Noninstantaneous $O(v/c)$ relativistic effects in bound states and a covariant Schrödinger equation

Paul M. Fishbane and J. M. Namyslowski*
Physics Department, University of Virginia, Charlottesville, Virginia 22901
(Received 12 June 1979)

The noninstantaneous and nonlocal interactions which follow from Weinberg-type dynamics are found to produce v/c relativistic corrections to potentials which are nonrelativistic limits of certain field-theoretic dynamics. Such corrections may be applicable in, for example, phenomenological models of charmonium. These interactions are driving terms in a covariant three-dimensional two-body integral equation derivable from field theory. For bound systems this equation is a fully covariant Schrödinger equation, with spacelike relative momentum and a proper angular resolution. We study this equation, including its systematic relativistic corrections, in various limits based on scalar-particle-exchange dynamics. We compare and contrast it to a related but different three-dimensional equation derivable from field theory which represents an equal-time projection. We also comment on other approaches to the relation between relativistic and nonrelativistic dynamics.

I. INTRODUCTION

The connections between relativistic quantum field theory and the nonrelativistic Schrödinger equation have always been an issue of great interest. The recent activity in charmonium[1] and other quarkonia, and in particular the necessity of understanding relativistic corrections to a nonrelativistic potential-model phenomenology, has made this issue of even greater importance. Independently of this, the extreme accuracy of measurement[2] of the positronium spectrum has demanded a systematic account of the theoretical relativistic corrections to the binding in high orders of the coupling.

Although we know the Lagrangian both of electrodynamics and of chromodynamics, and although both charmonium and positronium are essentially nonrelativistic, the similarities between the two cases mentioned above end there. The relevance of perturbation theory is doubtful, and we have at best only hints of how chromodynamics works at the long distances which govern the confinement of quarks and hence the dynamics of quarkonium. These hints motivate the choice of the nonrelativistic potentials which form the basis of quarkonium phenomenology. However, as we shall discuss in some detail, such a choice leads to unique relativistic corrections[3] only if the noninstantaneous aspects (i.e., the time dependence) and the nonlocal aspects of this potential are specified.

This general approach does not lead to a unique and well-defined set of kernels ordered by the coupling constant, as is the case in positronium. For positronium[4] perturbation theory is a useful tool, and the system is well enough understood to provide a useful cross check for our approach.

In quantum field theory the full Bethe-Salpeter equation[5] is the proper tool for dealing with the bound-state problem. Beyond the interpretational difficulties connected with the relative time variable, the necessity of performing the Wick rotation for an angular momentum decomposition and the existence of anomalous solutions[6] present problems. For these reasons, many methods have been developed over the years for reduction of the Bethe-Salpeter equation from a four-dimensional to a three-dimensional space.

In Sec. II we give a partial critical review of these methods, singling out the equal-time method proposed by Logunov and Tavkhelidze.[7] This method is truly three-dimensional; that is, correction terms contain no four-dimensional pieces. As we shall see, an equal-time interaction and equation, which follows from straightforward integration over energies of the Bethe-Salpeter equation, is quite different from an instantaneous interaction and equation, where time dependence is essentially dropped. This distinction is a crucial one and one which can lead to errors if not properly observed. Equal-time interactions are necessarily noninstantaneous.

Our focus is on the Green's functions, which are the primary objects in quantum field theory. From the full and free equal-time Green's function follow the equal-time interactions and t matrix.

Section III contains a systematic treatment of the equal-time method, with a primary example of scalar exchange as an underlying dynamics. The resulting equation, while of interest, has certain disadvantages. Among these, we have that, first, many of the interpretational disadvantages remain in that there does not appear to be a simple graphical version of the theory. Second, the nonspacelike character of the relative momentum makes an

angular analysis valid only for a given frame. However, a simple variant on the procedure, namely, integration over the minus rather than zero component of the momentum, i.e., an "equal-x_+ component" theory, eliminates these disadvantages without great cost by giving the Weinberg equation.[8] We call the resultant on-mass-shell dynamics *luminal* (referring to the fact that we project on the null plane), treat it in Sec. IV, and give a reformulation in Sec. V. This reformulation is a covariant form of the Schrödinger equation itself. The potential for this "Weinberg-Schrödinger" equation is generally nonlocal and noninstantaneous (energy dependent).

Systematic calculation of relativistic corrections to an infinite-mass nonrelativistic limit, where the potential becomes local and instantaneous, are possible and are discussed in Sec. VI. In addition to new $O((v/c)^2)$ corrections, we find $O(v/c)$ corrections whose origin is the noninstantaneous nature of the potential. Finally in Sec. VII we review the main results and discuss various questions which have been raised.

II. FROM FIELD THEORY TO POTENTIALS

In this section we review various attempts to reduce the bound-state problem in local quantum field theory (QFT) to a nonrelativistic form with relativistic corrections. Generally speaking the starting point for such a problem is an appropriate equation and its kernel; in nonrelativistic physics this is of course the Schrödinger equation and its corresponding potential. Systematic inclusion of relativistic corrections, however, requires us to go beyond the Schrödinger equation.

As shown by Gell-Mann and Low,[9] solution of the full Bethe-Salpeter (BS) equation is the correct solution to the bound-state problem of a QFT, where "full" means representation of both the particle propagators and irreducible kernels by the full Green's functions as well as use of an infinite series of these kernels. The resulting bound-state wave function depends on the relative energy-momentum (or equivalently the relative time-position), which describes the relative motion of the constituents. In practice, of course, we must approach the BS equation with various approximations in hand, such as free-particle propagators or a truncated set of kernels. To assess such approximations it is useful to have some point of reference. The Schrödinger equation can provide such criteria, as well as the reduction[10] to the Dirac or Klein-Gordon equations for a relativistic particle in a static field, i.e., the two-body problem with one body infinitely massive, or the case of two equally massive constituents.

The presence of the relative energy p_0 in the argument of the BS wave function makes it difficult to compare with the nonrelativistic wave function in which this variable is absent. Moreover, associated with this is the fact that the BS equation has extra anomalous solutions[6] which have no nonrelativistic counterparts. It is therefore important to have a way to eliminate this extra variable to make contact with the nonrelativistic problem. Salpeter and Bethe, in their classic paper, considered such a possibility by the definition of a one-time wave function,

$$\phi(\vec{p}) = \int_{-\infty}^{\infty} dp_0 \psi(\vec{p},p_0)\,, \tag{2.1}$$

where $\psi(\vec{p},p_0)$ is the full BS wave function. By itself, this is all that is required if the irreducible kernel is independent of p_0. Generally speaking this is not the case. Moreover, simply dropping the p_0 dependence in the kernel is correct neither in principle nor in practice. (For example, Salpeter and Bethe[5] found that in the loosely bound deuteron, the effect of this procedure was as large as 10% in an effective coupling constant or 50% in the binding energy.[11])

Salpeter[5] has studied the hydrogen atom as the solution to the BS equation. (This system, and in particular the one-photon-exchange kernel, has been a testing ground for many of the methods reviewed here.) By using the radiation gauge, he separated photon exchange into an instantaneous Coulomb part—independent of p_0—and a noninstantaneous transverse part. Schwinger[12] also gives a systematic account of the relativistic corrections to the H spectrum in a different formulation: the use of an equal-time free Green's function analogous to the one-time wave function in Eq. (2.1), and an intermediate Green's function obtained from the equal-time free Green's function and the instantaneous part of the interaction. The noninstantaneous part of the interaction is treated separately (see action 5–3 in Ref. 12).

What is relevant for our purposes in Salpeter[5] and Schwinger[12] is the observation that the noninstantaneous part of the interaction contributes to the binding $O(\alpha^4 m(m/M)^2)$ and $O(\alpha^5 m(m/M))$, where m and M are the electron and proton masses. This should be compared[12] to the spectrum from the Dirac equation with the Coulomb potential,

$$\begin{aligned} E_n &= m\left[1-\left(\frac{Z\alpha}{n}\right)^2\right]^{1/2} \\ &= m\left[1-\frac{1}{2}\left(\frac{Z\alpha}{n}\right)^2-\frac{1}{8}\left(\frac{Z\alpha}{n}\right)^4+\cdots\right]. \end{aligned} \tag{2.2}$$

Since the corresponding Schrödinger equation would give only the second term on the right-hand

side, the first relativistic corrections are $O(\alpha^4 m)$. Thus while in H the noninstantaneous part gives only a small correction, it is as important as the Coulomb part in computing the relativistic effects for equal-mass systems such as positronium. This fact has also been emphasized by Feldman *et al.*[4] and by Feinberg,[3] the latter in the context of speculations about quantum chromodynamics (QCD). In particular, for QCD extension of the above ideas may lead to the following interrelated problems: (i) An effectively large coupling constant may accentuate the role of all relativistic corrections, both instantaneous and noninstantaneous (nonlocal). (ii) The nonrelativistic ($m \to \infty$) limit of kernel iterations may not be iterations of the nonrelativistic kernel. (iii) The bound states of QCD may not be threshold bound states, which would allow Foldy-Wouthuysen transformations to determine a nonrelativistic Hamiltonian operator.

For noninstantaneous kernels, the required extension of the one-time wave function was provided by Logunov and Tavkhelidze.[7] It is based on the systematic equal-time treatment of both the free-particle and full Green's functions. We shall present details as well as applications of this approach in Sec. III. Here we note the following comments: (i) The equal-time approach is equivalent to the full BS solution if one includes all terms in the equal-time interaction, which is general is different from an instantaneous interaction. (ii) Just as for the BS photon ladder approximation, the equal-time photon ladder has the drawback that in the static limit, when one of the two constituents has infinite mass, it does not produce the Dirac equation result (or for a scalar exchange the Klein-Gordon equation result). (iii) For particles with spin the inverse of the equal-time Green's function does not exist.[13] To circumvent this problem[12] one can use the Breit-type free Hamiltonian, but one then must introduce an appropriate unitary transformation[12,27] on the states.

There are several more ways to deal with the relativistic bound system. Perhaps the method most closely related to the Logunov-Tavkhelidze method is that of Tamm and Dancoff,[14] which has continued[15] to provide useful insight.

A second way is to develop relativistic quantum dynamics as in recent work by Leutwyler and Stern.[16] They subdivide their approach into five subclasses, out of which the instant form, the point form, and front form are well known.[17]

A third way is to approach the bound-state problem by the use of old-fashioned perturbation theory, certainly satisfactory if one knows the Lagrangian and from it develops a definite sequence of kernels. Kadyshevsky[18] and his collaborators have studied this approach extensively. The major difficulty here lies in the very large number of diagrams in a given order, which, however, can be dramatically reduced by evaluation in the infinite-momentum frame.[8,19] This leads to the Weinberg equation,[8] which has many advantages for dealing with the relativistic bound system. Details and discussion of this scheme are given in Sec. IV.

A fourth way is the use of the BS equation with one intermediate particle always on the mass shell. This was originally proposed by Gross,[20] and more recently applied by Lepage[21] to the positronium spectrum. Such a procedure leads to the correct Klein-Gordon or Dirac equation in the static limit. While this scheme is certainly systematic, one must keep in mind that the equal-mass constituents are not treated symmetrically. In addition, one saves time-reversal invariance only if both positive and negative-energy states of the off-shell constituent are included. Finally, the relative momentum is no longer necessarily a spacelike four-vector, so that the angular momentum analysis is restricted to one frame of reference, usually taken to be the center-of-mass system. Thus manifest Lorentz invariance is lost, in spite of the fact that the mass-shell condition for one particle is itself invariant.

Blankenbecler and Sugar[22] developed a well-known method which is superficially like that of Logunov and Tavkhelidze.[7] However, in addition to a rather remote connection to field theory, off-shell continuation in this approach is not uniquely defined.[23,24] Woloshyn and Jackson[23] and Love[25] also have approaches which are however not truly three dimensional; rather there are correction terms which are four dimensional in nature. Finally we mention the work of Partovi,[26] whose approach seems to be applicable only for fermionic systems.

The recent literature[3] on the charmonium spectrum and its relativistic corrections contains many different applications of the ideas discussed above. Many of these have been quite valuable, although we shall concentrate on some of the negative aspects in the brief review of these. We emphasize that this criticism (as the applications themselves) is not based on any fundamental understanding of the QCD Lagrangian, but rather on lessons learned from model Lagrangians and in particular from QED.

In a formalism close to Schwinger's Celmaster and Henyey[27] argue that transverse (noninstantaneous) pieces of an effective dressed gluon exchange are unimportant to the first order of relativistic correction. This seems inconsistent with the results of Refs. 4, 5, and 12, which give $\alpha^4 m$ corrections to the binding energy in positronium

due to the transverse photon. [This originates in Ref. 4 from the noninstantaneous part of the interaction. See also the remark below Eq. (3.16) in Ref. 4.] This is not simply a matter of gauge choice, since the gauge used in Ref. 27 is the Coulomb gauge for one-photon exchange, and this is the gauge used in Ref. 12.

One could similarly argue against the nonappearance of noninstantaneous pieces for various phenomenological applications to quarkonium that exist in the literature.[1,3] In such cases it would not be possible to suggest just what these pieces should be like, since the starting point is the potential rather than a covariant interaction. It may or may not be justified, according to the field theory, to tailor off-shell continuations, or equivalently the choice of energy components, which in general lead to nonlocal or noninstantaneous interactions, to leave only local relativistic corrections, as in Grome's work.[28] For example, the gauge invariance of QED seems to make this possible up to $O((v/c)^2)$. On the other hand, as we shall see, a scalar field theory has nonlocality in the irreducible kernel at the level of the *first* relativistic correction. Thus we henceforth ask the reader to be aware of the potential importance of this kind of contribution.

III. EQUAL-TIME INTERACTION

The idea of an equal-time interaction can be traced to the classic paper[5] of Salpeter and Bethe, with the one-time wave function of Eq. (2.1). As stated in Sec. II, Schwinger[12] introduced an equal-time free Green's function, which he called a "one-time" Green's function. This name may be misleading, in that in the four-point Green's function the two-body relative times are zero in both the initial and final states; however, the common initial time differs in general from the common final time, and we prefer to refer to this situation as "equal time." Moreover, if in a covariant formalism we want to consider a Green's function corresponding to a given fixed value of the total energy of the two-body system, e.g., the bound-state problem, then more than one time is necessary, since a δ function in energy implies infinitely many values of both the initial and final common time.

Schwinger's[12] Green's function was defined only in terms of the instantaneous interaction. We are instead interested in the equal-time version of the full BS Green's function, given by Logunov and Tavkhelidze.[7] We shall review their derivation here and work out some examples which illustrate various points of interest. There are various forms of their procedure, all in principle equally good, all leading to genuinely three-dimensional equations. These equations, or their corresponding interaction kernels are frame dependent; normally one works in the center-of-mass frame, but by working in the infinite-momentum frame one can show[29] that one recovers the Weinberg equation from this procedure.

To derive the equal-time Green's function, we start from the BS equation for the full four-point Green's function G, written in operator form as

$$G = G_0 - G_0 I G , \tag{3.1}$$

where G_0 is the free Green's function, written as the product of two free particle propagators, and I represents some irreducible kernel. Let us denote the initial, intermediate, and final four-momenta as p_1, p_2; k_1, k_2; p_1', p_2', respectively. Figure 1 shows the equation and labels the lines. It is useful to define total and relative momenta as follows:

$$P = p_1 + p_2 , \quad p = (m_1 + m_2)^{-1}(m_2 p_1 - m_1 p_2), \tag{3.2a}$$

$$p_1 = p + m_1(m_1 + m_2)^{-1} P , \quad p_2 = -p + m_2(m_1 + m_2)^{-1} P , \tag{3.2b}$$

and similarly for the k_i and the p_i'.

From Eq. (3.1) we get after extraction of total four-momentum conservation $P = K = P'$ the momentum-space equation

$$\begin{aligned} \langle p' | G(P) | p \rangle = {} & (p_1^{\,2} - m_1^{\,2} + i\epsilon)^{-1} \\ & \times (p_2^{\,2} - m_2^{\,2} + i\epsilon)^{-1} \delta^{(4)}(p - p') \\ & - (p_1'^{\,2} - m_1^{\,2} + i\epsilon)^{-1} (p_2'^{\,2} - m_2^{\,2} + i\epsilon)^{-1} \\ & \times \int d^4k \langle p' | I | k \rangle \langle k | G(P) | p \rangle , \end{aligned} \tag{3.3}$$

where we have defined

$$\langle p' | G(P) | p \rangle \delta^{(4)}(P - P') = \langle p_1' p_2' | G | p_1 p_2 \rangle \tag{3.4}$$

and similarly for I.

We now define an equal-time Green's function $\tilde{G}$ which depends on relative three-momenta,

$$\langle \tilde{\mathbf{p}}' | \tilde{G}(P) | \tilde{\mathbf{p}} \rangle \equiv \int dp_0 dp_0' \langle p' | G(P) | p \rangle , \tag{3.5}$$

and similarly for the free Green's function. The name "equal time" refers to the fact that the Fourier transform of Eq. (3.5) corresponds to setting the initial and final relative times each equal to

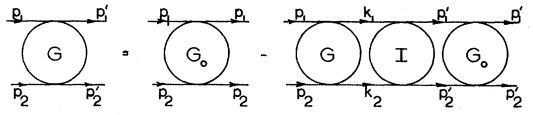

FIG. 1. Bethe-Salpeter equation for the four-point Green's function, with interaction I as the kernel.

zero. In position space this means

$$\tilde{G}(t',t) = G(t_1' = t_2' = t', t_1 = t_2 = t). \tag{3.6}$$

The name "one-time" may thus be misleading.

Consider the iterative solution of Eq. (3.1),

$$G = G_0 - G_0 I G_0 + G_0 I G_0 I G_0 - \cdots . \tag{3.7}$$

Applying the equal-time operation,

$$\tilde{G} = \tilde{G}_0 - [G_0 I G_0]^{\sim} + [G_0 I G_0 I G_0]^{\sim} - \cdots \tag{3.8}$$

(where for typographical reasons we use the notation $[A]^{\sim} \equiv \tilde{A}$), we define the equal-time interaction $\tilde{I}$ as the kernel such that $\tilde{G}$ is obtained from $\tilde{G}_0$ and $\tilde{I}$ by a BS-type equation,

$$\tilde{G} = \tilde{G}_0 - \tilde{G}_0 \tilde{I} \tilde{G}, \tag{3.9}$$

which will have the iterative solution

$$\tilde{G} = \tilde{G}_0 - \tilde{G}_0 \tilde{I} \tilde{G}_0 + \tilde{G}_0 \tilde{I} \tilde{G}_0 \tilde{I} \tilde{G}_0 - \cdots . \tag{3.10}$$

Comparison of Eqs. (3.10) and (3.8) gives an algorithm for calculating $\tilde{I}$ from I to any order of the interaction. In lowest order we find

$$\tilde{I} = \tilde{G}_0^{-1} [G_0 I G_0]^{\sim} \tilde{G}_0^{-1}, \tag{3.11}$$

or more explicitly in terms of the matrix elements,

$$\langle \vec{p}' | I | \vec{p} \rangle = [\langle \vec{p}' | G_0(P) | \vec{p}' \rangle]^{-1} \Big[\int dp_0 dp_0' (p_1'^2 - m_1^2 + i\epsilon)^{-1} (p_2'^2 - m_2^2 + i\epsilon)^{-1} \langle p' | I | p \rangle (p_1^2 - m_1^2 + i\epsilon)^{-1} \times (p_2^2 - m_2^2 + i\epsilon)^{-1} \Big] [\langle \vec{p} | \tilde{G}_0(P) | \vec{p} \rangle]^{-1}, \tag{3.12}$$

where

$$\langle \vec{p} | \tilde{G}_0(P) | \vec{p} \rangle = \int dp_0 dp_0' (p_1^2 - m_1^2 + i\epsilon)^{-1} (p_2^2 - m_2^2 + i\epsilon)^{-1} \delta^{(4)}(p' - p). \tag{3.13}$$

By integrating over poles in p_0 in Eq. (3.13), we can get an explicit expression for the matrix element of $\tilde{G}_0$. For example, for the case $m_1 = m_2 = m$ in the center-of-mass system where $P = (W, \vec{0})$,

$$\langle \vec{p} | \tilde{G}_0(P) | \vec{p} \rangle = -(2\pi i) E^{-1} (W^2 - 4E^2)^{-1} \delta^{(3)}(\vec{p}' - \vec{p}), \tag{3.14}$$

where

$$E = +(\vec{p}^2 + m^2)^{1/2}. \tag{3.15}$$

Similar techniques will be used to evaluate $\langle \vec{p}' | \tilde{I} | \vec{p} \rangle$ once I itself is specified.

Let us now consider some further applications of the above formalism. To start, suppose $\langle p' | I | p \rangle$ is independent of p_0' and p_0, as, for example, for the instantaneous Coulomb part of one-photon exchange. Then from Eq. (3.13) we see that in operator language

$$\begin{aligned} \tilde{I}^{\text{inst}} &= \tilde{G}_0^{-1} [G_0 I^{\text{inst}} G_0]^{\sim} \tilde{G}_0^{-1} \\ &= \tilde{G}_0^{-1} \tilde{G}_0 I^{\text{inst}} \tilde{G}_0 \tilde{G}_0^{-1} \\ &= I^{\text{inst}}. \end{aligned} \tag{3.16}$$

This is the case Schwinger studied.[12]

As a second example of $\tilde{I}$, we evaluate the case of scalar-meson exchange of mass μ, i.e.,

$$\langle p' | I | p \rangle = g^2 [(p_0' - p_0)^2 - (\vec{p}' - \vec{p})^2 - \mu^2 + i\epsilon]^{-1}, \tag{3.17}$$

for equal-mass ($m_1 = m_2 = m$) scattering in the center-of-mass frame. In addition to W, E, and E', defined above, we also define

$$\omega \equiv [(\vec{p}' - \vec{p})^2 + \mu^2]^{1/2}. \tag{3.18}$$

The pole structure for $\langle \vec{p}' | \tilde{I}(W) | p \rangle$ is more complicated, but not qualitatively different from that of $\tilde{G}_0$, and we find, using (3.12) and (3.14),

$$\langle \vec{p}' | I(W) | \vec{p} \rangle = -\tfrac{1}{4} g^2 (\omega W^2)^{-1} \{ (E + E' + \omega)^{-1} [(W - 2E)(W + 2E') + (W + 2E)(W - 2E')] + (W + E + E' + \omega)^{-1} (W - 2E')(W - 2E) - (W - E - E' - \omega)^{-1} (W + 2E')(W + 2E) \}. \tag{3.19}$$

This should be compared to the "instantaneous" analog of scalar exchange, Eq. (3.17) with $p_0' = p_0 = 0$,

$$\langle \vec{p}' | I^{\text{inst}} | \vec{p} \rangle = -g^2/\omega^2.$$

In addition, there is little resemblance between Eq. (3.19) and the corresponding old-fashioned perturbation-theory expression, which is of course associated with the fact that the equal-time inter-

action is not an on-mass-shell interaction. An important feature of $\langle \vec{p}' | \tilde{I}(W) | \vec{p} \rangle$ is the nonlocality, expressed in the separate dependence on E and E'. This is not a nonlocality which can be easily argued away by approximation, since in the integral equation an infinite range of all three-momenta is allowed, and the importance of the terms in Eq. (3.19) is comparable in different regions.

In addition, $\langle \vec{p}' | \tilde{I}(W) | \vec{p} \rangle$ is noninstantaneous, i.e., its Fourier transform has time dependence, manifested in the dependence on the total energy. This is connected with the cluster decomposition property present in field theory,[30] which for a two-body problem means that at infinite separation in space or time the two constituents do not interact. An instantaneous interaction violates this property in an arbitrary frame of reference, even if it is forced to hold true in one particular frame.

Finally let us consider the nonrelativistic limit of this example, by which we mean here $m \to \infty$, while all other quantities, including the binding energy B and the three-momenta $|\vec{p}|$ and $|\vec{p}'|$ remain finite. Then $W = 2m - B \to 2m$, and

$$\langle \vec{p}' | \tilde{I}(2m) | \vec{p} \rangle \to -g^2/\omega^2 . \tag{3.20}$$

At the same time the reader will recall our argument above that all regions of $\vec{k}$ in the integration may be important. Since at least for finite μ, ω is finite, the region $|\vec{k}| = 0$ is not a singular region of the integrand, and it is reasonable to take $m \to \infty$ before we do the $\vec{k}$ integration.

With the technique of picking up poles in p_0 and p_0' we could easily add some form factor $F((p'-p)^2)$ to our scalar exchange (3.17). If it had no poles in, say, the upper-half p_0, p_0' planes, then generalization of (3.19) is straightforward.

As a third example of $\tilde{I}$ we may consider the scalar exchange, but in a frame other than the center-of-mass frame. Here we refer to the paper of Feldman *et al.*,[29] where $\tilde{I}$ is evaluated in the infinite-momentum frame of reference. Their result gives the Weinberg equation,[8] just as the infinite-momentum limit of the old-fashioned perturbation theory is the Weinberg equation. In particular, in the infinite-momentum frame we get projective properties for the Green's function, which restrict the domain of internal momenta to positive $p_+ = p_0 + p_z$ for each constituent, including the intermediate states. We shall discuss the Weinberg equation in detail in the next section.

IV. LUMINAL INTERACTION

In this section we follow the reasoning and methods of Sec. III with the difference that the privileged component is not time but the light-cone[16,17,19,31] or luminal variable $x_+ \equiv z + t$. We shall see that this procedure leads us to the Weinberg equation,[8] which is a completely on-mass-shell equation, even for particles in the intermediate states.

Kinematic labeling is as in Sec. III, with the additional definition of the z axis by the direction of motion K_z of the bound state, i.e., $\vec{K}_\perp = 0$, where $\perp$ labels perpendicular components. Note $\vec{P}_\perp = \vec{P}_\perp' = 0$. We also introduce *luminal* variables $v_\pm = v_0 \pm v_z$, where v is a four-vector. These will apply in both configuration space or momentum space. We have the usual scalar product

$$u \cdot v = \tfrac{1}{2}(u_+ v_- + u_- v_+) - \vec{u}_\perp \cdot \vec{v}_\perp . \tag{4.1}$$

The luminal Green's function $\bar{G}$ is defined in complete analogy to the equal-time Green's function $\tilde{G}$,

$$\langle p_+', \vec{p}_\perp' | \bar{G}(P) | p_+, \vec{p} \rangle \equiv \int dp_-' dp_- \langle p' | G(P) | p \rangle , \tag{4.2}$$

or in configuration space

$$\bar{G}(x_+', x_+) = G(x_{1+}' = x_{2+}' = x_+', x_{1+} = x_{2+} = x_+) . \tag{4.3}$$

The luminal condition expressed in Eq. (4.3), namely

$$x_{1+} - x_{2+} = x_{1+}' - x_{2+}' = 0 , \tag{4.4}$$

is invariant under Lorentz boost in the z direction. This crucial property should be contrasted with the frame dependence of the zero-relative-time condition of Eq. (3.6). The z-boost invariance of Eq. (4.4) allows us to choose K_z at will; in particular we could just as easily choose the center-of-mass frame, where $K_+ = K_- = W$, as the infinite-momentum frame.

The luminal free Green's function can be evaluated from Eq. (3.13) by Cauchy's theorem for p_- poles. We find

$$\begin{aligned}\langle p_+' p_\perp' | \bar{G}_0(W) | p_+ p_\perp \rangle &= \int dp_- dp_-' (p_{1+} p_{1-} - p_{1\perp}^2 - m_1^2 + i\epsilon)^{-1} (p_{2+} p_{2-} - p_{2\perp}^2 - m_2^2 + i\epsilon)^{-1} \delta^{(4)}(p' - p) \\ &= -2\pi i W^{-1} \left[\left(p_+ - \frac{m_2}{m_1 + m_2} W \right) \left(p_+ + \frac{m_1}{m_1 + m_2} W \right) + p_\perp^2 + \frac{p_+}{W} (m_2^2 - m_1^2) + m_1 m_2 - i\epsilon \right]^{-1} \\ &\quad \times \theta\left(\frac{m_1}{m_1 + m_2} W + p_+ \right) \theta\left(\frac{m_2}{m_1 + m_2} W - p_+ \right) \delta^{(2)}(\vec{p}_\perp' - \vec{p}_\perp) \delta(p_+' - p_+) . \end{aligned} \tag{4.5}$$

The projective θ-function factors in Eq. (4.5) are now features of the luminal free Green's function. This property has been discussed previously[32] in connection with the use of the plus-minus variables. They appear because of the linear dependence of the propagator on p_+ and p_-, so that the poles in, say, p_- lie above or below the real line according to the sign of the θ-function factors. The θ functions mean that both constituents must have a positive plus component, as comparison with Eq. (3.2b) shows. This projective property allows us to reinterpret the fully off-mass-shell BS theory as an on-mass-shell, off-minus-component-shell Weinberg theory; note that for initial- and final-state on-mass-shell particles the plus component must be positive, and plus-component conservation at any vertex enforces positivity of the plus-component for the intermediate state.

Equation (4.5) is fully equivalent to the Weinberg two-body propagator.[8] To show this we need to define a dimensionless variable η as the ratio of the plus component of one constituent to the plus component of the total momentum, and then express η in terms of the relative momenta,

$$\eta = p_{1+} P_+^{-1} = \frac{m_1}{m_1+m_2} + p_+ W^{-1} . \tag{4.6}$$

The well-known Weinberg propagator,[8] which is written

$$\left[W^2\eta(1-\eta)\left(\frac{p_\perp^2+m_1^2}{\eta} + \frac{p_\perp^2+m_2^2}{1-\eta} - W^2 - i\epsilon \right) \right]^{-1} \theta(\eta)\theta(1-\eta)\delta(\eta'-\eta)\delta^{(2)}(\vec{p}_1' - \vec{p}_1) , \tag{4.7}$$

can then be easily shown to be Eq. (4.5).

We next evaluate the luminal interaction

$$\bar{I} = \bar{G}^{-1}\overline{G_0 I G_0}\,\bar{G}_0^{-1} , \tag{4.8}$$

where as an example we take I to be scalar exchange, Eq. (3.17). Using the same techniques, we get

$$\langle p_+', \vec{p}_1' | \bar{I} | p_+, \vec{p}_1 \rangle = -g^2 | p_+' - p_+ |^{-1} \left\{ \theta(p_+' - p_+) \left[\frac{(\vec{p}_1' - \vec{p}_1)^2 + \mu^2}{p_+' - p_+} + p_{1-} + p_{2-}' - W - i\epsilon \right]^{-1} \right.$$
$$\left. + \theta(p_+ - p_+') \left[\frac{(\vec{p}_1' - \vec{p}_1)^2 + \mu^2}{p_+ - p_+'} + p_{2-} + p_{1-}' - W - i\epsilon \right]^{-1} \right\} ,$$

which can be rewritten[11,33,34] as

$$\langle p_+' p_\perp' | \bar{I}(W) | p_+ \vec{p}_1 \rangle = -g^2 \left\{ -\left[p' - p + \frac{1}{2}\frac{m_1 - m_2}{m_1+m_2}(P'-P) \right]^2 + \mu^2 + | p_+' - p_+ | \tfrac{1}{2}(P_- + P_-' - 2W - i\epsilon) \right\}^{-1} . \tag{4.9}$$

In this expression the quantity in square brackets is a four-vector squared, but with the minus component determined by the parameters of the poles which lead to (4.9). In particular, as in Eqs. (3.2),

$$p_- = \frac{m_2}{m_1+m_2} P_{1-} - \frac{m_1}{m_1+m_2} P_{2-} , \quad P_- = p_{1-} + p_{2-} , \tag{4.10}$$

where

$$p_{1-} = \frac{p_\perp^2 + m_1^2}{p_{1+}} = (p_\perp^2 + m_1)^2 \left(\frac{m_1}{m_1+m_2} W + p_+ \right)^{-1} , \tag{4.11a}$$

$$p_{2-} = \frac{p_\perp^2 + m_2^2}{p_{2+}} = (p_\perp^2 + m_2^2) \left(\frac{m_2}{m_1+m_2} W - p_+ \right)^{-1} , \tag{4.11b}$$

and similarly for the primed component. Writing the luminal interaction in this way makes the complete equivalence of these results to the Weinberg equation, i.e., to the old-fashioned perturbation theory in the infinite-momentum frame,[29] or the x_+-ordered perturbation theory, more explicit. For example, Eqs. (4.11) are just mass-shell conditions for the constituents. The presence of the absolute value in Eq. (4.9) corresponds to the two standard x_+-ordered graphs, which we draw in Fig. 2. This equivalence thus carries with it the possibility of the reinterpretation discussed above in connection with the luminal free Green's function.

In this connection it may be appropriate to point out that this dynamics is *not* luminally instantaneous, because the result is not proportional to $\delta(x_+' - x_+)$. We could call the theory a "one-x_+" theory in the sense that the intermediate states

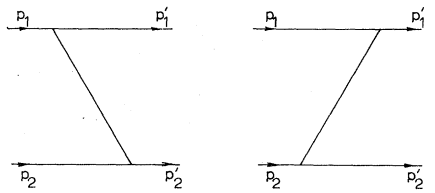

FIG. 2. Weinberg (x_+-ordered) graphs for scalar exchange. A vertical cut in the x_+ variable is implied.

considered have a given x_+ value, but in analogy with Sec. III it seems more fitting to call this approach an "equal-x_+" approach, corresponding to the differing x_+ scales in the initial and final states, as in Eq. (4.3).

We now consider two distinct large-mass limits of the luminal interaction: (1) the *static limit*, when $m_2 \to \infty$, while m_1, p_+, p'_+, $\vec{p}_\perp$, $\vec{p}'_\perp$, and $W - m_2$ are all fixed and comparable and (2) the *nonrelativistic limit*, when $m_1 = m_2 = m \to \infty$ while p_+, p'_+, $\vec{p}_\perp$, $\vec{p}'_\perp$, and $W - 2m$ remain fixed. Recall generally $W = m_1 + m_2 - B$. In the static limit we have, using Eqs. (4.10) and (4.11),

$$P_- = m_2 + \frac{p_\perp^2 + m_1^2}{m_1 + p_+}, \quad p_- = -m_1 + \frac{p_\perp^2 + m_1^2}{m_1 + p_+}. \tag{4.12}$$

Using this result in Eq. (4.9) yields

$$\bar{I} \underset{\substack{\text{static}\\ \text{limit}}}{\longrightarrow} -g^2\left\{\frac{1}{2}\frac{p_\perp'^2 + m_1^2}{m_1 + p_+}[|p'_+ - p_+| - (p'_+ - p_+)] + \frac{1}{2}\frac{p_\perp^2 + m_1^2}{m_1 + p_+}[|p'_+ - p_+| + (p'_+ - p_+)] + (\vec{p}'_\perp - \vec{p}_\perp)^2 + \mu^2 - |p'_+ - p_+|(m_1 - B)\right\}^{-1}. \tag{4.13}$$

In this limit $\bar{I}$ is not local in the variables $p_{+,\perp}$.

In the nonrelativistic limit, we note, however (see below for the $\mu = 0$ case), that $W^2 - 4m^2$ finite implies $(-4mB + B^2)$ finite, so that $B \sim 1/m$. In this case

$$P_- = 2m, \quad p_- = -p_+. \tag{4.14}$$

Equation (4.14) into Eq. (4.9) now gives

$$\bar{I} \underset{\substack{\text{nonrel.}\\ \text{limit}}}{\longrightarrow} -g^2[(\vec{p}'_\perp - \vec{p}_\perp)^2 + (p'_+ - p_+)^2 + \mu^2]^{-1}, \tag{4.15}$$

which is a Yukawa form in the new $(+, \perp)$ "three-vector," and is indeed local in these relative variables. [In the special case of a Coulomb force, where the mass of the constituents provides the only energy scale, the energy levels, the relative momentum, and B are all proportional to m. B is small compared to q, however, since $q \approx \alpha m$, $B \approx \alpha^2 m$. The luminal interaction (4.9) reduces to the ordinary local Coulomb force because the $(p' - p)^2$ term in (4.9) for equal masses is $O(\alpha^2)$ while the $|p'_+ - p_+|$ term is $O(\alpha^3)$.]

V. WEINBERG-SCHRÖDINGER EQUATION

In this section we recast the results of the previous section in terms of variables which are relativistic analogs of the nonrelativistic three-momenta. We end up with a three-dimensional equation for the off-energy-shell two-body dynamics, which is fully analogous to a Schrödinger equation with an energy-dependent nonlocal potential. We thus arrive at a relativistic version of the Schrödinger equation.

For this purpose we define[35] new relative momenta which have the advantage of being orthogonal to the timelike four-vectors P, K, and P'. Such vectors will be spacelike (in the BS equation, the relative momentum is *not* in general spacelike) and have an angular decomposition. Thus for $P = p_1 + p_2$, we define the Wightman-Gårding relative momentum[36] as

$$q \equiv (\tfrac{1}{2} - \nu)p_1 - (\tfrac{1}{2} + \nu)p_2, \tag{5.1}$$

where

$$\nu \equiv \tfrac{1}{2}(m_1^2 - m_2^2)/P^2. \tag{5.2}$$

When p_1 and p_2 are on shell, it is straightforward to verify that

$$q \cdot P = 0, \quad P^2 = [(-q^2 + m_1^2)^{1/2} + (-q^2 + m_2^2)^{1/2}]^2 \equiv M_q^2. \tag{5.3}$$

It is important to note that M_q^2 depends on q^2. We also note that

$$p_1 = q + (\tfrac{1}{2} + \nu)P, \quad p_2 = -q + (\tfrac{1}{2} - \nu)P, \tag{5.4}$$

and that for the invariant volume element we have

$$d^4p_1 d^4p_2 \delta(p_1^2 - m_1^2)\delta(p_2^2 - m_2^2) = \frac{1}{4}\frac{1}{p_{1+}p_{2+}} dp_{1+}dp_{2+}d^2p_{1\perp}d^2p_{2\perp} = 2\left|1 - (m_1^2 - m_2^2)^2 M_q^{-4}\right|^{-1} d^4q d^4P \times \delta(q \cdot P)\delta(P^2 - M_q^2). \tag{5.5}$$

We also define two more sets of these vectors, one for the final state, and one for the intermediate state. Following the notation of Sec. III, we have the following sets of quantities: $\{q, P, M_q\}$, $\{q', P', M_{q'}\}$, and $\{k, K, M\}$ for initial, final, and intermediate states, respectively.

Note finally conservation of plus and transverse components for the Weinberg dynamics,

$$P_+ = K_+ = P'_+, \quad \vec{P}_\perp = \vec{K}_\perp = \vec{P}'_\perp. \tag{5.6}$$

We next want to characterize the four-momenta involved in a manifestly covariant form. This is done by construction of a tetrad, a set of four-vectors, for each of the timelike four-vectors P, P', and K. For the benefit of the reader not familiar with this approach, we have included an appendix on the construction and properties of tetrads and on the invariant tetrad notation for four-vectors. The four four-vectors of the tetrads, or the tetrad axes, are labeled P, l, m, n; P', l', m', n'; and K, L, M, N for P, P', and K, respectively. In tetrad notation we have

$$(P_P, P_l, P_m, P_n) = (M_q, 0, 0, 0), \qquad (q_P, q_l, q_m, q_n) = (0, q_l, q_m, q_n), \tag{5.7}$$

similarly for the primed variables, and

$$(K_K, K_L, K_M, K_N) = (M, 0, 0, 0), \qquad (k_K, k_L, k_M, k_N) = (0, k_L, k_M, k_N). \tag{5.8}$$

We emphasize that the subscripted components above are invariants, in that they are projections on the appropriate tetrad, as in Eqs. (A3) and (A4).

Since in the tetrad notation the relative momenta have no first component, we write angular decompositions in terms of them. For example, we define[11,35] $\bar{q}$ and two proper angles θ and ϕ by

$$\vec{q} \equiv (q_l, q_m, q_n) \equiv (\bar{q}\sin\theta\cos\phi, \bar{q}\sin\theta\sin\phi, \bar{q}\cos\theta). \tag{5.9}$$

$\vec{q}$ is not a true three-vector but can be manipulated as if it were.

In order to express the Weinberg dynamics in tetrad notation we need to express four-vectors in terms of the tetrad projections of q and P. For this purpose we use the inversion formula (A6) to find

$$(q_+, q_l, q_2, q_-) = (q_n P_+ M_q^{-1}, q_l, q_m, -q_n P_+^{-1} M_q). \tag{5.10}$$

Given this expression for q_μ, we then project q_μ onto the tetrad $KLMN$. We find

$$(q_{K+N}, q_L, q_M, q_{K-N}) = (q_n P_{K+N} M_q^{-1}, q_l, q_m, -q_n P_{K-N} M_q^{-1}). \tag{5.11}$$

Moreover, since

$$(K_{K+N}, K_{K-N}) = (M, M), \tag{5.12}$$

and since the conservation laws, Eq. (5.6), hold in any particular tetrad projection, we have $P_{K+N} = P'_{K+N} = M$, and hence

$$(P_{K+N}, P_{K-N}) = (M, M_q^2 M^{-1}). \tag{5.13}$$

Equation (5.13) allows us to rewrite

$$(q_{K+N}, q_L, q_M, q_{K-N}) = (q_n M M_q^{-1}, q_l, q_m, -q_n M^{-1} M_q). \tag{5.14}$$

Similar expressions hold for the primed variables.

We now turn to the Weinberg dynamics. Our objective is to present it in the K tetrad. This will make our results manifestly covariant and will lead to a covariant form of the Schrödinger equation. We first write the interaction in terms of our new variables, and then in terms of the tetrad projections. The luminal interaction, Eq. (4.9), is written in terms of q rather than p by Eqs. (5.4) and (3.2). We find[34]

$$\bar{I} = -g^2[-(q' - q + v'P' - vP)^2 + \mu^2 + |q'_+ - q_+|\tfrac{1}{2}(P'_- + P_- - 2K_-)]^{-1}. \tag{5.15}$$

To express this in terms of our relative variables (q_l, q_m, q_n) and $(q'_{l'}, q'_{m'}, q'_{n'})$ and the mass parameters, we use Eqs. (5.13) and (5.14) to evaluate $\bar{I}$ in the K-tetrad frame. We have

$$-(q' - q + v'P' - vP)^2 = -(q' - q + v'P' - vP)_{K+N}(q' - q + v'P' - vP)_{K-N} + \sum_{i=L,M}(q' - q + v'P' - vP)_i^2$$
$$= (\vec{q}' - \vec{q})^2 - q'_{n'} q_n \frac{(M_{q'} - M_q)^2}{M_{q'} M_q} + (v' - v)(q'_{n'} M_{q'} - q_n M_q), \tag{5.16}$$

where we remind the reader of the definition (5.9) of the "three-vectors" $\vec{q}$ and $\vec{q}'$. Also

$$|q'_+ - q_+|\tfrac{1}{2}(P'_- + P_- - 2K_-) = \frac{1}{2}\left|\frac{q'_{n'}}{M'_q} - \frac{q_n}{M_q}\right|(M_{q'}^2 + M_q^2 - 2M^2). \tag{5.17}$$

These expressions can be written in terms of the angular variables and inserted into $\bar{I}$, Eq. (5.15).

Having written $\bar{I}$ in variables which allow a proper angular integration, we now manipulate the Weinberg equation itself to lead to a Schrödinger-type equation. This is a standard procedure[8,35] for such equations, and we only sketch the steps.

The required invariant volume element, Eq. (5.5), becomes

$$dV_p \equiv d^4p_1\, d^4p_2\, \delta(p_1^2 - m_1^2)\delta(p_2^2 - m_2^2) = (P_+ M_q)^{-1}\left|1 - (m_1^2 - m_2^2)M_q^{-1}\right|^{-1}\bar{q}^2\, d\bar{q}\, d(\cos\theta)\, d\phi \times dP_+\, dP_-\, d^2\vec{P}_\perp\, \delta(P_- - P_+^{-1}(P_\perp^2 + M_q^2)). \tag{5.18}$$

We next define from the standard two-body invariant amplitude $\mathfrak{M}$ another amplitude $\mathfrak{N}$ which is continued off shell in the minus component of the total momentum, in terms of which the S-matrix operator is

$$S = 1 - i(2\pi)^4\delta^{(4)}(P' - P)(2\pi)^{-6}\mathfrak{M} = 1 - 2\pi i P_+^{-1}\delta(P'_- - P_-)\mathfrak{N}. \tag{5.19}$$

The basis states are normalized as

$$\langle p_1' p_2' | p_1 p_2 \rangle = 4 p_{1+} p_{2+} \delta(p_{1+}' - p_{1+}) \delta(p_{2+}' - p_{2+}) \delta^{(2)}(\vec{p}_{1\perp}' - \vec{p}_{1\perp}) \delta^{(2)}(\vec{p}_{2\perp}' - \vec{p}_{2\perp}) . \tag{5.20}$$

The Weinberg equation is

$$\langle p_1' p_2' | \mathfrak{R} | p_1 p_2 \rangle = \langle p_1' p_2' | \bar{I} | p_1 p_2 \rangle - 2 \int dV_l [P_+ (l_{1-} + l_{2-} - K_- - i\epsilon)]^{-1} \langle p_1' p_2' | \bar{I} | l_1 l_2 \rangle \langle l_1 l_2 | \mathfrak{R} | p_1 p_2 \rangle , \tag{5.21}$$

where continuing our example the matrix element of $\bar{I}$ is, following Eqs. (3.4) and (4.2),

$$\langle p_1' p_2' | \bar{I} | p_1 p_2 \rangle = -4 m_1 m_2 \bar{g}^2 (2\pi)^{-3} P_+ \delta(P_+' - P_+) \delta^{(2)}(\vec{P}_\perp' - \vec{P}_\perp)$$
$$\times [-(q' - q + \nu P' - \nu P)^2 + \mu^2 + |q_+' - q_+|\, \tfrac{1}{2}(P_-' + P_- - 2K_-)]^{-1} . \tag{5.22}$$

One could, however, simply consider the quantity in square brackets in $\langle p_1' p_2' | \bar{I} | p_1 p_2 \rangle$ as given by some other dynamics. In (5.22) we evaluate the square brackets as in Eqs. (5.16) and (5.17). We have also defined a dimensionless coupling constant $\bar{g}$ by

$$g = 2\sqrt{m_1 m_2}\, \bar{g} , \tag{5.23}$$

where the factor $2\sqrt{m_1 m_2}$ is chosen to make the minimal relativity factor $g^2/M_{q'} M_q$ reduce to one in the nonrelativistic limit.

We separate relative motion[37] by reduced matrix elements, as follows:

$$\langle p_1' p_2' | \mathfrak{R} | p_1 p_2 \rangle = P_+ \delta(P_+' - P_+) \delta^{(2)}(\vec{P}_\perp' - \vec{P}_\perp) \{ M_{q'} M_q [1 - (m_1^2 - m_2^2)^2 M_{q'}^{-4}][(1 - (m_1^2 - m_2^2)^2 M_q^{-4}] \}^{1/2} \langle \bar{q}' \Omega | \mathfrak{R} | q \Omega \rangle , \tag{5.24}$$

and similarly for $\bar{I}$. The reduced matrix elements satisfy

$$\langle \bar{q}' \Omega | \mathfrak{R} | \bar{q} \Omega \rangle = \langle \bar{q}' \Omega' | \bar{I} | \bar{q} \Omega \rangle - 2 \int \frac{\bar{l}^2 d\bar{l}\, d\Omega_l}{M_l^2 - M^2 - i\epsilon} \langle \bar{q}' \Omega' | \bar{I} | \bar{l} \Omega_l \rangle \langle \bar{l} \Omega_l | \mathfrak{R} | \bar{q} \Omega \rangle , \tag{5.25}$$

with the kernel given by (5.24) and (5.22).

For M_q satisfying Eq. (5.3) we have

$$\bar{q}^2 = \tfrac{1}{4} M_q^2 - \tfrac{1}{2}(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 / 4 M_q^2 . \tag{5.26}$$

This equation can be used to verify the identity

$$M_l^2 - M^2 = 4(\bar{l}^2 - \bar{k}^2)[1 - (m_1^2 - m_2^2)^2 M_l^{-2} M^{-2}]^{-1} , \tag{5.27}$$

which we use in the definition of a T matrix,

$$\langle \bar{q}' \Omega' | T | \bar{q} \Omega \rangle = \frac{1}{4} \frac{m_1 + m_2}{m_1 m_2} \{ [1 - (m_1^2 - m_2^2)^2 M_{q'}^{-2} M^{-2}][1 - (m_1^2 - m_2^2)^2 M_q^{-2} M^{-2}] \}^{1/2} \langle \bar{q}' \Omega | \mathfrak{R} | \bar{q} \Omega \rangle , \tag{5.28}$$

satisfying, from (5.25), a Lippmann-Schwinger–type equation,

$$\langle \bar{q}' \Omega' | T | \bar{q} \Omega \rangle = \langle \bar{q}' \Omega' | V | \bar{q} \Omega \rangle - 2 \frac{m_1 m_2}{m_1 + m_2} \int \frac{\bar{l}^2 d\bar{l}\, d\Omega_l}{\bar{l}^2 - \bar{k}^2 - i\epsilon} \langle \bar{q}' \Omega' | V | \bar{l} \Omega_l \rangle \langle \bar{l} \Omega_l | T | q \Omega \rangle . \tag{5.29}$$

In this equation, the matrix elements of V and $\bar{I}$ have the same relation as those of T and N, Eq. (5.28). Equation (5.29) suggests the proper relativistic form of the relative energy, namely,

$$E = \tfrac{1}{2} \bar{k}^2 \frac{m_1 + m_2}{m_1 m_2} , \tag{5.30}$$

and for a *bound*-state problem of mass M an energy-shell variable

$$E_M = -\tfrac{1}{2} \bar{k}_M^2 \frac{m_1 + m_2}{m_1 m_2} , \tag{5.31}$$

where $\bar{k}_M$ satisfies $M = (-\bar{k}_M^2 + m_1^2)^{1/2} + (-\bar{k}_M^2 + m_2^2)^{1/2}$: note the minus signs in this expression and in (5.31).

The vertex function γ is defined in the usual way, namely near the bound-state pole

$$\langle \bar{q}' \Omega' | T | \bar{q} \Omega \rangle \approx \frac{\gamma(\bar{q}', \Omega') \gamma^*(\bar{q}, \Omega)}{E - E_M} , \tag{5.32}$$

satisfying the homogeneous equation

$$\gamma(\bar{q}, \Omega) = -2 \frac{m_1 m_2}{m_1 + m_2} \int \frac{\bar{l}^2 d\bar{l}\, d\Omega_l}{\bar{l}^2 + k_M^2} \langle \bar{q} \Omega | V | \bar{l} \Omega_l \rangle \gamma(\bar{l}, \Omega_l) . \tag{5.33}$$

Finally the bound-state wave function is related to the vertex function by

$$\psi(\bar{q}, \Omega) = \left(\frac{1}{2} \frac{m_1 + m_2}{m_1 m_2} \bar{q}^2 - E_M \right)^{-1} \gamma(\bar{q}, \Omega) , \tag{5.34}$$

satisfying the relativistic Schrödinger equation

$$\left(\frac{1}{2}\frac{m_1+m_2}{m_1m_2}\bar{q}^2 - E_M\right)\psi(\bar{q},\Omega)$$
$$= -\int \bar{l}^2 d\bar{l}\, d\Omega_l \langle \bar{q}\Omega | V | \bar{l}\Omega_l\rangle \psi(\bar{l},\Omega_l)\,. \tag{5.35}$$

For our particular example V is nonlocal and energy dependent. ψ is normalized by

$$1 = \int \bar{q}^2 d\bar{q}\, d\Omega\, |\psi(\bar{q},\Omega)|^2$$
$$-\int \bar{q}^2 d\bar{q}\, d\Omega\, \bar{l}^2 d\bar{l}\, d\Omega_l \psi^*(\bar{l},\Omega_l)$$
$$\times \left(\frac{\partial}{\partial \bar{E}}\langle \bar{l}\Omega_l | V | \bar{q}\Omega\rangle\Big|_{E=E_M}\right)\psi(\bar{q},\Omega)\,. \tag{5.36}$$

VI. RELATIVISTIC CORRECTIONS

Several methods can be used to estimate the relativistic corrections for a problem such as ours. We consider the case of equal-mass constituents, $m_1 = m_2 = m$, and let m become large. The interaction is a power series in m^{-1}. A natural way to extract this series is to expand integrands in m^{-1}, which is basically equivalent to expansion of these integrands in ratios of the magnitude of relative three-momentum to mass, e.g., $\bar{q}/m$. Such a ratio is locally the same as v/c, since the constituents' energy is proportional to $(\bar{q}^2+m^2)^{1/2}$, and $\bar{q}/(\bar{q}^2+m^2)^{1/2} = v/c$. However, even if the result of expanding in $\bar{q}/m$ under the integrand is finite, we must be wary of the result, since only if the basic dynamics holds the relative motion to the region $\bar{q}/m \ll 1$ is this procedure completely consistent in its interpretation. In discussing more details below, we assume this is indeed the case.

The care which must be taken in extracting relativistic corrections is illustrated by another possibility for the expansion parameter, namely the ratio of relative energy k_0 to relative three-momentum $|\vec{k}|$, where $k = \frac{1}{2}(k_1 - k_2)$ is the relative four-momentum used in Secs. III and IV. Here one must be certain that k_0 decreases to zero faster than $|\vec{k}|$ to avoid problems with the region $|\vec{k}| = 0$; we assume dynamics damps the large-k_0 region. To study the small-$|\vec{k}|$ region, let us for example consider the equal-time approach, discussed in Sec. III, where one has factors [see e.g., Eq. (3.19)] such as $W - 2E$, corresponding to one constituent on-shell, energy $E = (\vec{k}^2+m^2)^{1/2}$, and the other has energy $W - E$, determined by energy conservation. The relative energy is thus $k_0 = E - \frac{1}{2}W$. With $W = 2m - B$, we see

$$\frac{k_0}{|\vec{k}|} = \frac{m}{|\vec{k}|}\left[\left(\frac{\vec{k}^2}{m^2}+1\right)^{1/2} - 1 + \frac{B}{2m}\right] \underset{|\vec{k}|\to 0}{\sim} \frac{|\vec{k}|}{2m} + \frac{B}{2|\vec{k}|}\,.$$

A nonzero value of B thus leaves the region $|\vec{k}| \to 0$ a dangerous one, if one uses $k_0/|\vec{k}|$ as a small parameter.

In the literature on quarkonium, the relativistic corrections are discussed for local position-space potentials. Therefore it will be useful to consider the relativistic Schrödinger equation in position space and to start with a local potential. By this we mean

$$\langle \vec{q}' | V | \vec{q}\rangle = V(\vec{q}' - \vec{q})\,, \tag{6.1}$$

where we remind the reader that q' and q are evaluated in different tetrad bases, Eq. (5.9). Let us denote by $\vec{r} = (r_l, r_m, r_n)$ the variables conjugate to $\vec{q} = (q_l, q_m, q_n)$. Then in position space, we define

$$\langle \vec{r} | V | \vec{r}\rangle = (2\pi)^{-3}\int d^3q\, d^3q' \times \exp[-i(\vec{q}\cdot\vec{r} - \vec{q}'\cdot\vec{r}')]V(\vec{q}' - \vec{q})$$
$$= (2\pi)^{3/2}\delta^{(3)}(\vec{r}' - \vec{r})V(\vec{r})\,, \tag{6.2}$$

where

$$V(\vec{r}) = (2\pi)^{-3/2}\int d^3q \exp(i\vec{q}\cdot\vec{r})V(\vec{q})\,. \tag{6.3}$$

For $m_1 = m_2 = m$—assumed throughout this section—the Fourier transform of the relativistic Schrödinger equation, (5.35), is

$$\left[-\frac{1}{m}\nabla^2 + V(\vec{r})\right]\psi(\vec{r}) = E_M\psi(\vec{r})\,, \tag{6.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r_l^{\,2}} + \frac{\partial^2}{\partial r_m^{\,2}} + \frac{\partial^2}{\partial r_n^{\,2}}$$

and

$$\psi(\vec{r}) = (2\pi)^{-3/2}\int d^3q \exp(i\vec{q}\cdot\vec{r})\psi(\vec{q})\,. \tag{6.5}$$

This has the form of the ordinary Schrödinger equation, although r is now an invariant with tetrad notation. (Equivalently this is the ordinary Schrödinger equation in the rest frame of the bound state.) The spectrum and other physical quantities would then be that corresponding to the ordinary local potential $V(\vec{r})$, as in phenomenological models of quarkonium.

Let us turn to the potential corresponding to scalar exchange, as in Sec. V. According to Eqs. (5.29), (5.24), and (5.22),

$$\langle \vec{q}' | V | \vec{q}\rangle = -\bar{g}^2(2\pi)^{-3}2m(M_{q'}M_q)^{-1/2}(\mu^2+Q^2)^{-1}\,, \tag{6.6}$$

where

$$Q^2 = (\vec{q}' - \vec{q})^2 - \frac{q_3'q_3}{M_{q'}M_q}(M_{q'} - M_q)^2$$
$$+\left|\frac{q_3'}{M_{q'}} - \frac{q_3}{M_q}\right|\left[\tfrac{1}{2}(M_{q'}^{\,2}+M_q^{\,2}) - (2m-B)^2\right], \tag{6.7}$$

and $M_q^{\,2}$ is given by

$$M_q^2 = 4(\vec{q}^2 + m^2) . \tag{6.8}$$

This potential can be generalized by the inclusion of a form factor $F(Q^2)$, where the original BS exchange kernel has the form $F(Q^2)/(Q^2 - \mu^2 + i\epsilon)$, Q a four-vector, as long as the form factor neither cancels the singularity of the scalar exchange nor has itself any important singularities in, e.g., the integrations starting with Eq. (4.8). $F(Q^2)$ also serves to damp the relativistic region.

Such a potential is *nonlocal*. We distinguish in Eq. (6.6) two nonlocal factors. First, we have the "minimal relativity" factor[38] $2m(M_q M_{q'})^{-1/2}$, whose origin is the phase space of relativistic kinematics, and second, the last term in Q^2. The origin of this last factor is the noninstantaneous character of the interaction; as we have seen, noninstantaneous behavior is important for any luminal or equal-time scheme derived as a three-dimensional equation from the BS equation.

The effects of both of the nonlocal factors vanish in the limit $m \to \infty$, where we recover the nonrelativistic physics. We shall therefore expand them in $\bar{q}/m, \bar{q}'/m$ to measure the relativistic corrections. From Eq. (6.8) we have

$$2m(M_{q'} M_q)^{-1/2} \approx 1 - \frac{1}{4}\frac{\bar{q}^2 + \bar{q}'^2}{m^2} + \cdots , \tag{6.9}$$

and from Eq. (6.7),

$$Q^2 \approx (\vec{q}' - \vec{q})^2 + |q_3' - q_3| \left(\frac{\bar{q}'^2 + \bar{q}^2}{m} + 2B - \frac{1}{2}\frac{B^2}{m} \right) + \cdots . \tag{6.10}$$

If, according to the discussion above (4.14), $B = m^{-1}B_0^2$, B_0^2 fixed, then

$$Q^2 \approx (\vec{q}' - \vec{q})^2 + \frac{|q_3' - q_3|}{m} (\bar{q}'^2 + \bar{q}^2 + 2B_0^2) + \cdots . \tag{6.11}$$

Thus the correction in Q^2 is $O(1/m)$. We thus have

$$\frac{F(Q^2)}{\mu^2 + Q^2} \approx \frac{F((\vec{q}' - \vec{q})^2)}{\mu^2 + (\vec{q}' - \vec{q})^2} + \frac{|q_3' - q_3|}{m} (\bar{q}'^2 + \bar{q}^2 + 2B_0^2) \frac{d}{dQ^2} \left(\frac{F(Q^2)}{\mu^2 + Q^2} \right)_{Q^2 = (\vec{q}' - \vec{q})^2} + \cdots . \tag{6.12}$$

Equations (6.9) and (6.12) combined give the relativistic corrections in (6.6). A crucial observation is that the *leading* correction due to the noninstantaneous character of the interaction is $O(v/c)$, whereas that due to minimal relativity is $O((v/c)^2)$. Of course the noninstantaneous piece will also have $O((v/c)^2)$ terms in the potential. The presence of a v/c correction in the context of quarkonium, as opposed to $(v/c)^2$, would appear to be a qualitatively new result with nontrivial implications. [Even without the $O(v/c)$ piece, there are now additional $O((v/c)^2)$ terms.] If one believes that the zeroth-order local quarkonium potential is the nonrelativistic limit of some four-dimensional kernel derived from quantum field theory, then such $O(v/c)$ corrections may appear according to the underlying field theory. The equal-time scheme, interaction given by Eq. (3.19), also has corrections $O(B_0^2/m)$. It is interesting to note that in either scheme the presence of the binding-energy term B is only possible through the appearance of the noninstantaneous parameter W.

In closing the discussion of the relativistic corrections we remark that different terms in the expansion of the potential may cause quite different effects on the spectrum. One should therefore be quite careful in discarding, say the kinematic corrections in favor of noninstantaneous terms. We cite here[11] a numerical study of the binding of a scalar system by massive scalar exchange, and in particular the fact that the relativistic effects due to the minimal relativity factors can be comparable to the effects due to the noninstantaneous character of the luminal interaction. On the other hand, as the ratio of the constituent mass to the exchange mass increases, the relative size of the minimal relativity effects decreases, consistent with the results of this section.

We conclude this section with a brief discussion of the relation between relativistic corrections and dynamics based on a Lagrangian. For the exchange mass $\mu \neq 0$, consecutive kernels with an increasing number of exchange particles correspond in an average approximate way to decreasing range—the notion is vague because we must remember the forces are nonlocal. In addition to the small parameter q/m corresponding to relativistic corrections there are other parameters μ/m and coupling $\bar{g}$ in terms of which we can classify approximations to the kernel. The v/c correction we have discussed above for single-particle exchange is simply that of the longest range; multiparticle kernels will also have v/c corrections.

The case $\mu = 0$ is qualitatively different. In this case we have *a priori* the two small parameters $\bar{q}/m$ and $\bar{g}$, but as stated at the end of Sec. IV, $\bar{q}/m = O(\alpha)$, so there is only one parameter. The v/c correction from the one-particle exchange kernel is an $O(\alpha)$ correction, and thus the lowest-order correction in α. Multiparticle exchange kernels would have higher-order corrections in α.

However, we warn[25] the reader that this reasoning is consistent only if the one-particle kernel provides a binding energy analytic in α. For example, the first-order perturbation-theory expression for the energy should not have any $\ln\alpha$ terms. Such nonphysical behavior would have to be canceled by similar $\ln\alpha$ terms from multiparticle-exchange kernels; the Weinberg interaction may avoid the problems and we hope to address this question in more detail in separate work.

Of course, as we discussed in Sec. I. the situation in quarkonium phenomenology resembles neither situation above, in the sense that the phenomenological quarkonium potential is not tied to any one term in a systematic sequence of kernels, but rather phenomenologically subsumes the entire dynamics of confinement in chromodynamics. Nevertheless, we certainly expect the appearance of $O(v/c)$ corrections.

VII. CONCLUSIONS AND FINAL REMARKS

Systematic development of either equal-time or luminal projections of the BS equation leads to three-dimensional equations whose nonrelativistic limits and relativistic corrections can be systematically computed, given some BS kernel associated with a Lagrangian QFT. The luminal, or Weinberg, equation in particular possesses many attractive features for dealing with a relativistic two-body bound system, even near the nonrelativistic limit. This may be superficially surprising since the luminal interaction is equivalent to the equal-time interaction evaluated in the infinite-momentum frame. However, it has a derivation which makes it clear that the infinite-momentum frame is not especially favored for this equation.

Our main results may be summarized as follows:

(i) Relativistic corrections appear in $O(v/c)$.

(ii) There is a covariant form of the Schrödinger equation.

(iii) Interactions are noninstantaneous and nonlocal.

We discuss these below. The kernels associated with these equations are intrinsically nonlocal, although in zeroth order of relativistic correction v/c they reduce to ordinary local potentials. This fact has an important consequence: *nonlocal relativistic corrections which appear in* $O(v/c)$.

One cannot uniquely derive these corrections from the zeroth-order results. In other words, one can go from a fully relativistic BS kernel to an ordinary potential with its systematic relativistic corrections, but not from an ordinary potential to a full BS kernel. Unfortunately, phenomenological models of quarkonium[1] begin with just such a potential. Thus, we cannot write the relativistic corrections to quarkonium, but only warn that such corrections may first appear on the $O(v/c)$ level. Although the qualitative results of Beavis *et al.* (Ref. 3) are quite reasonable, the details would appear to be in question. On the other hand, if one began with, say, a BS kernel behaving at small momentum transfer q like q^{-4}, in other words with a kernel whose instantaneous limit corresponds to a linear confining potential at long distances, then the level of phenomenology could be raised a notch in the sense that all relativistic corrections are systematically given. Such a phenomenology would include the equation discussed here.

From the luminal projection of the BS equation, we have been able to formulate a covariant two-body bound-state Schrödinger-type equation, derivable from field theory, which has projective properties, and which has fully on-mass-shell dynamics. The equation, written in terms of invariant four-dimensional scalar products, has genuinely spacelike relative momentum, and hence a proper angular decomposition. We refer to this equation as the Weinberg-Schrödinger equation.

We have investigated the equation based largely on a single scalar-particle-exchange kernel and variants thereof. The nonlocal nature of the interaction persists in the static limit, where one of the constituent masses goes to infinity, but the interaction becomes local in the nonrelativistic limit, when both masses go to infinity.

We have also discussed the systematics of extraction of the relativistic corrections. Evidently, some care is necessary in this procedure.

Finally, although this work is in no sense a complete review, we have tried to give some idea of other approaches to the problem of nonrelativistic limits of relativistic equations.

Our formalism should be extended to the more realistic case of particles with spin. Thanks to recent study[39] of definite field theories with spin in the infinite-momentum frame, we have an idea of what the new features of luminal dynamics with spin $\frac{1}{2}$ and 1 will be: We expect a weakening of the interaction singularity due to gluon exchange, and contact terms in kernels. We also expect the gauge symmetry to play a role in the case of massless gluons. In particular, the gauge[40] $A_+ = 0$ seems to be singled out. In addition to clues about the range of allowed corrections to quarkonium, this formalism may then be useful for calculating the spectrum and transition rates for positronium itself.

A separate issue, which also deserves more study, is the angular momentum analysis of the solutions of the Weinberg-Schrödinger equation. Luminal dynamics has a different angular momen-

tum operator than equal-time dynamics, as recently explained in detail by Leutwyler and Stern.[16] In the case of noninteracting particles our variables $q=(q_l, q_m, q_n)$ are the appropriate ones for defining the angular momentum operator as $-i\vec{q}\times\partial/\partial\vec{q}$. The spherical harmonics of our proper angular variables θ and ϕ are then the eigenfunctions of this operator. However, for the case of interacting particles the angular momentum operator is more complicated, since it depends on the interaction. We can project the luminal interaction on the spherical-harmonic basis in both the final and initial state, but we find a nondiagonal representation. This can in turn be diagonalized to a different basis. We also note, that if we performed such a partial-wave analysis on the on-energy-shell t matrix as given in luminal dynamics, we would get the standard results.[41] For the off-shell case we can use an off-shell extension[37] of the Macfarlane partial-wave analysis,[41] which is done in terms of the Wightman-Gårding[34] relative momenta.

ACKNOWLEDGMENTS

One of us (J. M. N.) would like to acknowledge the hospitality of the University of Virginia Physics Department extended to him by R. V. Coleman and H. P. Kelly. We thank T. Fulton, G. Feldman, F. Gross, P. K. Kabir, G. P. Lepage, E. A. Remler, and H. J. Weber for stimulating discussions. This work was supported in part by the National Science Foundation under Contracts Nos. PHY 77-00151 and PHY 76-84374.

APPENDIX A

For the benefit of the reader who may not be familiar with their use, particularly in the context of the Bethe-Salpeter equation dynamics, we repeat here the use of tetrads.[34,35,42] A tetrad is a set of four four-vectors, which can be used to store information about a given four-vector, in a given frame, in an invariant way. In particular, for a timelike four-vector we build a tetrad based on the rest frame of the four-vector. This construction is based on knowing the Lorentz transformation which takes the four-vector to its rest frame. Let $V_\mu=(V_0, V_1, V_2, V_3)$ be this vector, such that $m_v^2=V_\mu V^\mu$. The Lorentz boost which carries V_μ to its rest frame, where $V_\mu=(m_v,0,0,0)$, is

$$L_\mu{}^0=L_0{}^\mu=V^\mu m_v^{-1}, \qquad L_i{}^j=\delta_i{}^j-V_i V^j[m_v(m_v+V_0)]^{-1}, \tag{A1}$$

where $i,j=1,2,3$. The rows of the Lorentz boost (A1) are the four contravariant tetrad vectors which in covariant form we call $V_\mu^{(k)}$, $V_\mu^{(l)}$, $V_\mu^{(m)}$ $V_\mu^{(n)}$. These four vectors have orthonormality properties $g_{\mu\nu}$, i.e.,

$$V^{(\alpha)}\cdot V^{(\beta)}=g_{\alpha\beta}, \quad \alpha,\beta=k,l,m,n. \tag{A2}$$

The "tetrad frame," which is the rest frame of the four-vector, is singled out as special, but it nevertheless allows an invariant characterization of any four-vector. If we have the four-vector A_μ, then we characterize A_μ by the four invariants $A\cdot V^{(\alpha)}$, $\alpha=k,l,m,n$, which we denote (A_k, A_l, A_m, A_n). It is most important to note that although this is written to resemble a four-vector and has many geometric properties like four-vectors, it is not a four-vector; its components are invariants.

In this notation (the "tetrad notation"), the four-vector which generates the tetrad is quite simple:

$$(V_k, V_l, V_m, V_n)=(m_v,0,0,0). \tag{A3}$$

Another related four-vector is the four-vector v_μ orthogonal to V_μ, $v\cdot V=0$. This purely spacelike vector has vanishing kth component in the tetrad frame, namely,

$$(v_k, v_l, v_m, v_n)=(0, v_l, v_m, v_n). \tag{A4}$$

Any scalar product can be expressed in terms of the tetrad projections:

$$A\cdot B=A_kB_k-A_lB_l-A_mB_m-A_nB_n. \tag{A5}$$

Thus the dot product for vectors in tetrad notation has the same formal expression as it does in terms of its ordinary coordinates.

Given a vector in tetrad notation it is also possible to use the Lorentz transformation (or the tetrad) to express the four-vector in ordinary notation. An important case is the four-vector v orthogonal to V, when the transverse coordinates V_1 and V_2 are zero. Then[34]

$$v_\mu=(v_0, v_1, v_2, v_3)=(v_nV_zm_v^{-1}, v_l, v_m, v_nV_0\,m_v^{-1}). \tag{A6}$$

In turn, once this inversion has been made, it is possible to find the projection of, say, v on some other tetrad in terms of the components v_l, v_m, v_n.

Finally, we can also define $\pm$ variables in tetrad notation, e.g., for a four-vector A we can define a tetrad-$\pm$ variable using the tetrad projections,

$$A_{k\pm n}\equiv A_k\pm A_n.$$

As an example, suppose we start in some frame where V_μ has no transverse component, $V_\mu=((m_v^2+V_z^2)^{1/2},0,0,v_z)$. Then the tetrad for V_μ is

$$V_\mu^{(k)}=m_v^{-1}((m_v^2+V_z^2)^{1/2},0,0,V_z),$$

$$V_\mu^{(l)}=(0,1,0,0),$$

$$V_\mu^{(m)}=(0,0,1,0),$$

$$V_\mu^{(n)}=m_v^{-1}(-V_z,0,0,m_v+V_z^2[m_v+(m_v^2+V_z^2)^{1/2}]^{-1}).$$

Using this tetrad, it is straightforward to verify the properties discussed above.

*On leave from the Institute of Theoretical Physics, Warsaw University, 00-681 Warsaw, ul. Hoża 69, Poland.

[1]For general reviews, see, e.g., T. Appelquist in *Particles and Fields*, edited by D. H. Boal and A. N. Kamal (Plenum, New York, 1977); C. Quigg, Fermilab Report No. 78/82-THY (unpublished); J. D. Jackson, in *Proceedings of the European Conference on Particle Physics, Budapest*, edited by L. Jenik and I. Montray (CRIP, Budapest, 1978); K. Gottfried, in *Proceedings of the* 1977 *International Symposium on Lepton and Photon Interactions*, edited by F. Gutbrod (DESY, Germany, 1977); J. Sucher, University of Maryland Report No. 78-196 (unpublished).

[2]See, e.g., M. Deutsch, Adv. Exp. Phys. 4, 63 (1975); also G. T. Bodwin and D. R. Yennie, Phys. Rep. 43C, 267 (1978).

[3]Relativistic corrections have been central in some recent work. A partial list of references would be J. Pumplin, W. Repko, and A. Sato, Phys. Rev. Lett. 22, 1538 (1975); H. J. Schnitzer, *ibid.* 22, 1540 (1975); F. L. Feinberg, Phys. Rev. D 17, 2659 (1978); D. Beavis, S. Y. Chu, B. R. Desai, and P. Kaus, *ibid.* 20, 743 (1979); A. B. Henriques, B. H. Kellett, and R. C. Moorhouse, Phys. Lett. 64B, 85 (1976); C. E. Carlson and F. Gross, *ibid.* 74B, 404 (1978).

[4]See, e.g., G. Feldman, T. Fulton and J. Townsend, Phys. Rev. A 8, 1149 (1973), and references therein.

[5]E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951); E. E. Salpeter, *ibid.* 87, 328 (1952).

[6]See N. Nakanishi, Progr. Theor. Phys. Suppl. 43, 1 (1969).

[7]A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento 29, 380 (1963).

[8]S. Weinberg, Phys. Rev. 150, 1313 (1966).

[9]M. Gell-Mann and F. E. Low, Phys. Rev. 84, 350 (1951).

[10]S. J. Brodsky, in *Atomic Physics and Astrophysics*, 1969 Brandeis Summer Institute in Theoretical Physics, edited by M. Chrétien and E. Lipworth (Gordon and Breach, New York, 1971).

[11]P. Danielewicz and J. M. Namyslowski, Phys. Lett. 81B, 110 (1979).

[12]J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass., 1973), Vol. II, Secs. 5–2, 3.

[13]R. N. Faustov, Nucl. Phys. 75, 669 (1966).

[14]I. Tamm, J. Phys. (USSR) 9, 449 (1945); S. M. Dancoff, Phys. Rev. 78, 382 (1950).

[15]M. Lévy, Phys. Rev. 88, 72 (1952); 88, 725 (1952); A. Klein, *ibid.* 90, 1101 (1953); G. Feldman *et al.*, Ref. 4.

[16]H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) 112, 94 (1978), and references therein.

[17]P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

[18]V. G. Kadyshevsky, Zh. Eksp. Teor. Fiz. 46, 654 (1964) [Sov. Phys. JETP 19, 443 (1964)]; V. G. Kadyshevsky, Nucl. Phys. B6, 125 (1968); C. Itzykson, V. G. Kadyshevsky, and I. T. Todorov, Phys. Rev. D 1, 2823 (1970)

[19]S. J. Chang and S. Ma, Phys. Rev. 180, 1506 (1969); S. J. Brodsky, R. Roskies, and R. Suaya, Phys. Rev. D 8, 4574 (1973); S. D. Drell, D. J. Levy, and T.-M. Yan, *ibid.* 1, 1035 (1970).

[20]F. Gross, Phys. Rev. 186, 1448 (1969).

[21]G. P. Lepage, Phys. Rev. A 16, 863 (1977).

[22]R. Blankenbecler and R. L. Sugar, Phys. Rev. 142, 1051 (1966).

[23]R. M. Woloshyn and A. D. Jackson, Nucl. Phys. B64, 269 (1973).

[24]A. Klein and T.-S. H. Lee, Phys. Rev. D 10, 4308 (1974); K. Erkelenz, Phys. Rep. 13C, 191 (1974).

[25]S. Love, Ann. Phys. (N.Y.) 113, 153 (1978).

[26]M. H. Partovi, Phys. Rev. D 12, 3887 (1975); 18, 2861 (1978); see also W. Krolkowski and J. Rzewuski, Nuovo Cimento 2, 203 (1955); 3, 260 (1955); 4, 1212 (1956).

[27]W. Celmaster and F. S. Henyey, Phys. Rev. D 17, 3268 (1978).

[28]D. Gromes, Nucl. Phys. B131, 80 (1977).

[29]G. Feldman, T. Fulton, and J. Townsend, Phys. Rev. D 7, 1814 (1973).

[30]E. H. Wichmann and J. H. Crichton, Phys. Rev. 132, 2788 (1963).

[31]L. Susskind, Phys. Rev. 165, 1535 (1968); K. Bardakci and M. B. Halpern, *ibid.* 176, 1686 (1968); J. Kogut and D. Soper, Phys. Rev. D 1, 2901 (1970); J. D. Bjorken, J. Kogut, and D. Soper, *ibid.* 3, 1382 (1971); R. A. Neville and F. Rohrlich, Nuovo Cimento 1A, 625 (1971); Phys. Rev. D 3, 1692 (1971); F. Rohrlich, in *Concepts in Hadron Physics*, proceedings of the X Schladming Conference on nuclear physics, edited by P. Urban (Springer, Berlin, 1971) [Acta Phys. Austriaca Suppl. 8 (1971)], p. 277.

[32]S. P. Kuleshov, A. N. Kvinikhidze, V. A. Matveev, A. N. Sissakian, and L. A. Slepchenko, Dubna Report No. E2-8128, 1974 (unpublished).

[33]J. M. Namyslowski, in *Few Body Systems and Nuclear Forces*, Lecture Notes in Physics, 1978, edited by Zingle, Haftel, and Zankel (Springer, Berlin, 1978), Vol. 82, p. 41.

[34]J. M. Namyslowski, University of Virginia report, 1979 (unpublished).

[35]See also J. M. Namyslowski, Phys. Rev. D 18, 3676 (1978).

[36]A. S. Wightman, in *Dispersion Relations and Elementary Particles*, edited by C. DeWitt and R. Omnès (Herman, Paris, 1960).

[37]See also J. M. Namyslowski and H. J. Weber, University of Virginia report, 1979 (unpublished).

[38]G. E. Brown, A. D. Jackson, and T. T. S. Kuo, Nucl. Phys. A133, 481 (1969); M. H. Partovi and E. L. Lomon, Phys. Rev. D 2, 1999 (1970).

[39]S.-J. Chang, R. G. Root, and T.-M. Yan, Phys. Rev. D 7, 1133 (1973); S.-J. Chang and T.-M. Yan, *ibid.* 7, 1145 (1973); T.-M. Yan, *ibid.* 7, 1760 (1973); 7, 1780 (1973).

[40]See also F. Rohrlich, Ref. 31.

[41]A. J. Macfarlane, Rev. Mod. Phys. 34, 41 (1962).

[42]R. N. Faustov, Ann. Phys. (N.Y.) 78, 176 (1973).