spurious cuts in the mapped planes, the conformal mapping used in II requires the knowledge of real zero trajectories about which experimental information is very meager. In this paper we have proposed a representation for diffraction scattering at high energies using conformal mappings of the unsymmetrically cut s and $\cos\theta$ planes, but without introducing any spurious cuts. To construct the conformal mapping no knowledge of zeros is necessary. However, the convergence of the series and the nature of the polynomials are not uniquely fixed, but vary with energy, at finite energies. For $s\to\infty$ the convergence of the series is maximum and the polynomials are uniquely the Laguerre polynomials.

It is argued that scaling of the differential-cross-section ratio $f(s,t)$ may occur at asymptotic energies in a variable $\chi(s,t)$. Several scaling variables exist in the literature and some of the successful ones are $t\sigma_{tot}^2/\sigma_{el}$ of Singh and Roy,[27] $t\sigma_{tot}$ of Dias de Deus,[33] $tb(s)$ of Auberson and Roy,[29] and $ut\sigma_{tot}(s)/[s\sigma_{tot}(s_0)]$ of Hansen and Krisch[34] which mainly apply for small $|t|$ and asymptotic values of s. All these variables reduce to the scaling variable $\tau = t(\ln s)^2$, obtained by Auberson, Kinoshita, and Martin.[26] If the unitarity upper bound is saturated for $s\to\infty$. The variable χ proposed here reduces to $tb(s)$ for high energies and for $|t|\ll 4m_\pi^2=0.078$ GeV^2 and has the potentialities to reduce to the variables $t\ln s$ or $t(\ln s)^2$, depending upon the asymptotic behavior of $b(s)$, but for $|t|\gg 0.078$ GeV^2, a condition which is satisfied by a large majority of the presently available differential-cross-section data at high energies, our varaible reduces to $b(s)\,(\ln t)^2$, which is a completely new prediction of our hypothesis on scaling. For the processes for which the slope parameter satisfies Regge-type asymptotic behavior or saturates the unitarity upper bound, $f(s,t)$ becomes an entire function of χ for $s\to\infty$.

The formula developed for the slope parameter has the potentiality to yield asymptotic behaviors of the type $\sim(\ln s)^n$, with $n=0,1,2$ and gives a very good account of the slope-parameter data for $\bar{p}p$, $\pi^\pm p$, and $K^\pm p$ scattering at high energies with theoretical elastic boundaries of spectral functions and less number of free parameters as compared to the fits of II, although it does not possess the potentialities to reproduce shrinkage-antishrinkage of forward peaks for $\bar{p}p$, K^-p, and $\pi^\pm p$ scattering observed at lower energies. The asymptotic behaviors of slope parameters for $\pi^\pm p$ and $K^\pm p$ are of $\sim\ln s$ type and the asymptotic behavior of the slope parameter for $\bar{p}p$ scattering is of $\sim$const type, although there is some weak indication[2] from the $\bar{p}p$ data that the asymptotic behavior can be of $\sim(\ln s)^2$ type. If the present formula is applied for the slopes of pp scattering,[35] the fit wil be exactly similar to that of I consistent with the asymptotic behaviors $\sim\ln s$ or $\sim(\ln s)^2$. These asymptotic behaviors supply the important information that for pp, $\pi^\pm p$, and $K^\pm p$ scattering, the differential-cross-section ratios are entire functions in the corresponding χ's. Such a conclusion is also true for the pp differential-cross-section ratio in the variable proposed in I. For very high energies and for $|t|\ll s$, the present variable reduces to the one proposed in I for pp scattering.[35] It may be remarked that no conclusion regarding the possible existence of entire functions can be drawn in the work of II, because of the existence of spurious cuts in the mapped planes for all the processes.

It follows[37] from exact results based upon principles laid down by axiomatic field theory that the forward slope parameter for particle-particle scattering should be the same as that for antiparticle-particle scattering. In the present analysis such a result is seen to be satisfied qualitatively for $K^\pm p$ scattering. From (4.8) and (4.9) it is clear that values of d_1 for K^+p and K^-p scattering are of the same order of magnitude. From the values of the parameter d_1 given in (4.5) and (4.6) it is clear that such a result also tends to be qualitatively satisfied for $\pi^\pm p$ scattering.[38] However, for pp and $\bar{p}p$ scattering there is no strong evidence of such a result being satisfied even qualitatively.

Whereas it was necessary to fit the slope-parameter data at both forward and nonforward angles to know the parameters in χ in II, in the present case the forward slopes are sufficient to supply information on the parameters. Scaling of the cross-section-ratio data for $\bar{p}p$, $\pi^{\pm}p$, and $K^{-}p$ occurs in the corresponding χ's almost in a similar manner as observed in II, except for the fact that at lower values of P_{lab} and higher values of $|t|$, the data deviate a little more from the scaling curve for $\pi^{+}p$ and $\bar{p}p$ scattering. For pp scattering, scaling is similar to that in I. For $K^{+}p$ scattering, the value of $|t|_{max}$, within which all the data for $P_{lab} \geq 3$ GeV/c scale, is nearly one fourth of the corresponding value in II. Significant deviations from the scaling curve are observed for the $K^{+}p$ data with $3 \leq P_{lab} < 50$ GeV/c and with $|t| > |t|_{max} = 0.5$ GeV^2. But all the data points for all the processes considered here with $P_{lab} \geq 50$ GeV/c lie on the same scaling curve as in II. The onset of scaling later in the energy scale for pp and $K^{+}p$ scattering and a little more deviation of the data for lower values of P_{lab} observed for $\bar{p}p$ and $\pi^{+}p$ scattering as compared to the analogous cases in II, may be due to the fact that the variable in II has been specifically designed to be applicable for all energies. But as we have already remarked, such a gain is possibly at the cost of the convergence for some values of χ. Even though scaling sets in a little later in the energy scale in $\bar{p}p$ and $\pi^{+}p$ scattering, and substantially later for pp and $K^{+}p$ scattering as compared to that of II, it is certainly better than the average scaling in the variable[14] $t\sigma_{tot}{}^2/\sigma_{el}$, and scaling in the variables[33] $t\sigma_{tot}$, at least as observed from Figs. 8, 10, 12, 14(a)–(c), and 16 in this paper, and Figs. 2(a)–3(b) of I. Cornille[28] has defined a class of scaling functions in which are included series in orthogonal polynomials including Laguerre, but excluding Hermite polynomails. In the present case we find that a series in Laguerre polynomials in the variable χ is a good candidate for scaling function for $s \to \infty$. From the analyses performed in this paper it is clear that scaling functions for various processes can be obtained with the help of (3.20) by replacing $P_n(2\chi)$ by $L_n(2\chi)$ and fitting the scaling curves. An alternative method would be to fit the data for a high value of P_{lab} above 50 GeV/c covering a sufficiently larger range of $|t|$. In this paper no attempt has been made to obtain scaling functions by actually fitting the data. Even without the knowledge of the scaling functions, it is possible to predict the values of $f(s,t)$ as a function of $|t|$ for any higher value of s from the knowledge of the scaling curve because of the simple structure of the variable $\chi(s,t)$. This problem, which will be discussed later in this section, will be attacked in a separate paper. In this respect the variable suggested in this paper has a definite advantage over that in II, by means of which no such predictions are possible.

B. Limitations of the method

Several limitations of this approach are summarized here. The first limitation of this approach is that the present approximation to the differential cross section fails and the simple picture of scaling hypothesized in this paper is spoiled if pole singularities contribute significantly. But experimental data near forward angles at high energies suggest that the absorptive part dominates scattering and the ratio of the real to the imaginary part is small. Since the absorptive part comes from cut contributions and poles contribute to the real part, the CPE suggested here is a good approximation to the cross-section ratio near forward angles. There is an enormous amount of literature which either proves[14,26–29,37] or hypothesizes[33] scaling by ignoring the real part completely as compared to the imaginary part. Even though poles themselves may not contribute significantly, their interference with the cut contributions may be significant away from forward angles. In our representation we have not been able to include such contributions.

Second, although it has been pointed out that the expansion in orthogonal polynomials in the mapped variable Z_0 (or equivalently in χ), for any physical value of s, is possible for all energies with an exponential weight function, the polynomials and hence their domains of convergence in the mapped planes are not the same for all energies.[25] In particular, there exists ambiguity and danger in using a Laguerre-polynomial expansion at finite energies because the coefficients in the expansion cannot be determined for lack of the correct physical region.[1] However, because of the choice of the exponential weight function, which comes in a natural way at asymptotic energies, the same formula for the slope parameter holds for all energies.

Third, it can be argued that the present model has been developed for scattering near forward angles and the representations (3.11) or (3.20) may not work for the data for larger values of $|t|$. There is no convincing reason why scaling is observed for the large angle data, but some heuristic plausibility arguments can be put forward. At larger angles the real parts may contribute significantly along with the absorptive part. As has been already mentioned earlier in this subsection, cut contributions to the real parts can still be represented by a part of the right-hand side of (3.11) or (3.20). In fact, one can adopt separate parabolic variables and construct CPE for the real and imaginary parts separately, as has been done by Chao.[4] But at high energies, the start of the cut for the real part is the same as that for the imaginary part and the mapping variables for both the parts are the same.[1,2] The alternative plausibility argument may be that real part effects may be negligible in the range of $|t|$ for which the data exhibit scaling. It is natural to believe that shorter-range forces may contribute to the absorptive part and the real part if it is also important. Although the distant branch point structures corresponding to shorter-range forces do not exist in our conformal mapping, their effects have been indirectly taken into account by conformal mapping by bringing the distant parts of the cuts to closer vicinity of the image of the forward direction in the mapped plane.

The fourth limitation of this method is that the scaling in the variable χ has not been derived from rigorous physical and mathematical reasonings, but rather scaling has been hypothesized. The main support in favor of a scaling hypothesis is the remarkable manner in which the cross-section-ratio data scale in χ. There exist other papers in which scaling in geometrical models has been either hypothesized or assumed.[33,34,39] But scaling of the cross-section-ratio data in other variables has been proved[26-30,37] rigorously using principles laid down by AFT.

C. Other applications of the method

The method outlined in this paper can be used to predict $f(s,t)$ as a function of $|t|$ at higher energies for elastic diffraction scattering processes. This method can also be used to account for the energy dependence and asymptotic behaviors of slope parameters for several inelastic diffractive and nondiffractive processes and to examine scaling of the cross-section-ratio data.

(*a*) *Predictions of $f(s,t)$ as a function of $|t|$*. The scaling variable χ constructed using conformal mapping and CPE in this paper is much simpler than the corresponding variable in II. Because of simplicity of the variable, it is possible to predict $f(s,t)$ for any higher value of s as a function of $|t|$ from the knowledge of the scaling curve obtained in this paper. Looking to the scaling curve for any of the processes the value of the ordinate $f(s,t)$ and the corresponding χ can be read from the graph. Since for large s and for $|t| \ll s$, within which scaling is observed at high energies, we can rewrite (2.26) as

$$w_0 = 1 - (t/t_R)\,, \tag{5.1}$$

and using (2.27) and (3.18) we can write

$$w_0 = \{\cosh[\chi/\alpha(s)]^{1/2}\}^2\,, \tag{5.2}$$

we get using (5.1) and (5.2)

$$|t| = t_R\{\sinh[\chi/\alpha(s)]^{1/2}\}^2\,. \tag{5.3}$$

For any of the processes considered, since the value of χ has been read out from the scaling graph corresponding to a known $f(s,t)$ value, $\alpha(s)$ is known from the fit to the slope-parameter data, and t_R is a known function of s, the value of $|t|$ corresponding to a given value of $f(s,t)$ is known from Eq. (5.3). In this manner a different χ and hence the value of $|t|$, for a given s, can be known corresponding to a different value of $f(s,t)$ and the curve of $f(s,t)$ vs $|t|$ can be plotted. In fact, this method can be adopted for any process, elastic or inelastic, in which scaling is exhibited by the cross-section-ratio data in the simple variable χ. It may be noted that such a prediction is not possible in the variable of II because of its complicated $|t|$ dependence, but such a prediction is possible in the variable of I since the variable proposed there is the same as the one proposed in this paper for high energies and values of $|t| \ll s$ (Ref. 35). Predictions of $f(s,t)$ as a function of $|t|$ will be carried out in a separate paper. Such predications can be verified by the results of furture experiments. Since the parameters in CPE are known to be stable against extrapolations to unknown regions,[5] scaling in χ and predictions of $f(s,t)$ as outlined here have definite advantages.

(*b*) *Applications to inelastic processes*. So far no rigorous proof based upon principles of AFT exists for scaling in inelastic nondiffractive processes, but recently, average scaling, in a weaker sense, has been proved and observed for the cross-section-ratio data for the inelastic charge-exchange processes[14] $\pi^- p \to \pi^0 n$ and $\pi^- p \to \eta n$. Foundations of model-independent results on scaling for elastic processes have been criticized and a generalized scaling law has also been proposed[15] which yields different scaling variables for both elastic and inelastic processes. Since the main requirements for the application of the present method is the Mandelstam analyticity of the s and $\cos\theta$ planes and the method has been generalized for asymmetric cut planes without developing any spurious cut, it will be very interesting to see if scaling of the cross-section-ratio data exists in a similar variable in inelastic charge-exchange scattering processes. It may be noted that it is very difficult to apply the method developed in II for inelastic processes, since that would require knowledge of real zeros, slope-parameter data even at lower energies, and both at forward and nonforward angles for the construction of the scaling variable. The present method is much simpler and more important is the fact that it does not introduce any spurious cut in the mapped plane.

The method developed in this paper has been applied to inelastic charge-exchange scattering processes such as $\pi^- p \to \pi^0 n$, $\pi^- p \to \eta n$, $K^+ n \to K^0 p$, $K^- p \to \bar{K}^0 n$, $K^+ p \to K^0 \Delta^{++}(1236)$, and $K^- n \to \bar{K}^0 \Delta^-$

in a separate paper.[19] It is found that the energy dependence of the slope parameters at high energies can be very well accounted for and the information on the asymptotic behaviors, and the existence of the entire functions can be usefully extracted by means of similar formulas developed here. In particular, scaling of the cross-section-ratio data are shown to be exhibited in a remarkable fashion.[19]

VI. CONCLUSION

From the results of the present paper it is very clear that the CPE proposed here is potentially useful in accounting for the energy dependence of the slope-parameter data for all the elastic diffraction scattering processes at high energies with less number of parameters, but without requiring any knowledge of zero trajectories, effective shapes of spectral functions, and at the same time without developing any spurious cut in the mapped plane. The formula developed for the slope parameter has been potentially useful in obtaining information on the asymptotic behaviors of forward slopes for several processes. Scaling of the cross-section-ratio data at high energies, in the variable χ, is exhibited in a remarkable fashion for all the elastic diffraction scattering processes considered. Our analysis yields further definite information that the cross-section ratios for the processes pp, $\pi^{\pm}p$, and $K^{\pm}p$ scattering are entire functions of the corresponding scaling variables. From the results of a subsequent paper,[19] we conclude that this method is very successful for similar purposes for several nondiffractive inelastic charge-exchange scattering processes also.

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$$\frac{du(w)}{dw} = \frac{1}{[(1-w^2)(1-k^2w^2)]^{1/2}}.$$

Presence of branch points in such conformal transformations has been discussed in text books. See, for instance, J. Mathews and R. L. Walker, *Mathematical Methods of Physics* (Benjamin, New York, 1970), Chap. 7, p. 206; F. Bowman, *Introduction to Elliptic Functions with Applications* (Dover, New York, 1961).

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[25]Let $\Gamma(s)$ be the length of the physical region for a given s in the Z_0 plane and $\gamma(s)=2\alpha\Gamma(s)$. If we know α, the orthogonal polynomials $\{P_n(x)\}$ occurring in (3.11), with $x=2\alpha Z_0$, can be constructed by using the orthogonality relation

$$\tfrac{1}{2}\int_0^{\gamma(s)} \exp(-x)P_n(x)P_m(x)dx=\delta_{nm}.$$

Since the length of the physical region varies with energy the nature of the polynomials also varies. However, for $s\to\infty$, $\Gamma(s)\to\infty$ and $\{P_n(x)\}\to\{L_n(x)\}$, the Laguerre polynomials in terms of which the expan-

sion (3.11) converges within the entire parabola in the Z_0 plane.

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[32]It has been stated on page 168 of Ref. 2 that the start of the left-hand cut, s_1, is determined by the root of the equation.

$$s-\Sigma+t_L(s)=0.$$

This has been also stated on p. 154 and used in Eqs. (10a) and (10b) of Ref. 1. But the above prescription is wrong since equations for $t_L(s)$ do not hold for all values of s. However, in the determination of parameters from data analysis throughout the rest of papers I and II, it has been assumed that s_1 is the same as the start of the left-hand cut of the whole amplitude. In the present paper we also assume that the s_1 is the same as the start of the left-hand cut of the whole amplitude. In the present paper we also assume absorptive part has the same left-hand cut as the whole amplitude in the s plane.

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[35]For high energies the formula for the slope parameter obtained in I is the same as that given by (4.2). As has been observed in I no effective shapes of spectral functions are needed to fit the high-energy data on forward slopes for pp scattering with $s>35$ GeV^2. Also for large s and for values of $|t|$ such that $4q^2+t_R(s) \gg |t|$, which inequality is satisfied by almost all the available data for pp scattering, the mapping variable Z in (2.2) which has been used in I is very nearly the same as Z_0. Therefore, the variable $\chi(s,t)$, as would be obtained using the formula (4.2) and Z_0, would be very nearly the same as the scaling variable of I at high energies. This would yield very nearly the same scaling curves for pp scattering as those obtained in I, if the present approach is applied.

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[38]From Eqs. (4.5) and (4.6)

$$d_1|_{\pi^+p}=0.100, \quad d_1|_{\pi^-p}=0.081.$$

But from Eqs. (32) and (33) of Ref. 2

$$d_1|_{\pi^+p}=0.1102, \quad d_1|_{\pi^-p}=0.0527.$$

Although asymptotically the same rate of growth is indicated from this analysis and from Ref. 2, quantitatively they do not appear to be approaching the same limit in Ref. 2. However, from the present analysis it is clear that both slopes can approach the same limit quantitatively, if the error in the value of d_1 is at least 0.01 for both scattering processes. In view of the large errors of the data for $s<40$ GeV^2, such an error is likely to occur in the parameters. The difference between the results of this analysis and those of Ref. 2 arises because inclusion of data at low energies in Ref. 2 might be obscuring information on the asymptotic behavior.

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