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Gauge fixing and operator ordering

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In a large class of gauges, including the Coulomb gauge, in non-Abelian gauge theories, an operator-ordering ambiguity exists in the canonically quantized Hamiltonian. In this paper, a method is described for resolving this ambiguity. It gives rise to an extra potential-like term of order $\hbar^2$.

I. INTRODUCTION

The quantization of non-Abelian gauge theories has traditionally been discussed in the language of the path integral. The derivation of diagrammatic rules for use in perturbative calculations is straightforward. In addition, the path integral is very useful in investigations of nonperturbative phenomena. In order to quantize the relevant degrees of freedom and avoid infinities arising from the gauge invariance of the action, Faddeev and Popov[1] introduced a method of choosing one configuration on each orbit generated by a gauge transformation. Although it has been shown that this procedure of fixing the gauge cannot be carried out globally in a continuous manner,[2] it is still valuable in the local sense. In addition, if the requirement of continuity is abandoned, the gauge can be fixed by patching together a set of different gauge surfaces so that each distinct orbit is intersected no more than once by a given surface, but every orbit is intersected by some surface.

The program of gauge fixing has also been carried out in a canonical framework.[3,4] In Ref. 3, this was done by performing a point transformation from the gauge potentials $\{A\}$ to a new set of coordinates $\{\hat{A},\hat{g}\}$, in which the gauge degrees of freedom $\hat{g}$ could be isolated and eliminated from the Hamiltonian. The gauge condition entered as a constraint on the remaining degrees of freedom, $F(\hat{A})=0$, where F depended only on the space components of $\hat{A}$. In the case of the Coulomb gauge, for example, this led to the usual Hamiltonian, which in conventional notation[5] reads

$$H=\tfrac{1}{2}\int d^3x(\vec{\mathrm{E}}_i^T\cdot\vec{\mathrm{E}}_i^T+\nabla_i\vec{\mathrm{f}}\cdot\nabla_i\vec{\mathrm{f}}+\vec{\mathrm{B}}_i\cdot\vec{\mathrm{B}}_i)\,. \tag{1}$$

Here $\vec{\mathrm{E}}_i^T$ is the transverse component of the electric field ($\nabla_i\vec{\mathrm{E}}_i^T=0$) and

$$f^\alpha(\vec{\mathrm{x}})=\int d^3y\,N^{-1}(\vec{\mathrm{x}},\vec{\mathrm{y}},\vec{\mathrm{A}}_i)^{\alpha\beta}f^{\beta\gamma\delta}A_k^\gamma(\vec{\mathrm{y}})E_k^{T\delta}(\vec{\mathrm{y}})\,, \tag{2}$$

where

$$N^{\alpha\beta}=\nabla^2\delta^{\alpha\beta}+f^{\alpha\beta\gamma}A_i^\gamma\nabla_i\,. \tag{3}$$

Upon quantization, of course, $\vec{\mathrm{E}}_i^T$ will not commute with $\vec{\mathrm{A}}_j$. Thus, the procedure of gauge fixing has given rise to the presence of operator-ordering ambiguities in the longitudinal part of the Hamiltonian. If the Hamiltonian is self-adjoint, then it is easy to see that the difference between any two choices of ordering is a potential-like term of order $\hbar^2$. Thus, one should expect that higher-order perturbative calculations in the Coulomb gauge should be sensitive to the quantum ordering effects. In principle, such calculations could be compared to experimentally determined quantities. It would be desirable to have a reasonable rule for choosing the correct ordering within the theory.

The issue of operator ordering is a long-standing problem in quantum mechanics.[6] In forming quantum-mechanical operators from classical functions, there are certain reasonable requirements which should be fulfilled. Among these are correspondence, uniqueness, and the association of self-adjoint operators with real classical functions. There have been many attempts to formulate a set of rules for forming operators from classical functions.[7] However, it is interesting to note that a satisfactory, consistent resolution of this quantum operator-ordering ambiguity has not been discovered since the beginnings of the quantum theory. It is perhaps a difficulty which, in general, can be overcome only by appeal to experiment in each individual case.

The ordering ambiguity present after gauge fixing in a non-Abelian gauge theory is of a specific type which seems more amenable to resolution within the theory. In the formalism of Ref. 3, the ordering problem arises only after a point transformation is performed from a theory in which no such difficulty is present. In Sec. II, this procedure is reviewed and a choice of ordering is made. The quantum ordering corrections to the operator Hamiltonian are derived. Section III contains some discussion of the procedure and the result.

II. DERIVATION OF THE HAMILTONIAN OPERATOR

Before quantizing a gauge theory, the physically irrelevant variables corresponding to the gauge freedom of the system must be eliminated. Since the time component of the gauge potential has no conjugate momentum, it is a dependent variable. Furthermore, Gauss's law implies that the momenta conjugate to the space components of the gauge potential are not all independent variables and that a further gauge condition can be imposed on the potentials. In Ref. 3, this procedure was carried out by starting in the temporal gauge, $\vec{A}_0=0$, and performing a point transformation into a new set of coordinates which expedited the elimination of the redundant degrees of freedom. In this section, this formalism is reviewed, with emphasis placed on quantization of the system via Dirac's method[8] for treating singular systems.

The classical Lagrangian density is

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}{}^{\alpha}F_{\mu\nu}{}^{\alpha}, \tag{4}$$

where subscripts are Lorentz indices and superscripts are group indices. The field strength tensor is given in terms of the gauge potentials as

$$F_{\mu\nu}{}^{\alpha}=\partial_{\mu}A_{\nu}{}^{\alpha}-\partial_{\nu}A_{\mu}{}^{\alpha}-ef^{\alpha\beta\gamma}A_{\mu}{}^{\beta}A_{\nu}{}^{\gamma}, \tag{5}$$

with e the coupling constant and $f^{\alpha\beta\gamma}$ the group structure constants. The momenta canonically conjugate to $A_{\mu}{}^{\alpha}$ are

$$E_i{}^{\alpha}=F_{0i}{}^{\alpha} \tag{6}$$

and

$$E_0{}^{\alpha}=0. \tag{7}$$

In Dirac's terminology, Eq. (7) represents primary constraints. The Hamiltonian density is constructed as usual by a Legendre transform,

$$\mathcal{H}=\tfrac{1}{2}E_i{}^{\alpha}E_i{}^{\alpha}+(D_iE_i)^{\alpha}A_0{}^{\alpha}+(\partial_0A_0{}^{\alpha})E_0{}^{\alpha}+\tfrac{1}{4}F_{ij}{}^{\alpha}F_{ij}{}^{\alpha}. \tag{8}$$

D_i is the gauge-covariant derivative. Consistency conditions are imposed on the primary constraint by demanding that Eq. (7) hold independently of time. The fundamental Poisson-bracket relation,

$$\{A_{\mu}{}^{\alpha}(\vec{x}),E_{\nu}{}^{\beta}(\vec{y})\}_P=\delta_{\mu\nu}\delta^{\alpha\beta}\delta^3(\vec{x}-\vec{y}), \tag{9}$$

is used to give

$$\partial_0E_0{}^{\alpha}=\left\{\int d^3x\,\mathcal{H},E_0{}^{\alpha}\right\}_P=(D_iE_i)^{\alpha}=0. \tag{10}$$

A secondary constraint has been derived, which is, in fact, simply Gauss's law. At this point, the temporal-gauge condition, $A_0{}^{\alpha}=0$, can be imposed so that the Hamiltonian density becomes

$$\mathcal{H}=\tfrac{1}{2}E_i{}^{\alpha}E_i{}^{\alpha}+\tfrac{1}{4}F_{ij}{}^{\alpha}F_{ij}{}^{\alpha}, \tag{11}$$

with $E_i{}^{\alpha}=\partial_0A_i{}^{\alpha}$. The Dirac brackets of the remaining variables are the same as their Poisson brackets since they commute with the second-class (noncommuting) constraints $A_0{}^{\alpha}$ and $E_0{}^{\alpha}$:

$$\{A_i{}^{\alpha}(\vec{x}),E_i{}^{\beta}(\vec{y})\}_D=\delta_{ij}\delta^{\alpha\beta}\delta^3(\vec{x}-\vec{y}). \tag{12}$$

These are translated upon quantization to the usual canonical equal-time commutation relations

$$[A_i{}^{\alpha}(\vec{x}),E_j{}^{\beta}(\vec{y})]=i\delta_{ij}\delta^{\alpha\beta}\delta^3(\vec{x}-\vec{y}). \tag{13}$$

As in Ref. 3, it is now possible to separate out the remaining gauge degrees of freedom by changing variables from the set $A_i{}^{\alpha}(\vec{x})$ to a new system of coordinates. To facilitate this procedure, a compact notation introduced in Ref. 3 will be used: $A_i{}^{\alpha}(\vec{x})$ is written as simply A_i so that, for example, the covariant derivative becomes

$$D_i{}^{\alpha}=\nabla_i{}^{\alpha}+ieT_{ij}{}^{\alpha}A_j, \tag{14}$$

where T^{α} are matrices generating the real unitary representation under which A_i transforms as

$$(A^g)_i=R_{ij}(g)A_i+\Lambda_i(g). \tag{15}$$

The inhomogeneous term Λ_i will not be important in this summary, so it is not written explicitly.

The new coordinates $\hat{A}_i$ and $\hat{g}$ are then defined by

$$A_i=R_{ij}(\hat{g})\hat{A}_j+\Lambda_i(\hat{g}), \tag{16}$$

with $\hat{g}$ parametrized in the fundamental representation as $\hat{g}=\exp(i\lambda^{\alpha}\omega^{\alpha})$ so that $R_{ij}(\hat{g})=(\exp(iT^{\alpha}\omega^{\alpha}))_{ij}$. In order that the number of degrees of freedom in the system is not increased by this point transformation, a gauge condition is then imposed on the new variables $\hat{A}$:

$$F^{\alpha}(\hat{A})=0. \tag{17}$$

For example, the Coulomb gauge would be recovered if $F^{\alpha}(\hat{A})=\nabla_i{}^{\alpha}\hat{A}_i$. Equation (16) induces a transformation on the electric field. As demonstrated in Ref. 3,

$$E_i=R_{ij}(\hat{g})[Q_{jk}(\hat{A})\Pi_k+eS^{\alpha\beta}(\hat{g})(N^{-1}(\hat{A}))^{\beta\gamma}F_j{}^{\gamma}(\hat{A})l^{\alpha}]. \tag{18}$$

Here,

$$Q_{ij}(\hat{A})=\delta_{ij}-D_j{}^{\alpha}(\hat{A})(N^{-1}(\hat{A}))^{\alpha\beta}F_i{}^{\beta}(\hat{A}), \tag{19}$$

where

$$N^{\alpha\beta}(\hat{A})=F_i{}^{\alpha}(\hat{A})D_i{}^{\beta}(\hat{A}) \tag{20}$$

is the Faddeev-Popov matrix,

$$F_i{}^{\alpha}(\hat{A})=\left.\frac{dF^{\alpha}(\hat{A})}{dA_i}\right|_{\hat{A}}, \tag{21}$$

and $S^{\alpha\beta}$ is the adjoint representation of the gauge group. Equation (18) can be inverted to give the

new momenta in terms of E_i:

$$l^\alpha = \frac{1}{e} D_i{}^\alpha(\hat{A}) E_i , \tag{22}$$

$$\Pi_i = P_{ij}(\hat{A}) R_{jk}(\hat{g}^{-1}) E_k . \tag{23}$$

Here,

$$P_{ij}(\hat{A}) = \delta_{ij} - F_i{}^\alpha(\hat{A})(M^{-1}(\hat{A}))^{\alpha\beta} F_j{}^\beta(\hat{A}) , \tag{24}$$

with $M^{\alpha\beta}(\hat{A}) = F_i{}^\alpha(\hat{A}) F_i{}^\beta(\hat{A})$. The Poisson brackets of the new coordinates and momenta are

$$\begin{aligned} &\{\hat{A}_i, \Pi_j\}_P = P_{ij} , \\ &\{\hat{g}, l^\alpha\}_P = -\lambda^\alpha \hat{g} , \\ &\{l^\alpha, l^\beta\}_P = f^{\alpha\beta\gamma} l^\gamma , \end{aligned} \tag{25}$$

with all others vanishing (if F is linear in $\hat{A}$). Note that in the new coordinate system, Gauss's-law constraint is $l^\alpha = 0$. The gauge condition $F^\alpha = 0$ is a new constraint which eliminates the gauge redundancy implied by Gauss's law. This new constraint is second class since it has nonvanishing Poisson brackets with l^α. Gauss's law is of course "demoted" to second class also. The Poisson brackets of the second-class constraints among themselves now read

$$\begin{aligned} &\{l^\alpha, l^\beta\}_P = f^{\alpha\beta\gamma} l^\gamma , \\ &\{F^\alpha, F^\beta\}_P = 0 , \\ &\{F^\alpha, l^\beta\}_P = \frac{1}{e} N^{\alpha\beta} . \end{aligned} \tag{26}$$

Dirac brackets corresponding to the fundamental Poisson brackets can then be defined:

$$\begin{aligned} &\{\hat{A}_i, \Pi_j\}_D = Q_{ji} , \\ &\{g, l^\alpha\}_D = 0 , \\ &\{l^\alpha, l^\beta\}_D = 0 . \end{aligned} \tag{27}$$

The projection operators P and Q form an interesting multiplication table:

$$\begin{aligned} &P^2 = P , \quad Q^2 = Q , \\ &PQ = P , \quad QP = Q . \end{aligned} \tag{28}$$

Upon quantization, the Dirac brackets are translated into commutation relations:

$$\begin{aligned} &[\hat{A}_i, \Pi_j] = i Q_{ji} , \\ &[\hat{g}, l^\alpha] = 0 , \\ &[l^\alpha, l^\beta] = 0 . \end{aligned} \tag{29}$$

The Hamiltonian can now be transformed into the new coordinates via Eq. (18) with the constraints implemented strongly. However, since Q and Π do not commute, a definite choice of operator ordering must be made. The most reasonable ansatz at this point is to use the symmetrized form of the momentum transformation,

$$E_i = \tfrac{1}{2} R_{ij}(\hat{g}) [Q_{jk}(\hat{A}) \Pi_k + \Pi_k Q_{jk}(\hat{A})] , \tag{30}$$

so that

$$H = \tfrac{1}{8} (Q_{ij}\Pi_j + \Pi_j Q_{ij})(Q_{ik}\Pi_k + \Pi_k Q_{ik}) + V(\hat{A}) . \tag{31}$$

In the classical limit, when $[\Pi, \hat{A}] \to 0$,

$$H \to \tfrac{1}{2} \Pi_j (Q^T)_{ji} Q_{ik} \Pi_k + V(\hat{A}) . \tag{32}$$

This reduces to the usual expression as in Eq. (1). To see this, decompose $Q\Pi$ into transverse and longitudinal pieces:

$$Q_{ij}\Pi_j = E_i^T + E_i^L , \tag{33}$$

$$E_i^T = P_{ij} Q_{jk} \Pi_k = \Pi_i , \tag{34}$$

$$E_i^L = (1 - P)_{ij} Q_{jk} \Pi_k . \tag{35}$$

Then

$$H = \tfrac{1}{2} E_i^T E_i^T + \tfrac{1}{2} E_i^L E_i^L + V(\hat{A}) , \tag{36}$$

where

$$E_i^L = D_j^\alpha (N^{-1})^{\alpha\beta} F_i{}^\beta \Pi_j . \tag{37}$$

To compare the ordered operator Hamiltonian in Eq. (31) to the usual quantum-mechanical system derived via the path integral, it is important to note that the symmetric (midpoint) path integral corresponds to the Weyl-ordered operator Hamiltonian.[9] So the naive path integral which uses the classical function in Eq. (32) corresponds to the theory defined by the operator Hamiltonian:

$$\begin{aligned} H_W = \tfrac{1}{2} (&Q^T_{ji} Q_{ik} \Pi_j \Pi_k + 2 \Pi_j Q^T_{ji} Q_{ik} \Pi_k + \Pi_j \Pi_k Q^T_{ji} Q_{ik}) \\ &+ V(\hat{A}) , \end{aligned} \tag{38}$$

where the subscript W refers to the Weyl ordering. Thus, to compare the Hamiltonian H with this naive Hamiltonian, the factors can be appropriately reordered to give

$$\begin{aligned} H &= H_W - \tfrac{1}{8} [Q_{ij}, \Pi_k][Q_{ik}, \Pi_j] \\ &= H_W + \Delta V(\hat{A}) . \end{aligned} \tag{39}$$

$\Delta V(\hat{A})$ is an extra potential-like term in the theory which adds new interaction vertices to the standard Feynman rules derived via the naive path-integral approach. The calculation has been presented with $\hbar = 1$, but if Planck's constant is restored, this term is of order $\hbar^2$. The commutator is easily evaluated to be

$$[Q_{ij}, \Pi_k] = ie\hbar Q_{km} T^\alpha_{ml} Q_{lj} (N^{-1})^{\alpha\beta} F_i{}^\beta , \tag{40}$$

so that

$$\Delta V(\hat{A}) = \frac{e^2 \hbar^2}{8} Q_{jk} T^\alpha_{kl} Q_{lm} T^\gamma_{mj} (N^{-1})^{\alpha\beta} F_i{}^\beta (N^{-1})^{\gamma\delta} F_i{}^\delta . \tag{41}$$

In conventional notation, for the Coulomb gauge, Eq. (41) reduces formally to

$$\Delta V(\hat{A}) = -\frac{e^2\hbar^2}{8}\delta^3(0)\int d^3x\, d^3y \nabla_k(\vec{x})(N^{-1}(\vec{x},\vec{y},\hat{A}))^{\alpha\beta} f^{\beta\delta\tau}\nabla_k(\vec{x})(N^{-1}(\vec{x},\vec{y},\hat{A}))^{\alpha\lambda} f^{\lambda\tau\delta}\,. \tag{42}$$

III. DISCUSSION

As pointed out recently by Gervais and Sakita,[10] the procedure of gauge fixing in a gauge theory is very similar to that of separating out collective coordinates in a theory with a soliton. In the former case, one transforms to new coordinates in which the gauge symmetry is isolated and then imposes constraints to eliminate this redundancy in the variables. In the latter case, a transformation is made in which the collective coordinates of the soliton are isolated and constraints are imposed to eliminate the zero-frequency modes which correspond to symmetries associated with the collective coordinates. The operator-ordering problem dealt with in this paper was in fact encountered in the quantization of two-dimensional scalar field theories about their soliton solutions. There, quantization was carried out in both the path-integral[11] and canonical formalisms.[12] Gervais, Jevicki, and Sakita[13] proved that a very careful treatment of the definition of the path integral was necessary for a fully consistent version of the quantum theory. In fact, their method gave rise to certain extra potential-like terms of order $\hbar^2$ in the Hamiltonian. Canonical quantization with attention paid to the ordering of noncommuting factors gave complete agreement if it was recognized that the path integral corresponds to the Weyl-ordered Hamiltonian operator. Tomboulis, in his canonical treatment of the problem,[12] chose the ordering by symmetrizing the momentum transformation to the collective coordinates. It is his approach which is analogous to the canonical treatment of gauge theories presented here. Indeed, Gervais and Sakita[10] point out that a careful path-integral quantization of the non-Abelian gauge theory would give these extra potential-like terms, but they do not calculate them explicitly. In fact it would seem rather difficult to do so in that formalism.

The importance of an extra potential-like piece in the Hamiltonian is, first of all, that it gives new interaction terms in the Feynman rules. Since ΔV is of order $\hbar^2$, calculations in, say, the Coulomb gauge would be affected if they were to be pursued to a high enough order. For example, if one were to calculate two-loop corrections to the static $q\bar{q}$ potential, such terms might play an important role, In the calculation of the two-loop correction to the mass of the soliton in two-dimensional scalar field theories, the contribution of this extra term was important in canceling an infrared divergence present in the naive path-integral method.[13,14] Also, in the case of soliton quantization, it was noted by Tomboulis in his canonical approach,[12] that the extra term derived by symmetrizing noncommuting factors in the momentum transformation was necessary to prove the Lorentz covariance of the theory. It is interesting to point out the similarity between ΔV in Eq. (42) and the potential-like terms of order $\hbar^2$ derived by Schwinger by requiring that the non-Abelian theory in the Coulomb gauge be operator Lorentz covariant.[15] In summary, an ambiguity in the ordering of noncommuting factors exists in the quantum non-Abelian gauge theories in certain gauges. Its resolution is important in calculations in these gauges and possibly in more formal questions. Although not the only possible choice, the ordering presented in this paper is reasonable, convenient, and somewhat justified by the analogy with soliton theories. The question of ordering ambiguities in other gauges (especially the covariant ones) has not been addressed here. It is possible that such difficulties do not arise in a covariant gauge. In that case, a gauge-invariant S-matrix element could be unambiguously calculated in a covariant gauge, and the result could be used to help choose the operator ordering in the noncovariant gauge, where calculation must, of course, yield the same answer.

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