Comments on radiating fluid spheres in general relativity

Selçuk Ş. Bayin
Department of Physics, University of Victoria, Victoria V8W 2Y2, British Columbia, Canada
(Received 13 November 1979)

We discuss the boundary conditions for interior solutions of general-relativistic radiating fluid spheres which were found by Vaidya, and others which were found by us. We also present a class of new solutions that could be connected to an exterior solution analytically.

I. INTRODUCTION

The existence of objects with large energy output, either in the form of photons or neutrinos or both in some phases of their evolution, is well known. In this regard we have recently studied radiating fluid spheres in general relativity where the effects of changing gravitional fields of the matter as well as the forces on the matter due to escaping radiation were considered, and four new analytic solutions corresponding to interiors of such objects were given.[1] The energy-momentum tensor we used was that for a perfect fluid plus the energy-momentum tensor for radially expanding radiation. The second tensor was originally derived by Tolman[2] and generalized to curvilinear coordinates by Vaidya.[3] Even though this tensor is derived for plane-polarized radiation it is also true on the average for incoherent unpolarized radiation, both for photons and neutrinos.[4]

In general, to solve the problem of radiating fluid spheres in general relativity we need two auxiliary relations to add to the field equations: One corresponding to an equation of state, and the other representing the law of energy generation within the object. However, to obtain analytic solutions we used the method discussed by us.[1] One problem we faced concerned the boundary conditions. In general, the coordinate system in which it is advantageous to obtain analytic interior solutions is not so useful in representing the exterior solution.

In this short communication we discuss boundary conditions and present a class of new solutions that could be connected to a suitable exterior solution analytically.

II. BOUNDARY CONDITIONS

The metric used by Vaidya[5] and us[1] is of the form

$$ds^2 = -h(r)^2 m(t)^2 dr^2 - l(r)^2 m(t)^2 (r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + f(r)^2 g(t)^2 dt^2 , \qquad (2.1)$$

where $e^\alpha = h^2 m^2$, $e^\beta = l^2 m^2$, and $e^\gamma = f^2 g^2$ in Vaidya's notation. Note that in general, time-dependent, spherically symmetric space-times can be reduced to a form with only two unknowns. However, introducing this additional freedom allows us to separate the field equations in r and t and give the following differential equations to be solved[1]:

$$h'\left(-\frac{f'}{f} - \frac{l'}{l} - \frac{1}{r}\right) = -\left(\frac{l''}{l} - \frac{l'^2}{l^2} + \frac{f''}{f} - \frac{f'l'}{fl} - \frac{f'}{fr} - \frac{1}{r^2}\right)h + \left(s\frac{f'}{f^2}\right)h^2 - \left(\frac{1}{l^2 r^2}\right)h^3 \qquad (2.2)$$

and

$$2\dot{m}/g = s . \qquad (2.3)$$

As we mentioned, one has to supply two relations among f, h, and l corresponding to an equation of state and the law of energy generation. By assuming various relations we gave four solutions in the previous article.

From (2.2) it is also apparent that if we assume

$$\frac{f'}{f} + \frac{l'}{l} + \frac{1}{r} = 0 , \qquad (2.4)$$

the system will be solved immediately, giving h and f in terms of l as

$$h(r) = \left(\frac{l'}{l} + \frac{1}{r}\right)\left(-\frac{l^2 r^2}{2}\right)\left[\frac{slr}{c_0} \mp \left(\frac{s^2 l^2 r^2}{c_0^2} - \frac{8}{l^2 r^2}\right)^{1/2}\right], \qquad (2.5)$$

$$f(r) = c_0 / lr , \qquad (2.6)$$

where c_0 is an integration constant and s is a separation constant. Hence, if we assume $l(r)$ as our second relation all the physical quantities will be determined.

This solution has to be matched to Vaidya's radiating exterior solution, which could be given as

$$ds^2 = -\left(1 - \frac{2M}{r'}\right)^{-1} dr'^2 - r'^2 (d\theta^2 + \sin^2\theta d\phi^2) + \frac{\dot{M}^2}{F^2}\left(1 - \frac{2M}{r'}\right) dt^2 , \qquad (2.7)$$

where $F(M)$ is an arbitrary function of M given by

$$M'\left(1-\frac{2M}{r'}\right)=F(M) \tag{2.8}$$

and M is a function of r' and t.

This metric could also be expressed in a more convenient form by introducing a retarded time coordinate u as[6]

$$ds^2=\left[1-2\frac{M(u)}{r'}\right]du^2+2du\,dr' - r'^2(d\theta^2+\sin^2\theta\, d\phi^2)\,. \tag{2.9}$$

We will use the form obtained by replacing $u=t-r'$ in (2.9) which is[7]

$$ds=\left(1-\frac{2M}{r'}\right)dt+2\left(\frac{2M}{r'}\right)dt\,dr'-\left(1+\frac{2M}{r'}\right)dr'^2 - r'^2(d\theta^2+\sin^2\theta\, d\phi^2)\,. \tag{2.10}$$

The line element (2.1) can be transformed into the above form, most easily, without the necessity of an integration factor as follows: If we define a new radial marker as

$$r'=l(r)m(t)r\,, \tag{2.11}$$

the metric (2.1) will be cast into the form

$$ds=-\frac{h^2}{(l'r+l)^2}(dr')^2-r'^2d\theta^2-r'^2\sin^2\theta\, d\phi^2 +\left[f^2g^2-\frac{h^2l^2r^2\dot{m}^2}{(l'r+l)^2}\right]dt^2+\frac{2h^2lr\dot{m}}{(l'r+l)^2}dr'\,dt\,, \tag{2.12}$$

where from (2.11) r as a function of r' has to be substituted into the metric (2.12). This metric could further be transformed into the form of (2.7) or (2.9), but that would require evaluating an integration factor, which might not be possible to do analytically. Note that after the transformation (2.11) space and time coordinates are now mixed in the new metric.

From the field equation (2.2) it is seen that in the solutions there will be five independent integration constants. $f(r)$ and $l(r)$ each will have two, while $h(r)$ will have only one constant. Requiring the continuity of the metric, the continuity of the pressure, and the luminosity at the surface, the five integration constants can be determined in terms of the radius, the mass, and the luminosity.

The radius R of the star is defined as the radius of the surface over which pressure drops to zero. The definition of luminosity is somewhat difficult, due to the fact that σ in $T^{\mu\nu}_{\rm rad}$ is only determined up to a Lorentz transformation. Following Misner *et al.*[7] we define σ measured by an observer moving with the four-velocity v^μ so that $w^\mu=(1,1,0,0)$ as

$$\sigma=v^\mu v^\nu T^{\mu\nu}_{\rm rad}\,, \tag{2.13}$$

where $T^{\mu\nu}_{\rm rad}=\sigma w^\mu w^\nu$ and $w_\mu w^\mu=0$. From this it can be shown that the luminosity measured by an observer at rest at infinity is just

$$L_\infty=-\frac{dM}{du}\,. \tag{2.14}$$

The luminosity at finite r' will be

$$L(r',u)=4\pi r'^2\sigma=-\frac{dM}{du}\frac{1}{(\gamma+v^1)^2}\,, \tag{2.15}$$

where

$$\gamma=\left(1+(v^1)^2-2\frac{M}{r'}\right)^{1/2},\quad v^1=\frac{dr'}{d\tau}\,. \tag{2.16}$$

If we transform u to the normal time coordinate t defined by $u=t-r'$, we get

$$L(r',t)=-\frac{dM}{dt}[1+v^1(\gamma+v^1)]\frac{1}{(\gamma+v^1)^2}\,. \tag{2.17}$$

This has to be matched to the luminosity obtained from the interior solution at R'. We can define the interior luminosity in analogy to (2.15) as

$$L_{\rm int}(r',t)=4\pi r'^2\sigma_{\rm int}(r',t)\,. \tag{2.18}$$

σ in terms of the old coordinates was given as[1]

$$\sigma(r,t)=-e^{-(3\alpha-\gamma)/2}T_1^{\ 4}\,, \tag{2.19}$$

where $w^1=1$ according to our normalization and $T_1^{\ 4}$ will be obtained from the field equation as[1]

$$8\pi T_1^{\ 4}(r,t)=sf'/f^3g^2\,, \tag{2.20}$$

where $r'=l(r)m(t)r$.

Hence, continuity of the luminosity at the surface will be achieved by the continuity of $\sigma(r',t)$:

$$\sigma_{\rm int}(R',t)=\frac{L(R',t)}{4\pi R'^2}\,. \tag{2.21}$$

$L(R',t)$ is the luminosity at the surface and can be given in terms of the luminosity observed at infinity as

$$L(R',t)=L_\infty[1+v_s^{\ 1}(\gamma_s+v_s^1)]\frac{1}{(\gamma_s+v_s^1)^2}\,, \tag{2.22}$$

where a subscript s denotes the value of that quantity at the surface. The velocity of the observer will simply follow from definition (2.13) and normalization of w^μ, as $w^\mu=(1,1,0,0)$,

$$\sigma=v^\mu v^\nu\sigma w_\mu w_\nu\,. \tag{2.23}$$

Considering radially moving observers this gives two conditions on v^μ:

$$-v^1+v^4=1 \tag{2.24}$$

and

$$v^1 v_1 + v^4 v_4 = 1\,. \tag{2.25}$$

This determines v^μ as

$$v^1 = \frac{-g_{44} - g_{41} + (g_{41}{}^2 - g_{44}g_{11} + g_{11} + g_{44} + 2g_{41})^{1/2}}{g_{11} + g_{44} + 2g_{41}}\,, \tag{2.26}$$

$$v^4 = \frac{+g_{41} + g_{11} + (g_{41}{}^2 - g_{44}g_{11} + g_{11} + g_{44} + 2g_{41})^{1/2}}{g_{11} + g_{44} + 2g_{41}}\,, \tag{2.27}$$

at infinity $v^1 \to 0$ and $v^4 \to 1$.

In our paper[1] we have used a different normalization of w^μ. However, we prefer this new definition because it is easier to evaluate the integration constants in terms of the luminosity observed at infinity. Note that with the old definition, luminosity for the interior is independent of time. For a different way of handling boundary conditions we refer the reader to Misner[8] and Misner and Sharp.[9] We have also shown that in comoving coordinates the time dependence of the pressure and the density is unique for separable solutions and has the form e^{st} where s is a separation constant.

For the sake of completeness we would like to define a mass function within the fluid as[10]

$$M(r,t) = m(t)^3 \int l^2[(l'r + l)(\rho + \sigma E^2/f^2g^2) + \tfrac{1}{2}s\sigma l r E^2 h/f^3g^2]4\pi r^2 dr\,.$$

Partial derivatives of $M(r,t)$ are given as

$$M' = 4\pi R^2(T_4{}^4R' - T_1{}^4\dot{R})\,, \quad \text{where } R = lmr$$

and

$$\dot{M} = 4\pi R^2(T_1{}^1\dot{R} - T_4{}^1R')\,.$$

Finally, using Eqs. (2.5) and (2.6) we give a new solution that has a sufficient number of arbitrary constants and can be connected to the exterior solution analytically for special cases. We take

$$l(r) = \frac{1}{r^{k+1}}(a_0 + a_1 r^q)^p\,,$$

which leads to

$$f(r) = \frac{c_0 r^k}{(a_0 + a_1 r^q)^p}\,,$$

$$h(r) = -\frac{s(a_0 + a_1 r^q)^{3p}}{2c_0 r^{3k+1}}\left[\frac{Ar^q + B}{(a_0 + a_1 r^q)} + 1\right] \times \left[1 \mp \left(1 - \frac{8c_0{}^2 r^{4k}}{s^2(a_0 + a_1 r^q)^{4p}}\right)^{1/2}\right],$$

where

$$A = a_1 pq - a_1(k+1)\,, \quad B = -a_0(k+1)\,.$$

We have a_0, a_1, q, p, k, c_0, and s as constants; five of these will be determined from the boundary conditions leaving the rest arbitrary.

I would like to thank Professor F. I. Cooperstock for a critical reading of the manuscript.

[1] S. Ş. Bayin, Phys. Rev. D 19, 2838 (1979).

[2] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, New York, 1934), p. 272.

[3] P. C. Vaidya, Proc. Indian Acad. Sci. A33, 264 (1951).

[4] J. B. Griffiths, Gen. Relativ. Gravit. 5, 453 (1974).

[5] P. C. Vaidya, Phys. Rev. 83, 10 (1951).

[6] P. C. Vaidya, Astrophys. J. 144, 943 (1966).

[7] R. W. Lindquist, R. A. Schwartz, and C. W. Misner, Phys. Rev. 137, B1364 (1965).

[8] C. W. Misner, Phys. Rev. 137, B1360 (1965).

[9] C. W. Misner and D. H. Sharp, Phys. Rev. 136, B571 (1964).

[10] M. E. Cahill and G. C. McVittie, J. Math. Phys. 11, 1382 (1970).