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Quantization of gauge theories in a finite volume without constraints or A_0 ambiguity

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We argue that, in a gauge field theory defined in a finite-volume box, some set of constraints should be imposed on the surface fields in order to get the proper equations of motion of the fields on the surface. For the pure SU(2) gauge theory we present four such sets. Each one of them predicts a different consistent theory, in which the ambiguity in A_0 is removed, and all the constraints become an identity of the theory. In each one of these theories some of the total electric and/or magnetic charges vanish identically. We realize these constraints and "Gauss's law" ($\dot{\pi}_0^a=0$) in a remodified axial gauge: $\partial_3 A_3=0$. Such an analysis and procedure can be done similarly for other gauge theories and/or other gauges.

I. INTRODUCTION

In the last few years, the interest in non-Abelian gauge theories has grown steadily. The hope is that such a theory would appear to be a good model for the strong interaction[1] as an unbroken one and for the weak interaction[2] as a broken one. The curious point in this hope is that these theories are, nowadays, not completely understood and will be only through hard work. In this paper we would like to clear up the situation slightly concerning the ambiugity of unbroken gauge theories.[3] The source of the ambiguity in these gauge theories is that they are defined by fields A_i^a and their conjugate momenta π_i^a which are not physically observable, and only some combinations of them have physical meanings. As a result of this situation nonphysical operators and states appear in many ways and produce many uncertainties about the validity of the calculation and interpretation[4] of the predictions of these theories (in the cases where some kind of predictions are possible). When the conventional Lagrangian or Hamiltonian formalism with the Poisson brackets (1.1) is used, the ambiguity problem presents itself by the vanishing of the π_0^a and the appearance of a local constrained operation $\dot{\pi}_0^a(\vec{\pi},\vec{A})=0$. This is the non-Abelian version of Gauss's law, which ensures that π_0^a will remain zero. Both constraints are in contradiction with (1.1):

$$\{A_\mu^a(\vec{x},t),\pi_\nu^b(\vec{y},t)\}=\delta_{\mu,\nu}^{a,b}\delta^3(\vec{x}-\vec{y})\,. \tag{1.1}$$

The conventional way of handling this embarrassing situation is to use Gauss's law to gauge away some combination of the fields $\{A_\lambda^a\}^G$ (e.g., axial gauge, Coulomb gauge). In this way Gauss's law becomes solvable for the vanishing fields[1] conjugate momenta $\{\pi_i^a\}^G$, and thus instead of being a constraint on the theory it becomes an identity. The fields A_0^a should be solved now from the equations $\{\dot{A}_i^a\}^G=0$ and expressed by the remaining fields. Such a procedure should lead to a well-defined theory, presented by a set of fields ($\{A_\mu^a\}-\{A_i^a\}^G-A_0^a$) and their canonical conjugate momenta, which are not subject to any kind of constraints. In this way the contradiction between (1.1) and the two rules $\pi_0=0$, $\dot{\pi}_0(\vec{\pi},\vec{A})=0$ disappears. Anyhow, it seems now that the second step in this procedure, namely the expression of A_0^a in terms of other fields, is not complete and leaves some dynamical degrees of freedom in A_0^a.[3] Taking into account that π_0^a is zero identically, the contradiction with (1.1) thus remains. This situation is most confusing, and may lead to all kinds of unreliable theories and predictions, arising from the contribution of nonphysical states.[4]

In order to improve this situation, one should find a way to define A_0^a completely in a way consistent with the whole theory. At this point, we leave for a moment this A_0^a-ambiguity problem and turn our attention to another embarrassing problem in these theories.

Taking a conventional field theory described by a set of fields ϕ^i and their canonical conjugate momenta π^i, in a box of volume $(2L)^3$, the Hamiltonian contains the kinetic energy term $\frac{1}{2}\int(\partial_j\phi^i)^2\,dv$, and thus one finds that the ordinary equations of motion (EOM) of π^i are not satisfied on the surface S of the box. Instead, one finds that $\pi^i(s)$ is linearly dependent on $\partial_n\phi^i(s)$ (the tangential derivative on the surface). If the fields ϕ^i describe a massive particle, one can neglect this problem if L is taken to be large enough. Anyhow, considering the problem of unbroken non-Abelian gauge theories with long-range forces, which is crucially dependent on the solutions of the EOM which falls as r^{-1}, this problem is in no way negligible. It was suggested[6] that in these theories this surface problem should be solved by adding to the theory more dynamical degrees of freedom on the surface.

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In this way when one introduces some constraints on the states the proper EOM are produced on the surface. We will not take this point of view for the following reasons. First, we do not see the point in the introduction of new dynamical degrees of freedom, which do not have any physical meaning. Second, the sets of constraints presented there [(6) a-15, (6) c-12] are not self-consistent for finite-volume "bags" unless various other constraints are satisfied; in particular, $E_{\parallel}(s)=0$. However, we cannot find any justification for such "universal" forces. In the following, we present four new types of bags, free from such restrictive conditions.

In this paper we suggest that in order to get the proper EOM on the surface the fields $\vec{A}^a(s)$ and their conjugate momenta $\vec{\pi}^a(s)$ should be subject to some set of constraints; these constraints, like Gauss's law, should be realized and become identities of the theory. We present in Sec. II, for example, four different sets of this kind, for the pure local gauge theory of SU(2). Each one implies some kind of constraints on $A_0(s)$ which exactly removes the ambiguity which is left after the realization of Gauss's law. In this way each set of constraints can be used to define a proper quantized gauge theory in a box of finite volume $(2L)^3$. The problem of presenting these constraints in an open space where $L\to\infty$ is left to further publications.

Defining the total electric and magnetic charges as $Q_E=\int\vec{E}\cdot d\vec{s}$, $Q_B=\int\vec{B}\cdot d\vec{s}$, the four theories are characterized as follows ($a=1,2,3$):

Theory I: $Q_E^3=Q_B^3=0$,

Theory II: $Q_E^a=Q_B^a=0$,

Theory III: $Q_E^a=0$,

Theory IV: $Q_E^1=Q_B^2=Q_B^3=0$.

It is not clear to us whether each set of constraints leads to a different actual physical prediction (e.g., confinement, spectra, scattering, amplitudes, etc.). Neither is it clear whether other sets of constraints are needed in order to describe some physical environments. In any case, one should go on investigating these problems and try to learn the physics which these theories predict.

In order to realize these theories we present in Sec. III a modified axial gauge defined by (1.2):

$$\partial_3 A_3^a=0 \quad (a=1,2,3). \tag{1.2}$$

It is necessary to define this gauge, as we find the commonly used axial gauge[7] $A_3=0$ too restrictive and not well defined.

In Sec. IV we realize Gauss's law and the surface constraints. The theory is thus well defined, leaving aside the order problem, and we can quantize.

That is done by replacing the Poisson brackets between the canonical variables by commutation relations in Sec. V.

As was mentioned above, we do not include fermions or scalars in our discussion. This procedure can be carried out with them in a similar fashion, by enlarging the set of constraints in a proper way. In environments such as the MIT bag theory, where the fermion's currents vanish on the surface, such an enlargement is trivial and our sets of constraints are sufficient for this purpose.

We reserve a last comment for the problem of the Hamiltonian which naively seems to imply an infinite energy.[8] We do not think that this problem is a consequence of the A_0 ambiguity only, and this should not motivate the identification of A_0.[9] Moreover, such an approach does not remove the infinite energies, which are of local nature. It seems to us that the correct approach to this problem should be similar to that in QED, namely to find the solution with finite energy and forget about all others.

II. SURFACE CONSTRAINTS

The common Lagrangian density which is used to describe pure non-Abelian SU(2) gauge theory is

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}=-\tfrac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{2.1}$$

$$\begin{aligned} F_{\mu\nu}^a &= \partial_\mu A_\nu^a-\partial_\nu A_\mu^a+g\epsilon^{abc}A_\mu^b A_\nu^c,\\ F_{\mu\nu} &= \tfrac{1}{2}\tau^a F_{\mu\nu}^a=\partial_\mu A_\nu-\partial_\nu A_\mu-ig[A_\mu,A_\nu]. \end{aligned} \tag{2.2}$$

[a is an isospin index, μ, ν, are Lorentz indices, and τ^a ($a=1,2,3$) are Pauli matrices.] The conjugate momenta of the A_μ^a fields are

$$\frac{\partial\mathcal{L}}{\partial\dot{A}_j^a}\equiv\pi_j^a=F_{0j}^a=-E_j^a \quad \left(B_i^a=\tfrac{1}{2}\epsilon_{ijk}F_{jk}^a\right), \tag{2.3}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{A}_0^a}\equiv\pi_0^a=0 \tag{2.4}$$

where the presence of too many degrees of freedom reflects itself in the vanishing π_0^a.

In order to continue with a Hamiltonian quantization procedure, one has to define the equal-time nonvanishing Poisson brackets of the theory to be given by (1.1). The contradiction between (2.4) and (1.1) for π^0 should be resolved by expressing A_0 explicitly in terms of other fields, and eliminating both A_0 and π_0 from the theory as independent degrees of freedom. The Hamiltonian of a theory, defined in a finite box of volume $(2L)^3$, is given by

$$H=\int_V d^3x[\tfrac{1}{2}\vec{\pi}^a\cdot\vec{\pi}^a+\tfrac{1}{2}\vec{B}^a\cdot\vec{B}^a+A_0^a(\vec{\nabla}\cdot\vec{\pi}^a+g\epsilon^{abc}\vec{\pi}^b\cdot\vec{A}^c)]$$
$$-\int_s d\vec{s}\cdot\vec{\pi}^a A_0^a. \tag{2.5}$$

The equations of motion (EOM) are derived from the Hamiltonian (2.5) by using the (1.1) Poisson bracket. Inside the box one gets the EOM (2.6), (2.7):

$$\dot{\vec{A}}=\{\vec{A},H\}=\vec{\pi}-\vec{\partial}A_0+ig[A_0,\vec{A}], \tag{2.6}$$

$$\dot{\vec{\pi}}^a=\{\vec{\pi}^a,H\}$$
$$=-\vec{\nabla}\times\vec{B}^a+g\epsilon^{abc}\vec{\pi}^bA_0^c+g\epsilon^{abc}\vec{B}^b\times\vec{A}^c. \tag{2.7}$$

The operator π_0 should vanish for all times; hence one gets the SU(2) Gauss's law constraints. Inside the box one gets the constraints

$$\dot{\pi}_0=\{\pi_0,H\}=-ig[\vec{\pi},\vec{A}]+\vec{\nabla}\cdot\vec{\pi}. \tag{2.8}$$

(As $\{A_i^a,\pi_j^b\}=0$ for $i\neq j$, the brackets $[\vec{A},\vec{\pi}]$ are well defined and are equal to $\epsilon^{abc}\frac{1}{2}\tau^a\vec{A}^b\cdot\vec{\pi}^c$.) Inside the box (2.6), (2.7) are the proper EOM and (2.8) is the proper Gauss's law constraint, which together define a Yang-Mills gauge theory.

The contradiction between (2.8) and (1.1) is resolved by realization of Gauss's law. That will be done in Sec. III.

Contrary to the proper EOM for $\vec{A}$, $\vec{\pi}$, and the $\pi_0=0$ constraints given by (2.6), (2.7), (2.8), which are obtained inside the box, the resulting EOM [obtained from (2.6) and (1.1)] for $\pi_1(S)$ [the parallel to the surface components of $\vec{\pi}(S)$] and the constraints $\pi_0(S)$ are different from (2.7) and (2.8) and thus are wrong.

In the following we will discuss the situation on two facing sides of the box, a surface we denoted by S_3. The two faces $S_3^\pm$ are defined by $\{x_3=\pm L\,|\,S_3^\pm\ni\vec{x}\}$. The following discussion is valid also for the other axes of the box (surfaces S_1 and S_2). In these cases, instead of the third space direction, each time one should take the normal direction to the surface, and instead of the first and second space directions, the analog parallel directions to the surface should be taken.

The resulting EOM for $\pi_\alpha(\vec{X}_\alpha^\pm)$ [we denote the surface points $(X_1,X_2,\pm L)$ by $(\vec{X}_\alpha^\pm)$; $\alpha=1,2$] and the constraints $\pi_0(X_\alpha^\pm)$ are given by

$$\dot{\pi}^a_{1(2)}(\vec{X}_\alpha^\pm)=(\cdots)(\mp)\int_{s+}d\vec{y}_\alpha^+\{A^b_{1(2)}(\vec{y}_\alpha^+),\pi^a_{1(2)}(\vec{X}_\alpha^\pm)\}B^b_{2(1)}(\vec{y}_\alpha^\pm)\,(\pm)\int d\vec{y}_\alpha^-\{A^b_{1(2)}(\vec{y}_\alpha^-),\pi_{1(2)}(\vec{X}_\alpha^\pm)B^b_{2(1)}(\vec{y}_\alpha^-)$$
$$=(\cdots)(\pm)B^a_{2(1)}(\vec{X}_\alpha^\pm)\delta(0), \tag{2.9}$$
$$\dot{\pi}_0^a(\vec{X}_\alpha^\pm)=(\cdots)\pm\pi_3^a(\vec{X}_\alpha^\pm)\delta(0). \tag{2.10}$$

[The three dots in (2.9) and (2.10) represent the correct EOM of $\pi^a(\vec{X}_\alpha^\pm)$ and the $\dot{\pi}_0^a(\vec{X}_\alpha^\pm)=0$ constraint, as they appear in (2.7) and (2.8), correspondingly.] It is seen that these are not the proper EOM and constraint which we get inside the box. Therefore, we should expect to get these wrong equations on the surface, as the Hamiltonian in the box (2.5) does not contain enough information to treat properly the fields on the surface.

In order to get out from this embarrassing situation, one should enforce some kind of constraints on the surface fields, which will change this situation. By such a method, some of the dynamical degrees of freedom are removed from the theory by these contraints in a way that the remaining ones form a consistent theory with the proper EOM.

In the following, we present some sets of such constraints. Each one of the sets leads to a different environment in which the proper EOM are obtained from a well-defined Hamiltonian and a set of Poisson brackets.

In general, it seems that there are basically two kinds of classes of such constraints. The first contain constraints which enforce the vanishing of some of the fields, whereas the second contain constraints which enforce definite relations between the fields on the surface. In this paper we will present some simple examples of both these types, while more sophisticated examples will hopefully be given in the future.

To begin with we look at six kinds of basic constraints which we would like to apply to the fields on the surface:

$$\begin{aligned}
&(\mathrm{a})\ A_i^a(\vec{X}_\alpha^\pm)=0,\quad \pi_i^a(\vec{X}_\alpha^\pm)=0,\\
&(\partial\mathrm{a})\ \partial_3A_i^a(\vec{X}_\alpha^\pm)=0,\quad \partial_3\pi_i^a(\vec{X}_\alpha^\pm)=0,\\
&(\mathrm{b})\ A_i^a(\vec{X}_\alpha^+)=A_i^a(\vec{X}_\alpha^-),\quad \pi_i^a(\vec{X}_\alpha^+)=\pi_i^a(\vec{X}_\alpha^-),\\
&(\partial\mathrm{b})\ \partial_3A_i^a(\vec{X}_\alpha^+)=\partial_3A_i^a(\vec{X}_\alpha^-),\quad \partial_3\pi_i^a(\vec{X}_\alpha^+)=\partial_3\pi_i^a(\vec{X}_\alpha^-),\\
&(\mathrm{c})\ A_i^a(\vec{X}_\alpha^+)=-A_i^a(\vec{X}_\alpha^-),\quad \pi_i^a(\vec{X}_\alpha^+)=-\pi_i^a(\vec{X}_\alpha^-),\\
&(\partial\mathrm{c})\ \partial_3A_i^a(\vec{X}_\alpha^+)=-\partial_3A_i^a(\vec{X}_\alpha^-),\quad \partial_3\pi_i^a(\vec{X}_\alpha^+)=-\partial_3\pi_i^a(\vec{X}_\alpha^-).
\end{aligned} \tag{2.11}$$

Each component of the fields will be subject to different constraints. Nevertheless, in all the simple examples which will be given below we demand that the fields $A_1^a(\vec{X}_\alpha^\pm)$, $A_2^a(\vec{X}_\alpha^\pm)$, $\pi_1^a(\vec{X}_\alpha^\pm)$,

$\pi_2^a(\vec{X}_\alpha^\pm)$ satisfy the same constraints at S_3 (for each α index) for each one of the above six constraints does not hold on its own. In order to fix it, one must require that it be a constant of the theory; thus other constraints arise. In this way the constraints on different components of the fields are correlated and subsidiary constraints arise. In the following, we examine these correlations between the different constraints. First, we take for simplicity the $g=0$ theory, and discuss the three U(1) theories which are obtained. The time derivatives of $\vec{\pi}(\vec{X}_\alpha^\pm)$ and $\vec{A}(\vec{X}_\alpha^\pm)$ in U(1) gauge theory are given by

$$\dot{\vec{\pi}} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}, \tag{2.12}$$

$$\dot{\vec{A}} = \vec{\pi} - \vec{\nabla}A_0. \tag{2.13}$$

These EOM imply that in order that the constraints of types (a) or (b) or (c) acting on $\pi_\alpha(\vec{X}_\alpha^\pm)$, $A_\alpha(\vec{X}_\alpha^\pm)$ have zero time derivatives, the field $A_0(\vec{X}_\alpha^\pm)$ would be constrained in the same way and $\pi_3(\vec{X}_\alpha^\pm)$ should be constrained by (∂a), or (∂b), or (∂c), correspondingly.

Furthermore, one finds that the time derivative of the above constraint on $\pi_3(\vec{X}_\alpha^\pm)$, namely, ($\partial$a), or ($\partial$b), or ($\partial$c), vanishes if the fields $A_\alpha(\vec{X}_\alpha^\pm)$, $\pi_\alpha(\vec{X}_\alpha^\pm)$, and $A_0(\vec{X}_\alpha^\pm)$ are constrained by (∂^2a), or (∂^2b), or (∂^2c) types of constraints, correspondingly. Continuing in this way, one concludes that six proper self-consistent sets of constraints on the two sets of fields $\{A_\alpha(\vec{X}_\alpha^\pm), \pi_\alpha(\vec{X}_\alpha^\pm), A_0(\vec{X}_\alpha^\pm); A_3(\vec{X}_\alpha^\pm), \pi_3(\vec{X}_3)\}$ are

$$\begin{aligned}
&(1)\ \{\partial_3{}^{2m}a;\ \partial_3{}^{2m+1}a\}_m,\\
&(2)\ \{\partial_3{}^{2m}b;\ \partial_3{}^{2m+1}b\}_m,\\
&(3)\ \{\partial_3{}^{2m}c;\ \partial_3{}^{2m+1}c\}_m,\\
&(4)\ \{\partial^{2m+1}a;\ \partial^{2m}a\}_m,\\
&(5)\ \{\partial^{2m+1}b;\ \partial^{2m}b\}_m,\\
&(6)\ \{\partial^{2m+1}c;\ \partial^{2m}c\}_m.
\end{aligned} \tag{2.14}$$

[Each set is constructed from two different subsets (m is a non-negative integer), and when one operator of the first subset is acting on one of the fields $A_\alpha(\vec{X}_\alpha^\pm)$, $\pi_\alpha(\vec{X}_\alpha^\pm)$, $A_0(\vec{X}_\alpha^\pm)$, it should vanish. Similarly, when one operator of the second subset is acting on the fields $A_3(\vec{X}_\alpha^\pm)$, $\pi_3(\vec{X}_\alpha^\pm)$, it should vanish.]

Turning back now to the wrong EOM of $\pi_\alpha(\vec{X}_\alpha^\pm)$, $\pi_0(X_\pm)$, we see that a theory constrained by the above (1) or (2) set of constraints will predict the correct EOM on the surface. Each one of the other sets of constraints is not sufficient for this purpose. Thus the second (4) and the fifth (5) sets together, and the third (3) and the sixth (6) together, predict the correct EOM. Hence in U(1) gauge theory each one of the following four sets of constraints which vanish on the two sets of fields: $\{A_\alpha(\vec{X}_\alpha^\pm), \pi_\alpha(\vec{X}_\alpha^\pm), A_0(\vec{X}_\alpha^\pm); A_3(\vec{X}_\alpha^\pm), \pi_3(\vec{X}_\alpha^\pm)\}$ is sufficient to ensure that the proper EOM will be obtained on the surface

$$\{\partial^{2m}a;\ \partial^{2m+1}a\}_m, \tag{2.15a}$$

$$\{\partial^{2m+1}a;\ \partial^{2m}a\}_m, \tag{2.15b}$$

$$\{\partial^m b;\ \partial^m b\}_m, \tag{2.15c}$$

$$\{\partial^m c;\ \partial^m c\}_m. \tag{2.15d}$$

TABLE I. Definition of four SU(2) gauge theories, by the constraints that each one of the isospins index fields satisfies.

Field of isospin index	1	2	3
Theory I	(2.15d)	(2.15d)	(2.15c)
Theory II	(2.15c)	(2.15c)	(2.15c)
Theory III	(2.15b)	(2.15b)	(2.15b)
Theory IV	(2.15b)	(2.15a)	(2.15a)

Turning back to an SU(2) gauge theory with $g \neq 0$, we find that only four different theories can be defined by the (2.15) kind of constraints, regarding EOM with $g\neq 0$ [Eqs. (2.6), (2.7)].

In each one of these theories the fields satisfy the proper EOM given by (2.6), (2.7), and the Gauss's law constraint given by (2.8) in the whole box. Each one of the theories is subject to a different set of constraints. These theories are defined in Table I, where in each column the constraints on one of the isospin index fields are given.

As A_0 should be solved from a second-order differential equation, the ambiguity which is left in its identification would be eliminated in each one of these constrained theories. Hence we expect that in each one of these four theories, A_0 does not suffer from any ambiguity. We will not give a full proof of it, but will show it explicitly in a specific gauge, namely the modified axial gauge "$\partial_3 A_3 = 0$." This is done in Sec. IV for each one of these four theories, where in Sec. III we present this modified axial gauge. It should be noted that each theory given above describes a different physical environment in a finite volume of dimension $(2L)^3$. At this stage it is not clear to us if the limiting theories ($L\to\infty$) are physically equivalent concerning actual problems. It should be clear that it might not be the case and only part of them might be correlated with experiments (spectra, scattering amplitudes, etc.). Hence one should find principally all the possible kinds of constraints and then choose the proper ones according to their physical predictions.

III. MODIFIED AXIAL GAUGE: $\partial_3 A_3 = 0$

The main task of this paper is to get in a box of volume $(2L)^3$ a quantized Yang-Mills field theory defined by an Hamiltonian which is constructed from a set of free canonical dynamical variables with well-defined Poisson brackets. The word "free" in the last sentence means that the set of the dynamical variables is not subject to any kind of constraint. In order to present a gauge theory in this way, one has to fix π_0 and Gauss's law to be operationally identically zero. The common way of doing it is by using all the local constraints of Gauss's law to generate a specific gauge transformation. In this way one eliminates some components of $\vec{A}$ from the theory (e.g., A_T, A_3), solves Gauss's law for its conjugate momenta (e.g., π_T, π_3), and stops treating them as canonical variables. In this way Gauss's law becomes an identity, and is in no way in conflict with the Poisson brackets of the remaining free dynamical variables. The field A_0 should be solved now from the EOM of the vanishing $\vec{A}$ field, and $\pi_0 \equiv 0$ is consistent with (1.1). (For excellent consistent review of this procedure, see Regge *et al.*, in Ref. 5.) It is far from being trivial to prove that all field configurations can be transformed in this way to one specific gauge. Thus we take the following approach. If Gauss's law can be solved for some component of $\vec{\pi}$ when its canonical conjugate field is determined to be zero, the resulting gauge is a consistent one. In such a theory the inconsistency between (1.1) and (2.8) is resolved, and A_0 can be solved apart from the ambiguity mentioned above. A proof that all such formally different gauge theories are physically equivalent is still most desirable.

In this paper we choose to work in a modified axial gauge $\partial_3 A_3 = 0$. We choose the $\partial_3 A_3 = 0$ gauge instead of the usual axial gauge[7] $A_3 = 0$ because Gauss's law is not solvable for the constant part of $\pi_3(X)$, denoted by ${}^0\pi_3(\vec{X}_\alpha)$. Thus there is no way to gauge A_3 to zero. Moreover, a theory with $A_3 = 0$ and independent constant part of π_3 contains explicit contradiction to the Poisson bracket (1.1)—solving it by elimination of ${}^0\pi_3(\vec{X}_\alpha)$ is not trivial as ${}^0\pi_3(\vec{X}_\alpha)$ interacts with the other fields.

In the $\partial_3 A_3 = 0$ space, A_3 is a function of $\vec{X}_\alpha$ only, and we will denote it by ${}^0A_3$. The differential equation for π_3 is

$$\partial_3 \pi_3 = \rho_3 + ig[\pi_3, {}^0A_3],$$

where

$$\rho_3 = -\partial_\alpha \pi_\alpha + ig[\pi_\alpha, A_\alpha]. \tag{3.1}$$

Defining for each matrix operation $Q = Q^a \frac{1}{2}\tau^a$ an operator $\tilde{Q}$ by

$$\tilde{Q} \equiv e^{+ig{}^0A_3X_3} Q e^{-ig{}^0A_3X_3}.$$

Gauss's law can be written as in (3.2),

$$\partial_3 \tilde{\pi}_3 = \tilde{\rho}_3. \tag{3.2}$$

Hence

$$\begin{aligned}\tilde{\pi}_3(\vec{X}) = {}^0\pi_3(\vec{X}_\alpha) + \frac{1}{2L}\tilde{Q}_{3(1)}(\vec{X}_\alpha) &+ \int_{-L}^{+L} e^{ig{}^0A_3X_3'} \rho_3(\vec{X}_\alpha, X_3') \\ &\times e^{-ig{}^0A_3x_3'} \epsilon(x_3 - x_3') dx_3', \\ \pi_3(\vec{X}) = e^{-ig{}^0A_3X_3} \tilde{\pi}_3(\vec{X}) e^{+ig{}^0A_3X_3}&\end{aligned} \tag{3.3}$$

$$\left(\tilde{Q}_{3(1)}(\vec{X}_\alpha) = \int_{-L}^{L} X_3' \rho_3(\vec{X}_\alpha, X_3') dx_3'\right)$$

[see the definition of the operators ${}^0Q$, ${}^*Q$ in (4.1)]. At this stage ${}^0A_3$ and ${}^0\pi_3$ are the only independent degrees of freedom which were left from A_3, π_3, and the only nonvanishing equal-time Poisson brackets for ${}^0\pi_3^a$ are

$$\{{}^0A_3^a(\vec{X}_\alpha), {}^0\pi_3^b(\vec{y}_\alpha)\} = \frac{1}{2L}\delta^{a,b}\delta^2(\vec{X}_\alpha - \vec{y}_\alpha). \tag{3.4}$$

After solving Gauss's law, we now solve A_0 by the EOM of A_3:

$$\pi_3 = \dot{A}_3 + \partial_3 \dot{A}_0 - ig[A_0, {}^0A_3]. \tag{3.5}$$

Thus

$$\pi_3 = \dot{A}_3 + e^{-ig{}^0A_3X_3}(\partial_3 A_0) e^{+ig{}^0A_3X_3}. \tag{3.6}$$

Hence

$$\tilde{\pi}_3 = \tilde{T} + \partial_3 \tilde{A}, \tag{3.7}$$

where T is a constant matrix. The solution for A_0 is

$$A_0(\vec{X}) = e^{-ig{}^0A_3}\left\{{}^0\pi_3 X_3 + \frac{X_3}{2L}\tilde{Q}_{3(1)} + \tfrac{1}{2}\int_{-L}^{+L} [\rho(\vec{X}_\alpha, X_3')\,|X_3 - X_3'| + 2\tilde{T}\epsilon(X_3 - X_3') dx_s'] + V\right\} e^{+ig{}^0A_3X_3}. \tag{3.8}$$

It is seen that although Gauss's law was solved and becomes an operator identity, A_0 given by (3.8) is not completely solved and some part of it T, V, remain as independent operators.

It should be noticed that the physical reason for this situation is that the number of constraints given by $\pi_0=0$ is greater than the number of the constraints needed to fix π_0 in time, namely, the Gauss's law $\dot{\pi}_0=0$. This is seen most easily by counting the number of degrees of freedom which are frozen by these two different laws, on a finite lattice.

The ambiguity which is left in A_0 reflects itself in many ways. For example, the Poisson bracket $\{A_0, \pi_0\}$ does not vanish, although π^0 is zero. This specific problem can be resolved by constraining the so-called "physical states." Nevertheless, the unphysical states manifest themselves in other ways. One of the most important ones seems to be the unphysical singularities which appear in the field propagator in some gauges. The interpretation of this situation is far from clear. All kinds of speculation about confinement,[4] etc, which rely on it, stand on shaky ground as the role of unphysical states in such an ambiguous theory is not understood yet.

Hence it seems to us that in order to get a consistent theory, one should try to get rid of the constraints on the states, and A_0 should be solved completely, even if that means specifying the environment in which the theory is defined.

Turning back to our expression (3.8) for A_0 we see that A_0 is already quite solved. The remaining degrees of freedom T and V represent the freedom of A_0 on the two surfaces $X_3=\pm L$. Hence it seems that one has to look for other kinds of constraints besides Gauss's law, which should hold on the surface and fix A_0 completely. In our opinion the starting point of this search should be whether or not the proper EOM can be obtained on the surface. It seems to us that the four theories which were defined in Sec. II are good examples of how to solve this problem as they define A_0 completely and predict the proper EOM.

IV. FOUR SU(2) GAUGE THEORIES IN THE $\partial_3 A_3=0$ GAUGE

In this section we realize Gauss's law by gauging the field $A_3(X)$ to $\partial_3 A_3=0$ and solving π_3 and A_0 in the four theories defined in Sec. III. As it appears, we can use the constraints on the surface in each theory in order to rotate $A_3(\vec{X}_\alpha)$ to be $A_3^3(\vec{X}_\alpha)\frac{1}{2}\tau^3 \equiv \bar{A}_3(\vec{X}_\alpha)$. To simplify the notation we will define the two operators ${}^0Q(\vec{X}_\alpha)$, ${}^*Q(\vec{X})$ for each local operator $Q(\vec{X})$ in the following way:

$$\begin{aligned} &Q(\vec{X}) \equiv {}^*Q(\vec{X}) + {}^0Q(\vec{X}_\alpha)\,, \\ &{}^0Q(\vec{X}_\alpha) = \frac{1}{2L}\int_{-L}^{+L} Q(\vec{X}_\alpha, X_3)dx_3\,, \\ &\Rightarrow \int_{-L}^{+L} {}^*Q(\vec{X}_\alpha, X_3)dX_3 = 0\,. \end{aligned} \tag{4.1}$$

In the same manner we define the two operators ${}^{*,0/*}Q(\vec{X})$, ${}^{0,0/*}Q(\vec{X})$ for the operators ${}^{0/*}Q(\vec{X})$:

$$\begin{aligned} &{}^{0/*}Q(\vec{X}) \equiv {}^{*,0/*}Q(\vec{X}) + {}^{0,0/*}Q(X_1, X_3)\,, \\ &{}^{0,0/*}Q(X_1,X_3) = \frac{1}{2L}\int_{-L}^{+L,\,0/*} Q(\vec{X}')\,dX_2'\,, \\ &\Rightarrow \int_{-L}^{L} dx_2\ {}^{*,0/*}Q(\vec{X}) = 0\,. \end{aligned} \tag{4.2}$$

A. Theory I

Beginning with the theory I, defined by the constraints [Eqs. (2.15a)–(2.15d)] in the $\partial_3 A_3=0$ gauge, we first solve Gauss's law for π_3. In this theory ${}^0A_3(\vec{X}_\alpha)$ does not vanish in the third isospin direction only the other two components ${}^0A_3^{\alpha}(\vec{X}_\alpha)$ and their conjugate momenta are eliminated by the surface constraints. The field π_3 is given by (4.3), (4.4):

$$\pi_3^3(\vec{X}) = {}^0\pi_3^3(\vec{X}_\alpha) + \frac{1}{2L}Q_{3(1)}^3(\vec{X}_\alpha) + \int \rho_3^3(\vec{X}^1)\epsilon(X_3 - X_3')dX_3'\,, \tag{4.3}$$

$$\pi_3^{\alpha}(\vec{X}) = \int e^{ig\bar{A}_3(X_3'-X_3)}\rho_3^{\underline{\alpha}}(\vec{X}')e^{-ig\bar{A}_3(X_3'-X_3)} \times \epsilon(X_3 - X_3')dX_3' \quad (Q^{\underline{\alpha}} = \tfrac{1}{2}\tau^1 Q^1 + \tfrac{1}{2}\tau^2 Q^2)\,, \tag{4.4}$$

where $Q_{3(1)}^3$ are defined by (4.5):

$$Q_{3(n)}^i(\vec{X}_\alpha) = \int \rho_3^i(\vec{X}_\alpha, X_3)X_3^n dx_3\,. \tag{4.5}$$

At this stage one has to take care that the condition $\pi_3^3(\vec{X}_\alpha, +L) = \pi_3^3(\vec{X}_\alpha, -L)$ be satisfied. That implies the constraint

$$\rho_3^3(\vec{X}_\alpha, x_3)dx_3 = 0\,. \tag{4.6}$$

The other two constraints $\pi_3^{\alpha}(\vec{X}_\alpha, +L) = -\pi_3^{\alpha}(\vec{X}_\alpha, -L)$ are satisfied identically.

In order to realize the constraint (4.6) we gauge ${}^0A_2^3(\vec{X}_\alpha)$ to $\partial_2\, {}^0A_2^3=0$, and then solve (4.6) for its conjugate momenta ${}^0\pi_2^3(\vec{X}_\alpha)$ in (4.7):

$${}^0\pi_2^3(\vec{X}_\alpha) = {}^{0,0}\pi_2^3 + \frac{1}{2L}Q_{3,2(1)}^3 + \int {}^0\rho_{3,2}^3(x_1, x_2')\epsilon(x_2 - x_2')dx_2'\,, \tag{4.7}$$

$$\begin{aligned} {}^0\rho_{3,2}^3(X_\alpha) &= -\int (\partial^0\pi_2^3 + [\pi_1^{\underline{\alpha}}, A_1^{\underline{\alpha}}] + [\pi_2^{\underline{\alpha}}, A_2^{\underline{\alpha}}])(\vec{X}_\alpha, X_3)dx_3^3 \\ &= -(\partial^0\pi_2^3 + {}^0[\pi_1^{\underline{\alpha}}, A_1^{\underline{\alpha}}] + {}^0[\pi_2^{\underline{\alpha}}, A_2^{\underline{\alpha}}])\,. \end{aligned} \tag{4.8}$$

The operator $Q_{3,2(2)}^3$ is defined by (4.9):

$$Q_{3,2(2)}^3(x_n) = \int {}^0\rho_{3,2}(x_1, x_2)x_2^n\, dx_2\,. \tag{4.9}$$

The field ${}^0A_2^3$ is given now by ${}^{0,0}A_2^3$ and satisfies

$$\partial_2 {}^0A_2^3 = 0\,. \tag{4.10}$$

Continuing this procedure, we have to employ the conditions ${}^0\pi_2^3(X_1, X_2 = L) = {}^0\pi_2^3(X_1, X_2 = -L)$, which results in the condition

$$\int {}^0\rho^3_{3,2}(x_1, x_2)dx_2 = 0\,. \tag{4.11}$$

The realization of (4.11) is achieved by gauging away the operator $\partial_1 {}^{0,0}A_1^3(X_1)$ [Eq. (4.13)] and solving for its conjugate momenta (4.12):

$$ {}^{0,0}\pi_1^3(x_1) = {}^{0,0,0}\pi_1^3 + \frac{1}{2L}Q^3_{3,2(1)} + \int {}^{00}\rho_{3,2,1}(x_\alpha)\epsilon(x_1 - x_1')dx_1'\,, \tag{4.12}$$

$${}^{0,0}\rho^3_{3,2,1}(x_1) = -\int {}^0([\pi_1^\alpha, A_1^\alpha] - [\pi_2^\alpha, A_2^\alpha])(x_1, x_2)dx_2\,,$$

$$Q^3_{3,2,1}(x) = -\int {}^{00}\rho^3_{3,2,1}(x_1)x\,dx_1, \quad \partial_1 {}^{0,0}A_1^3(x_1) = 0\,. \tag{4.13}$$

To ensure the last constraint ${}^{0,0}\pi_1^3(X_1 = L) = {}^{0,0}\pi_1^3(X_1 = -L)$, one has to realize also the global condition (4.14) which is the last one in this theory,

$$Q_E^3 = 0 = \int {}^{0,0}\rho^3_{3,2,1}(x_1)\,dx_1\,. \tag{4.14}$$

Thus, this is a global constraint, and it should not have explicit effects on the dynamics of the theory. Thus we leave it as a constraint on the states which should have zero global electric charge in the third isospin direction. After realizing Gauss's law and all the local boundary constraints, we are going to solve for the A_0 operator. We begin with the two isospin components 1, 2. As A_3^α vanishes identically, the differential equation for A_0^α is

$$\pi_3^\alpha(\vec{x}) = \partial_3 A_0^\alpha + ig[A_0^\alpha, \bar{A}_3]\,. \tag{4.15}$$

The solution for A_0^α is given by

$$A_0^\alpha(\vec{X}) = \exp(-ig\bar{A}_3x_3)\left(\tfrac{1}{2}\int \rho_3^\alpha(\vec{X}_\alpha, X_3')\,|x_3 - x_3{}^1|\,dx_3{}^1 + V^\alpha(\vec{X}_\alpha)\right)\exp(+ig\bar{A}_3X_3)\,. \tag{4.16}$$

As we claim, Gauss's law leaves in A_0^α, the undefined operator $V^\alpha(\vec{X}_\alpha)$, which should be fixed now by the surface constraint $A_0^\alpha(\vec{X}_\alpha + L) = -A_0^\alpha(\vec{X}_\alpha, -L)$. This constraint is given now by (4.17) and has the solution (4.18):

$$e^{-ig\bar{A}_3L}[\tfrac{1}{2}L\hat{Q}^\alpha(\vec{X}_\alpha) - \tfrac{1}{2}\hat{Q}^\alpha_{(1)}(\vec{X}_\alpha) + V^\alpha(\vec{X}_\alpha)]e^{+ig\bar{A}_3L} = -e^{+ig\bar{A}_3L}[\tfrac{1}{2}L\hat{Q}^\alpha(\vec{X}_\alpha) + \tfrac{1}{2}\hat{Q}^\alpha_{(1)}(\vec{X}_\alpha) + V^\alpha(\vec{X}_\alpha)]e^{-ig\bar{A}_3L}\,, \tag{4.17}$$

$$\begin{aligned} V^1(\vec{X}_\alpha) &= -\tfrac{1}{2}\hat{Q}^2_{(1)}\tan(g\bar{A}_3L) - \tfrac{1}{2}L\hat{Q}^1(\vec{X}_\alpha)\,,\\ V^2(\vec{X}_\alpha) &= \tfrac{1}{2}\hat{Q}^1_{(1)}\tan(g\bar{A}_3L) - \tfrac{1}{2}L\hat{Q}^2(\vec{X}_\alpha) \quad \left(\hat{Q}^2_{(1)} = \int e^{+ig\bar{A}_3x_3}\rho_3^i(\vec{X}_\alpha, x_3)e^{-ig\bar{A}_3x_3}x_3^1\,dx_3\right). \end{aligned} \tag{4.18}$$

We solve now for $A_0{}^3$. The differential equation which defines it is

$$\pi_3^3 = \dot{A}_3^3 + \partial_3 A_0^3\,. \tag{4.19}$$

Before solving the above equation we would like to use the surface constraint. We integrate (4.19) over X_3 from $-L$ to $+L$ and employ the condition $A_0^3(\vec{X}_\alpha, +L) = A_0^3(\vec{X}_\alpha, -L)$ to get the more definitive equation (4.20) for $A_0{}^3$:

$$\pi_3^3(\vec{x}) - {}^0\pi_3^3(\vec{x}) = \partial_3 A_0^3(\vec{x})\,. \tag{4.20}$$

The solution to (4.20) is given by (4.21); it defines ${}^*A_0{}^3$:

$${}^*A_0^3(\vec{x}) = \tfrac{1}{2}\int \rho_3^3(\vec{x}_\alpha, x_3')\left(|x_3 - x_3'| - \frac{L}{2} - \frac{x_3'^2}{2L}\right)dx_3' + \frac{X_3}{2L}Q^3_{3(1)}\,. \tag{4.21}$$

This representation of ${}^*A_0^3(X)$ by (4.21) satisfies the above conditions on the surface. The freedom which is left in A_0^3, namely ${}^0A_0^3(\vec{X}_\alpha)$, should be solved through the EOM of ${}^0A_2^3$ and ${}^{0,0}A_1^3$.

The gauging of ${}^0A_2^3$ [(4.10)] implies

$${}^0\pi_2^3 = {}^0\dot{A}_2^3 + \partial_2 {}^0A_0^3 + ig\,{}^0[A^\alpha, A_2^\alpha]\,. \tag{4.22}$$

We integrate (4.22) over X_2 from $-L$ to L and using the boundary constraint ${}^0A_0^3(X_1, +L) = {}^0A_0^3(X_1, -L)$ to get the more definitive differential equation for ${}^0A_0^3$:

$${}^0\pi_2^3 - {}^{0,0}\pi_2^3 = \partial_2 A_0^3 + ig\,{}^{*,0}[A_0^\alpha, A_2^\alpha]\,. \tag{4.23}$$

With the solution for ${}^{*,0}A_0^3$,

$${}^{*,0}A_0^3 = \tfrac{1}{2}\int \rho^3_{3,2}(X_1, X_2')\left(|X_2 - X_2'| - \frac{L}{2} - \frac{X_2'^2}{2L}\right)dx_2' + \frac{X_2}{2L}Q^3_{3,2(1)} - ig\int^{*,0}[A_0^\alpha, A_2^\alpha](X_1, X_2')\left[\epsilon(X_2 - X_2') + \frac{x_2'}{2L}\right]dX_2'\,. \tag{4.24}$$

The remaining freedom of A_0^3 at this stage is ${}^{0,0}A_0^3$ which should be solved from the EOM of ${}^{0,0}A_1^3$.

Applying the boundary condition ${}^{0,0}A_0^3(X_1=L) = {}^{0,0}A_0^3(X_1=-L)$ we get

$$ {}^{0,0}\pi_1^3(X_1) - {}^{0,0,0}\pi_1^3 = \partial_1^{0,0}A_0^3 + ig^{*,0,0}[A_0^{\alpha}, A_1^{\alpha}] . \tag{4.25}$$

The solution for ${}^{*,0,0}A_0^3$ is then given by

$$ {}^{*,0,0}A_0 = \tfrac{1}{2}\int \rho^3_{3,2,1}(X_1')\left(|x_1 - x_1'| - \frac{l}{2} - \frac{x_1'^2}{2L}\right)dx_1' + \frac{x_1}{2L}Q^3_{3,2,1(1)} - ig\int^{*,0,0}[A_0^{\alpha}, A_2^{\alpha}](x_1')\left(\epsilon(X_1 - X_1') + \frac{x_1'}{2L}\right)dX_1' . \tag{4.26}$$

The complete solution for A_0^3 is given by

$$ A_0^3(\vec{x}) = {}^*A_0^3(\vec{x}) + {}^{*,0}A_0^3(\vec{x}_{\alpha}) + {}^{*,0,0}A(X_1) + M^3 . \tag{4.27}$$

As we claim, the realization of Gauss's law plus the surface constraints determines A_0 completely, and in a consistent way. We left one global constraint as a symmetry of the theory, without realization and thus A_0 contains one global undefined operator M^3 and the physical states are subject to the constraints $Q_E^3 = 0$. We will discuss this global symmetry in a future paper. Nevertheless, we do not expect it to influence the dynamics of the theory.

B. Theory II

In theory II which is constrained by the set (2.15c) all the components of ${}^0A_3(\vec{X}_{\alpha})$ do not vanish. We use these surface constraints to generate an additional X_3 independent rotation which rotates ${}^0A_3(\vec{X}_{\alpha})$ to the third axis. Thus $A_3(\vec{X}_{\alpha}) = \bar{A}_3{}^3$ and the solution for π_3 is given by

$$ \pi_3^3(\vec{x}) = {}^0\pi_3^3(x_{\alpha}) + \frac{1}{2L}Q^3_{3(1)} + \int \rho_3^3(\vec{x}_{\alpha}, x_3')\epsilon(x_3 - x_3')dX_3' . \tag{4.28}$$

The condition $\pi_3(\vec{X}_{\alpha'}+L) = \pi_3(\vec{X}_{\alpha'}-L)$ implies that

$$ \int \rho_3^3(\vec{x}_{\alpha}, X_3)\,dx_3 = 0 . \tag{4.29}$$

It is the same condition as (4.6) in the first theory and we solve it in the same manner with Eqs. (4.7), (4.10), and (4.12)–(4.14). The conjugate π_3^{α} are given by

$$ \pi_3^{\alpha}(\vec{x}) = e^{+ig\bar{A}_3X_3}\left({}^0\pi_3^{\alpha} + \int \tilde{\rho}(\vec{x}_{\alpha}, x_3')\epsilon(x_3 - x_3')dx_3'\right)e^{-ig\bar{A}_3x_3} . \tag{4.30}$$

The constraint $\pi_3^{\alpha}(\vec{X}_{\alpha'}+L) = \pi_3(\vec{X}_{\alpha'}-L)$ determines

$$ {}^0\pi_3^2 = \tfrac{1}{2}\hat{Q}_3^1\cot(gA_3^3L) , \quad {}^0\pi_3^1 = -\tfrac{1}{2}\hat{Q}_3^2\cot(gA_3^3L) . \tag{4.31}$$

Solving A_0, we begin with A_0^{α}, which is defined by the EOM of A_3^{α}:

$$ \pi_3^{\alpha} = \partial_3 A_0^{\alpha} + ig[A_0^{\alpha}, \bar{A}_3] . \tag{4.32}$$

The solution to (4.32) is given by

$$ A_0^{\alpha}(x) = e^{-ig\bar{A}_3x_3}\left(\tfrac{1}{2}\int \tilde{\rho}(\vec{x}_{\alpha}, x_3')|x_3 - x_3'|dx_3' + {}^0\pi_3^{\alpha}(\vec{x}_{\alpha})X_3 + V^{\alpha}(\vec{x}_{\alpha})\right)e^{+ig\bar{A}_3x_3} . \tag{4.33}$$

The surface condition implies that $A_0^{\alpha}(\vec{X}_{\alpha'}-L) = A_0^{\alpha}(\vec{X}_{\alpha'}+L)$, i.e.,

$$ e^{-ig\bar{A}_3L}(\tfrac{1}{2}L\hat{Q}_3^{\alpha} - \tfrac{1}{2}\hat{Q}^{\alpha}_{3(1)} + {}^0\pi_3^{\alpha}L + V^{\alpha})e^{+ig\bar{A}_3L} = e^{+ig\bar{A}_3L}(\tfrac{1}{2}L\hat{Q}_3^{\alpha} + \tfrac{1}{2}\hat{Q}^{\alpha}_{3(1)} - {}^0\pi_3^{\alpha}L + V^{\alpha})e^{-ig\bar{A}_3L} . \tag{4.34}$$

This equation defines V^{α} uniquely, with the solution given by

$$ \begin{aligned} V^1(\vec{x}_{\alpha}) &= -\tfrac{1}{2}[L + \cot^2(gA_3^3L)]\,\hat{Q}_3^1 + \tfrac{1}{2}\hat{Q}^2_{3(1)}\cot(gA_3^3L) , \\ V^2(\vec{x}_{\alpha}) &= -\tfrac{1}{2}[L + \cot^2(gA_3^3L)]\,\hat{Q}_3^2 - \tfrac{1}{2}\hat{Q}^1_{3(1)}\cot(gA_3^3L) . \end{aligned} \tag{4.35}$$

The operator A_0^3 is, again solved in the same manner as in the theory I and is thus given by (4.21), (4.24), (4.26), and (4.27).

C. Theory III

Theory III is defined by the set of conditions [Eq. (2.15b)]. It is the only one which belongs also to the $A_3 = 0$ gauge theory. We will not discuss here whether the $A_3 = 0$ gauge theory implies as a rule the whole set of constraints (2.15b) or whether some other set of constraints will do. Nevertheless, the following solution, the set (2.15b), is relevant also to the conventional axial gauge theory. The solution for π_3 in theory III is given by

$$ \pi_3(\vec{x}) = \int \rho(\vec{x}_{\alpha}, X_3')\epsilon(X_3 - x_3')dx_3' . \tag{4.36}$$

The constraint $\pi_3(\vec{X}_{\alpha}, \pm L) = 0$ implies the constraint

$$ {}^0\rho_3(\vec{x}_{\alpha}) = 0 . \tag{4.37}$$

In order to realize the constraint (4.37), we gauge ${}^0A_2(\vec{X}_{\alpha})$ to be zero [taking into account the boundary condition at $\vec{X}_{\alpha} = (X_1, \pm L)$] and solve for ${}^0\pi_2(\vec{X}_{\alpha})$:

$$ {}^0A_2 = 0 , \tag{4.38}$$

$$ {}^0\pi_2(\vec{x}_{\alpha}) = \int {}^0\rho(x_1, x_2')\epsilon(X_2 - X_2')dx_2' . \tag{4.39}$$

As $\pi_2(X_1, \pm L)=0$, we get the new constraint

$$^{0,0}\rho_{3,2}=0\,. \tag{4.40}$$

This constraint, with the conditions $^{0,0}A_1(X_1=L) = {}^{0,0}A_1(X_1=-L)$ and $^{0,0}\pi_1(X_1=L)=\pi_1(X_1=-L)$, are realized by gauging away $^{0,0}A_1(X_1)$ and solving for (4.41) and (4.42):

$$^{0,0}A_1=0\,, \tag{4.41}$$

$$^{0,0}\pi_1(x_1)=\int {}^{0,0}\rho_{3,2,1}(x_1')\epsilon(x_1-x_1')dx_1'\,. \tag{4.42}$$

The above constraint $^{0,0}\pi_1(X_1=L)={}^{0,0}\pi_1(X_1=-L)$ implies also the global condition

$$Q_E=0={}^{0,0,0}\rho_{3,2,1}\,. \tag{4.43}$$

As for the other theories we are not going to realize these three global constraints and leave them as harmless constraints on the states, which should not influence the dynamics of the theory. The equation and solution for $^*A_0$, after the realization of the Gauss's law constraints, is given by

$$\begin{aligned}&\pi_3=\partial_3{}^*A_0\,,\\ &^*A_0=\tfrac{1}{2}\int\rho(\vec{x}_\alpha,x_3')\left(|x_3-x_3'|-\frac{L}{2}-\frac{x_3'^2}{2L}\right)dx_3'\,.\end{aligned} \tag{4.44}$$

The realization of (4.37) implies Eq. (4.45) for $^{*,0}A_0$:

$$^0\pi_2=\partial_2{}^{*,0}A+ig^0[{}^*A_0,{}^*A_2]\,. \tag{4.45}$$

Thus the solution for $^{*,0}A_0(X)$ is given by

$$\begin{aligned}{}^{*,0}A(\vec{x}_\alpha)=\tfrac{1}{2}\int^0\rho_{3,2}(x_1,x_2')\left(|x_2-x_2'|-\frac{L}{2}-\frac{x_2'^2}{2L}\right)dx_2'\\ -ig\int{}^0[{}^*A_0,{}^*A_2](x_1,x_2')\left(\epsilon(x_2-x_2')+\frac{x_2'}{2L}\right)dx_2'\,.\end{aligned} \tag{4.46}$$

The remaining freedom of A_0, namely $^{0,0}A_0$, should be solved from the vanishing EOM of $^{0,0}A_1$ given by (4.47), with the solution to $^{0,0}A_0$ given by (4.48):

$$^{0,0}\pi_1(x_1)=\partial_1{}^{0,0}A_0+ig^{0,0}[A_0,A_1]\,, \tag{4.47}$$

$$\begin{aligned}{}^{0,0}A_0(x_1)=\tfrac{1}{2}\int^{0,0}\rho_{3,2,1}(x_1')\left(|x_1-x_1'|-\frac{L}{2}-\frac{x_1'^2}{2L}\right)dx_1'\\ -ig\int\Big({}^{0,0}[{}^*A_0,{}^*A_1](x_1')\\ +{}^{0,0}[{}^{*,0}A_0,{}^{*,0}A_1]\\ \times(x_1')\left[\epsilon(x_1-x_1')+\frac{x_1'}{2L}\right]\Big)dx_1'+M\,.\end{aligned} \tag{4.48}$$

The global operator M should be solved in accordance with the constraint (4.43); we leave it here as a harmless free operator.

D. Theory IV

The set of constraints of theory IV implies that $A_3^1=0$. We will use the surface constraint and rotate $A_3(\vec{X}_\alpha)$ to be in the third isospin direction only, where $^0A_3=\bar{A}_3$, $(^0A_3^1={}^0A_3^2=0)$. Thus one should express $\pi_3^\alpha(x)$ completely by the independent variables of this theory. The solution for π_3^3 is given by (4.49):

$$\begin{aligned}\pi_3^3(\vec{x})={}^0\pi_3^3(x_\alpha)+(1/2L)Q_1^3(\vec{x}_\alpha)\\ +\int\rho_3^3(\vec{x}_\alpha,x_3')\epsilon(x_3-x_3')dx_3'\,.\end{aligned} \tag{4.49}$$

The boundary constraint at $X_3=\pm L$ on $\partial_3\pi_3^3$, (4.50), is satisfied as an identity of the theory, as $\pi^\alpha(x_3=\pm L)=A^\alpha(x_3=\pm L)=0$:

$$\partial_3\pi_3^3|_{x_3=\pm L}=\partial_3(\partial_1\pi_1^3+\partial_2\pi_2^3+[\vec{\pi}_\alpha^{\alpha}\vec{A}_\alpha^{\alpha}])|_{x_3=\pm L}=0\,. \tag{4.50}$$

The solution for $\pi_3^\alpha(X)$ is given by

$$\begin{aligned}\pi_3^{\alpha}(\vec{x})=e^{-ig\bar{A}_3x_3}\left({}^0\pi_3^{\alpha}(\vec{x}_\alpha)+\int\tilde{\rho}^{\alpha}(x_\alpha,x_3')\epsilon(x_3-x_3')\right)\\ \times e^{+ig\bar{A}_3x_3}\,.\end{aligned} \tag{4.51}$$

The conditions on $\pi_3^2(\vec{X})$ is $\partial_3\pi_3^2(X_\alpha,X_3=\pm L)=0$, which yields the conditions (4.52) in the second isospin direction;

$$(e^{\mp ig\bar{A}_3L}[\bar{A}_3,{}^0\pi_3^{\alpha}\pm\hat{Q}_3^{\alpha}(\vec{x}_\alpha,\pm L)]e^{\pm ig\bar{A}_3L})^{(2)}=0. \tag{4.52}$$

Equation (4.52) defines $^0\pi_3^{\alpha}$ in

$$\begin{aligned}&^0\pi_3^1(\vec{x}_\alpha)=\hat{Q}^1(x_\alpha)\cot(gA_3^3L)\,,\\ &^0\pi_3^1(\vec{x}_\alpha)=\hat{Q}^2(X_\alpha)\tan(gA_3^3L)\,.\end{aligned} \tag{4.53}$$

In this way the conjugate momenta to A_3^α are completely defined. The condition $\pi_3{}^1(\vec{X}_\alpha,\pm L)=0$ implies Eq. (4.54) in the first isospin direction:

$$[e^{\mp ig\bar{A}_3L}({}^0\pi_3^{\alpha}\pm\hat{Q}^{\alpha})e^{\pm ig\bar{A}_3L}]^{(1)}=0\,. \tag{4.54}$$

Substitute the expression (4.53) for $^0\pi_3^\alpha$ into (4.54) and Eq. (4.54) is satisfied automatically.

Going on to solve A_0, we begin with $A_0{}^3$ which should satisfy

$$\pi_3^3=\dot{A}_3^3+\partial_3A_0^3\,. \tag{4.55}$$

Thus A_0^3 should be a solution of (4.56) with T yet to be defined:

$$\pi_3^3=T^3+\partial_3A_0^3\,. \tag{4.56}$$

The solution to (4.56) is given by

$$\begin{aligned}A_0^3=\tfrac{1}{2}\int\rho_3^3(\vec{x}_\alpha,x_3')|X_3-X_3'|dx_3'\\ +{}^0\pi_3^3X_3+T^3x_3+V^3\,.\end{aligned} \tag{4.57}$$

The boundary condition on A_0^3 in the $X_3=\pm L$ surfaces is given by (4.58). It defines the operators T and V which are given by (4.59):

$$A_0(\vec{x}_\alpha, \pm L) = \frac{1}{2}\int \rho_3^3(x_\alpha, x_3')(L \mp x_3')dx_3 \pm T^3 L + V = 0. \tag{4.58}$$

Thus

$$T^3 = \frac{1}{2L}\int \rho_3^3(\vec{x}_\alpha, x_3')x'dx_3' - {}^0\pi_3^3, \qquad V(\vec{x}_\alpha) = \frac{1}{2}\int \rho_3^3(x_\alpha, x_3')dx_3'. \tag{4.59}$$

As $A_3^\alpha = 0$, the solution for A_0^α is given by

$$A_0^\alpha(\vec{x}) = e^{-ig\bar{A}_3 x_3}\left(\frac{1}{2}\int \bar{\rho}^\alpha(x_\alpha, x_3')|x_3 - x_3'|dx_3' + {}^0\pi_3^\alpha x_3 + V^\alpha\right)e^{+ig\bar{A}_3 L}. \tag{4.60}$$

The condition $A_0^2(\vec{X}_\alpha, X_3 = \pm L) = 0$ defines V^α in (4.62) through (4.61):

$$A_0^2(\vec{x}_\alpha, \pm L) = [e^{\mp ig\bar{A}_3 L}(\tfrac{1}{2}L\hat{Q}^\alpha \mp \tfrac{1}{2}\hat{Q}^\alpha_{(1)} \pm {}^0\pi_3^\alpha L + V^\alpha)e^{\pm ig\bar{A}_3 L}]^{(2)} = 0, \tag{4.61}$$

$$V^1(\vec{x}_\alpha) = -\tfrac{1}{2}L\hat{Q}^1 + (-{}^0\pi_3^2 L + \tfrac{1}{2}\hat{Q}^2_{(1)})\cot(gA_3^3 L), \qquad V^2(\vec{x}_\alpha) = -\tfrac{1}{2}L\hat{Q}^2 + (-{}^0\pi_3' L + \tfrac{1}{2}\hat{Q}^1_{(1)})\tan(gA_3^3 L). \tag{4.62}$$

The condition on A_0^1 on the $X_3 = \pm L$ boundaries is given by

$$0 = \partial_3 A_0^1 = \pm\{[\bar{A}_3, (\tfrac{1}{2}LQ^\alpha \mp Q^\alpha_{(1)} \pm {}^0\pi_3^\alpha L + V^\alpha)]\}^{(1)}. \tag{4.63}$$

This equation is satisfied automatically with V^α given by (4.62).

V. QUANTIZATION OF THE FOUR SU(2) GAUGE THEORIES

In this section we pass from the classical theory to the quantum theory. In the quantized theory we define all the dependent operators to have the same expressions as were given in Sec. IV and new commutation relations (CR) to replace the Poisson brackets. Thus we skip the order ambiguity in this transition.

The new CR between the independent fields and their conjugate momenta should take into account the boundaries constraints which were defined above. Thus the new CR are not the usual canonical ones. In order to present them in a closed form we define in Eqs. (5.1)–(5.4) some distributional functions, in the finite space volume $-L \le x_i, y_i \le +L$ $(i = 1, 2, 3)$:

$$V^3(\vec{x} - \vec{y}) \equiv \sum_{i=1,2,3} \delta(|x_i - y_i| - 2L)\delta(x_j - y_j) \times \delta(x_k - y_k) \quad (i \ne j \ne k \ne i), \tag{5.1}$$

$$\delta_\pm{}^3(\vec{x} - \vec{y}) \equiv \delta^3(\vec{x} - \vec{y}) \pm V^3(\vec{x} - \vec{y}) \tag{5.2}$$

[the definition of $\delta_\pm^2(\vec{x}_\alpha - \vec{y}_\alpha)$ and $\delta_\pm^1(x - y)$ are done in an analogous way],

$$\delta_{0,i}(X_i, y_i) \equiv -[\delta(x_i + y_i - 2L) + \delta(x_i + y_i + 2L)] + \delta(X_i - y_i). \tag{5.3}$$

The distribution function $\delta_{\partial,i}(X_i, y_i)$ is defined by giving all its derivatives in the interval $-L \le X_i, y_i \le L$ (m is a non-negative integer):

$$\partial_{x_i(y_i)}{}^m \delta_{\partial,i}(X_i, y_i) = \partial_{x_i(y_i)}{}^m \delta(X_i - y_i) - \tfrac{1}{2}[(-1)^m - 1]\partial_{x_i(y_i)}{}^m \times \delta(2L - |y_i + x_i|). \tag{5.4}$$

The independent fields and their nonvanishing equal-time CR in the four theories are given below (neglecting the undefined global operators). Theories I and II are presented by the fields

$${}^0A_3^3(\vec{x}_\alpha),\ {}^*A_2^3(\vec{x}),\ {}^{0,0}A_2^3(x_1),\ {}^*A_1^3(\vec{x}),\ {}^{*,0}A_1^3(\vec{x}_\alpha),\ {}^{0,0,0}A_1^3,\ A_\beta^\alpha(\vec{x}_\alpha)\ (\alpha, \beta = 1, 2),$$

and their conjugate momenta. The nonvanishing, equal-time CR in these theories are given by

$$[{}^0A_3^3(\vec{x}_\alpha), {}^0\pi_3^3(\vec{y})] = \frac{1}{2L}\delta_+^2(\vec{x}_\alpha - \vec{y}_\alpha), \tag{5.5}$$

$$[{}^*A_\beta^3(\vec{x}), {}^*\pi_\alpha^3(\vec{y})] = \left[\delta_+^3(\vec{x} - \vec{y}) - \frac{1}{2L}\delta_+^2(\vec{x}_\alpha - \vec{y}_\alpha)\right]\delta_{\alpha,\beta}, \tag{5.6}$$

$$[{}^{0,0}A_2^3(x_1), {}^{0,0}\pi_2^3(y_1)] = \left(\frac{1}{2L}\right)^2\delta_+^1(x_1 - y_1), \qquad [{}^{0,0,0}A_1^3, {}^{0,0,0}\pi_1^3] = \left(\frac{1}{2L}\right)^3, \tag{5.7}$$

$$[{}^{*,0}A_1^3(\vec{x}_\alpha), {}^{*,0}\pi_1^3(\vec{y}_\alpha)] = \frac{1}{2L}\delta_+^2(\vec{x}_\alpha - \vec{y}_\alpha) - \left(\frac{1}{2L}\right)^2\delta_+^1(x_1 - y_1), \tag{5.8}$$

$$[A_\delta^\alpha(\vec{x}), \pi_{\beta'}^{\alpha'}(\vec{y})] = \delta_{\beta,\beta'}^{\alpha,\alpha'}\delta_-^3(\vec{x} - \vec{y}), \quad \text{Theory I}, \tag{5.9a}$$

$$[A_\delta^\alpha(\vec{x}), \pi_{\beta'}^{\alpha'}(\vec{y})] = \delta_{\beta,\beta'}^{\alpha,\alpha'}\delta_+^3(\vec{x} - \vec{y}), \quad \text{Theory II}. \tag{5.9b}$$

Theory III. The independent fields are ${}^*A_\alpha^a$, ${}^{*,0}A_1^a$, and their conjugate momenta. Their nonvanishing equal-time CR are given by

$$[{}^*A_1^a(\vec{x}), {}^*\pi_1^b(\vec{y})] = \delta^{a,b}\delta_{0,1}(x_1,y_1)\delta_{\partial,2}(x_2,y_2)\delta_{\partial,3}(x_3,y_3)\,, \tag{5.10}$$

$$[{}^*A_2^a(\vec{x}), {}^*\pi_2^b(\vec{y})] = \delta^{a,b}\left[\delta_{\partial,1}(x_1,y_1)\delta_{0,2}(x_2,y_2)\delta_{\partial,3}(x_3,y_3) - \left(\frac{1}{2L}\right)\delta_{\partial,1}(x_1,y_1)\delta_{0,2}(x_2,y_2)\right], \tag{5.11}$$

$$[{}^{*,0}A_1^a(\vec{x}_\alpha), {}^{*,0}\pi_1^b(\vec{y}_\alpha)] = \delta^{a,b}\left[\left(\frac{1}{2L}\right)\delta_{0,1}(x_1,y_1)\delta_{\partial,2}(x_2,y_2) - \left(\frac{1}{2L}\right)^2\delta_{0,1}(x_1,y_1)\right]. \tag{5.12}$$

Theory IV. The independent fields are ${}^0A_3^3, A_\alpha^a$, and their conjugate momenta. Their nonvanishing equal-time CR are given by

$$[{}^0A_3^3(\vec{x}_\alpha), {}^0\pi_3^3(\vec{y}_\alpha)] = \frac{1}{2L}\delta_{0,1}(x_1,y_1)\,\delta_{0,2}(x_2,y_2)\,, \tag{5.13}$$

$$[A_1^2(\vec{x}), \pi_1^2(\vec{y})] = \delta_{\partial,1}(x_1,y_1)\delta_{0,2}(x_2,y_2)\,\delta_{0,3}(x_3,y_3)\,, \tag{5.14}$$

$$[A_2^2(\vec{x}), \pi_2^2(\vec{y})] = \delta_{0,1}(x_1,y_1)\delta_{\partial,2}(x_2,y_2)\,\delta_{0,3}(x_3,y_3)\,, \tag{5.15}$$

$$[A_1^1(\vec{x}), \pi_1^1(\vec{y})] = \delta_{0,1}(x_1,y_1)\delta_{\partial,2}(x_2,y_2)\,\delta_{\partial,3}(x_3,y_3)\,, \tag{5.16}$$

$$[A_2^1(\vec{x}), \pi_2^1(\vec{y})] = \delta_{\partial,1}(x_1,y_1)\delta_{0,2}(x_2,y_2)\,\delta_{\partial,3}(x_3,y_3)\,. \tag{5.17}$$

The Hamiltonian of these theories is given by (2.5), and should be expressed by the free dynamical variables only. The EOM are obtained then by computing the CR $[H, A_i^a]$ and $[H, \pi_i^a]$ by using the CR given above for the independent variables. The correct EOM [Eqs. (2.6) and (2.7)] are thus obtained by a straightforward, although tedious, calculation in all the four theories and in the whole box.

VI. DISCUSSION AND SPECULATION

It is clear that further work is needed. One should look at other kinds of constraints which can predict the correct EOM on the surface, and remove the A_0 ambiguity. Thus one conclusion seems to be very likely; not all kinds of solution for these EOM exist in one theory. As an example in some theories one cannot introduce one quark states, and has to consider quark-antiquark states, or two quark states only. In other theories the instanton is an illegitimate solution and one has to consider instanton-anti-instanton solutions, etc. That should reflect itself in computing vacuum-to-vacuum tunneling amplitudes, etc. It is interesting to notice that by the boundary conditions the SU(2) symmetry is broken in theories I and IV. We learn that in pure non-Abelian gauge theories it is impossible to take antiperiodical boundary conditions in all the isospin directions. Also, it seems that in these theories only the MIT bag can be predicted (theory III) where the outside magnetic permeabilities μ^a are taken to be infinity. It is interesting to learn also that the analogous technique taking all the dielectric constants ϵ^a to be infinities is forbidden in these theories.

It is clear that much work has to be done in order to learn these theories. Among other problems one has to find out the propagation of wave packets in these boxes; to find how their behavior depends on L, and if some phase transition occurs in which one theory is changed to another. The theory in an open space should be a limit of the theories, where $L \to \infty$. Therefore, it might be that before reaching this limit some critical value of L occurs which can help in understanding the confinement phenomena and other features of strong interactions.

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