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Monte Carlo study of quantized SU(2) gauge theory

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Using Monte Carlo techniques, we evaluate path integrals for pure SU(2) gauge fields. Wilson's regularization procedure on a lattice of up to 10^4 sites controls ultraviolet divergences. Our renormalization prescription, based on confinement, is to hold fixed the string tension, the coefficient of the asymptotic linear potential between sources in the fundamental representation of the gauge group. Upon reducing the cutoff, we observe a logarithmic decrease of the bare coupling constant in a manner consistent with the perturbative renormalization-group prediction. This supports the coexistence of confinement and asymptotic freedom for quantized non-Abelian gauge fields.

I. INTRODUCTION

Gauge theories currently dominate our understanding of elementary particle physics. Indeed, we now conceive that all interactions represent ramifications of underlying local symmetries. The elegant inclusion of the strong nuclear force into this picture demands the phenomenon of confinement; indeed, physical hadrons should be gauge-singlet bound states of the fundamental quark and gluon degrees of freedom. At our present level of knowledge, confinement appears to play a role solely for the unbroken non-Abelian gauge theory of strong interactions.

Theoretical evidence for quark confinement by gauge fields is remarkably sparse. Renormalization-group arguments imply that perturbation theory may be inapplicable at large distances,[1] thus dismissing the lack of perturbative evidence for confinement of quarks.[2] Studies of large orders in the weak-coupling expansion[3] as well as semiclassical treatments[4,5] all suggest important nonperturbative effects in non-Abelian gauge theories.

Any true nonperturbative analysis requires a means of controlling the ultraviolet divergences of field theory in a manner independent of Feynman diagrams. Wilson's formulation of gauge theory on a lattice provides such a cutoff scheme.[6] This particular regulator also preserves an exact local symmetry. With the cutoff in place, Wilson derived a strong-coupling expansion in terms of quarks connected by strings. In this picture confinement arises naturally; however, to take the continuum limit one must leave the strong-coupling domain and the expansion could fail to converge. Balian, Drouffe, and Itzykson have presented arguments that in a sufficient number of space-time dimensions, the lattice theory will exhibit a phase transition between the strong-coupling region of confinement and the weak-coupling perturbative regime.[7] Such a transition is essential for the lattice formulation of conventional electrodynamics where photons and electrons exist as free particles.

Renormalization-group analysis implies that for short-distance phenomena the effective coupling of non-Abelian gauge theories becomes small and perturbative results become valid.[2] If this "asymptotic freedom" is to arise in the confining phase of Wilson's formulation, then four space-time dimensions must be inadequate to support the transition of Ref. 7. As evidence for this, Migdal has presented an approximate nonperturbative recursion relation between different values of the cutoff parameter.[8] He finds a close analogy between d-dimensional gauge theories and $(d/2)$-dimensional nearest-neighbor spin systems of statistical mechanics. On this basis he concludes that four dimensions represents a critical case where gauge theories based on non-Abelian groups only possess the confining phase, whereas the Abelian group U(1) of electrodynamics possesses a peculiar transition similar to that occurring in the two-dimensional "XY" model.[9]

Recently, Monte Carlo techniques have proven to be a powerful nonperturbative tool for analysis of quantized gauge fields.[10,11] We have seen clear confinement–spin-wave phase transitions for U(1) lattice gauge theory in four space-time dimensions and for the SU(2) theory in five dimensions.[10] In contrast, this transition appears to be absent for the four-dimensional SU(2) model. These results support the Migdal arguments on the existence of phase transitions; however, the observation of first-order transitions in Z_2, Z_3, and Z_4 lattice gauge theories rather than the predicted second-order critical points shows that Migdal's approximate recursion relations may misidentify the nature of the transition.[11]

In this paper we extend our analysis of the four-dimensional SU(2) theory. Working on lattices of

up to 10^4 sites, we emphasize the continuum limit of the theory. Renormalizing such that the string tension, the asymptotic linear part of the quark-antiquark potential, remains fixed,[12] we obtain the cutoff dependence of the bare coupling constant. Passing out of the strong-coupling regime at large lattice spacing, we observe the onset of a logarithmic decrease of the bare change as the continuum limit is approached. This decrease is at a rate consistent with the prediction of asymptotic freedom. As the renormalization scheme is based on confinement, this provides strong evidence that quantized SU(2) Yang-Mills fields simultaneously exhibit confinement and asymptotic freedom. This unifies the nonperturbative lattice formulation and the more conventional perturbative treatments of continuum non-Abelian gauge theory.

As discussed in Ref. 10, our Monte Carlo algorithm consists of successively touching a heat bath to each gauge variable in the system. After setting up the theory, we describe in Sec. II how special features of the group SU(2) allow the efficient execution of this procedure. Section III contains a review of the asymptotic-freedom prediction for the cutoff dependence of the bare charge. Section IV presents our numerical results connecting the bare coupling, the string tension, and the cutoff. In Sec. V we draw some further conclusions from the analysis.

II. THE ALGORITHM

The model is Wilson's lattice gauge theory with gauge group SU(2).[6] A link variable U_{ij}, which is an element of SU(2), is associated with each nearest-neighbor pair of sites i and j on a four-dimensional simple hypercubic lattice. The reversed link is associated with the inverse element

$$U_{ij}=(U_{ji})^{-1}. \tag{2.1}$$

To eliminate surface effects we impose periodic boundary conditions. Regarding the connection with Minkowski-space formulation as well established, we work entirely in Euclidian space.[13,14] The path integral

$$Z=\int\left(\prod_{(i,j)} dU_{ij}\right)e^{-\beta S(U)} \tag{2.2}$$

defines the quantum theory. Here the integral includes all independent link variables and uses the invariant group measure. The action S is a sum over all elementary squares or plaquettes $\square$ in the lattice,

$$S(U)=\sum_{\square} S_{\square}, \tag{2.3}$$

where

$$S_{\square}=1-\tfrac{1}{2}\mathrm{Tr}(U_{ij}U_{jk}U_{kl}U_{li}). \tag{2.4}$$

Here i, j, k, and l label the sites circulating about the square $\square$. Our normalization is such that any plaquette contributes a number between zero and two to the action. In the next section we review the classical continuum limit of this theory and show that it reduces to the conventional Yang-Mills system with coupling e_0 given by

$$e_0^2=\frac{4}{\beta}. \tag{2.5}$$

Equation (2.2) represents the partition function of a statistical system at temperature $T=1/\beta$. We treat the system by obtaining an ensemble of configurations which simulates an ensemble in equilibrium at this temperature. Starting in some initial configuration, we successively touch a heat bath to each link variable. Each U_{ij} in turn is replaced with a new element U'_{ij} chosen randomly from the entire group with probability density proportional to the Boltzmann factor,

$$dP(U')\sim\exp[-\beta S(U')]dU', \tag{2.6}$$

where $S(U')$ is the action evaluated with the given link having value U'_{ij} and all other links fixed at their previous values. In what follows, one iteration refers to one application of this procedure to every link in the lattice.

This algorithm satisfies the detailed-balance requirements which ensure that any ensemble will eventually be brought to equilibrium.[15] Of all possible Monte Carlo algorithms which vary only a single spin at a time, the above method leads to equilibrium in the least number of iterations. This follows because repeated application of any valid algorithm to a single spin eventually simulates the heat bath.

The action of lattice gauge theory carries an exact local gauge symmetry. The action is unchanged by the replacement

$$U_{ij}\to g_i U_{ij} g_j^{-1}, \tag{2.7}$$

where the g_i represent arbitrary group elements associated with each site i. This symmetry allows gauge fixing; thus, in an axial gauge all U_{ij} on links parallel to some particular axis would be set to the identity and held fixed in the path integral.[14] This procedure does not affect gauge-invariant correlation functions. Gauge fixing is an essential first step in the conventional perturbative expansion about the classical ground state.[16] Nonetheless, with Wilson's cutoff the theory is well defined without going into a particular gauge. Furthermore, the process of gauge fixing introduces long-range interactions which are not well simulated by a local Monte Carlo algorithm. Although the final results are the same, we have found that a fixed gauge increases the convergence

time for a simulation. Thus, the numerical results of this paper include an integral over gauges.

We now discuss in detail the algorithm for generation of group elements with weighting given in Eq. (2.6). We parametrize SU(2) elements in the form

$$U = a_0 I + i\vec{a}\cdot\vec{\sigma}\,, \tag{2.8}$$

where a_μ is a real four-vector of unit length

$$a_0^2 + \vec{a}^2 = 1. \tag{2.9}$$

The 2×2 identity matrix is denoted I and the three Pauli matrices satisfy

$$\begin{aligned} &\sigma^\alpha\sigma^\beta = i\epsilon^{\alpha\beta\gamma}\sigma^\gamma + \delta^{\alpha\beta} I\,, \\ &\mathrm{Tr}(\sigma^\alpha) = 0\,. \end{aligned} \tag{2.10}$$

Here, $\epsilon^{\alpha\beta\gamma}$ is totally antisymmetric with $\epsilon^{123}=1$ and $\sigma^{\alpha\beta}$ is the Kronecker function

$$\delta^{\alpha\beta} = \begin{cases} 1, & \alpha=\beta \\ 0, & \alpha\neq\beta\,. \end{cases}$$

Our group elements are stored in the computer memory as the four numbers a_μ for each link. In this notation the invariant group measure takes the simple form

$$dU = \frac{1}{2\pi^2}\,\delta(a^2-1)d^4a\,. \tag{2.11}$$

While working on a particular link $\{i,j\}$, we need consider only the contribution to the action coming from the six plaquettes containing that link. If we denote by $\tilde{U}_\alpha$, $\alpha = 1,\ldots,6$, the six products of three link variables which interact with the link in question, then Eq. (2.6) assumes the form

$$dP(U) \sim dg\,\exp\left[\tfrac{1}{2}\beta\,\mathrm{Tr}\left(U\sum_{\alpha=1}^{6}\tilde{U}_\alpha\right)\right]. \tag{2.12}$$

A useful property of elements of the group SU(2) is that any sum of them is proportional to another SU(2) element. In particular, it follows from representation (2.8) that

$$\sum_{\alpha=1}^{6}\tilde{U}_\alpha = k\bar{U}\,, \tag{2.13}$$

where k is given by the determinant

$$k = \left|\sum_{\alpha=1}^{6}\tilde{U}\right|^{1/2}, \tag{2.14}$$

and $\bar{U}$ is an element of SU(2). The utility of this observation appears when we use the invariance of the group to measure to write

$$\begin{aligned} dP(U\bar{U}^{-1}) &\sim dU\exp(\tfrac{1}{2}\beta k\,\mathrm{Tr}U) \\ &= \frac{1}{2\pi^2}\,\delta(a^2-1)d^4a\exp(\beta k a_0)\,. \end{aligned} \tag{2.15}$$

The problem reduces to generating points randomly on the surface of the unit sphere in four dimensions with exponential weighting along the a_0 direction. Generating an element U in this manner, we replace the link variable on the lattice with the product

$$U'_{ij} = U\bar{U}^{-1}\,. \tag{2.16}$$

To generate the appropriately weighted points on the four-dimensional sphere, first note that the integration over $|\vec{a}|$ can be done using the δ function

$$\begin{aligned} \delta(a^2-1)d^4a\exp(\beta k a_0) &= \tfrac{1}{2}\,da_0 d\Omega(1-a_0^2)^{1/2} \\ &\quad\times\exp(\beta k a_0)\,, \end{aligned} \tag{2.17}$$

where $d\Omega$ is the differential solid angle of $\vec{a}$ and $\vec{a}$ has length $(1-a_0^2)^{1/2}$. Thus we need to generate a_0 stochastically in the interval $[-1,+1]$ with probability

$$P(a_0) \sim (1-a_0^2)^{1/2}\exp(\beta k a_0)\,, \tag{2.18}$$

and the direction of $\vec{a}$ is chosen totally randomly. Our algorithm for the a_0 selection begins with a trial a_0,

$$a_0 = 1 + \frac{1}{\beta k}\ln(x)\,, \tag{2.19}$$

where x is a random number uniformly distributed in the region

$$e^{-2\beta k} < x < 1\,. \tag{2.20}$$

This generates a_0 distributed with exponential weight $e^{\beta k a_0}$. To correct for the factor $(1-a_0^2)^{1/2}$ in Eq. (2.18), reject this a_0 with probability $1-(1-a_0^2)^{1/2}$ and select a new trial a_0. Repeat this until an a_0 is accepted. We leave it to the interested reader to design his own scheme for randomly selecting the direction for $\vec{a}$. This completes the algorithm.

Once the lattice is in equilibrium, one can measure any desired correlation function. One simple quantity we shall use extensively is the average action per plaquette:

$$P = \langle S_\square\rangle\,. \tag{2.21}$$

This quantity is equivalent to the "internal energy" of the equivalent statistical-mechanical system. As β runs from zero to infinity, P falls from one to zero. In addition to P we will study Wilson loops, which we define in the next section.

For initial conditions, we either select all elements randomly from the group or we set them all to the identity. Agreement of simulations from these independent starting configurations is a test of convergence. Other "mixed" starting configurations as described in Refs. 8–10 are of no particu-

lar value for this system which appears to lack any phase transition at finite β.

III. THE CONTINUUM LIMIT AND THE RENORMALIZATION GROUP

We begin this section with a review of the classical continuum limit of the lattice theory.[6,7] We do this in order to establish the relation of β to the charge of conventional perturbation theory, a low-temperature expansion about this limit. We then present the renormalization-group prediction for the cutoff dependence of the bare coupling for the quantized theory. Finally we define our renormalization prescription for the continuum limit of the quantum theory.

The classical SU(2) Yang-Mills theory follows from the Lagrangian density[17]

$$\mathcal{L} = \tfrac{1}{4} F^{\alpha}_{\mu\nu} F^{\alpha}_{\mu\nu}, \tag{3.1}$$

where the internal-symmetry index α runs from one to three, and $F^{\alpha}_{\mu\nu}$ is defined in terms of potentials

$$F^{\alpha}_{\mu\nu} = \partial_\mu A^{\alpha}_{\nu} - \partial_\nu A^{\alpha}_{\mu} - e_0 \epsilon^{\alpha\beta\gamma} A^{\beta}_{\mu} A^{\gamma}_{\nu}. \tag{3.2}$$

Using the Pauli matrices of Eq. (2.10) gives a convenient matrix formulation

$$A_\mu = \tfrac{1}{2} \sigma^{\alpha} A^{\alpha}_{\mu}, \tag{3.3}$$

$$A^{\alpha}_{\mu} = \mathrm{Tr}(\sigma^{\alpha} A_\mu), \tag{3.4}$$

$$F_{\mu\nu} = \tfrac{1}{2} \sigma^{\alpha} F^{\alpha}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i e_0 [A_\mu, A_\nu], \tag{3.5}$$

$$\mathcal{L} = \tfrac{1}{2} \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}). \tag{3.6}$$

To connect the lattice theory with this Lagrangian, identify

$$U_{ij} = \exp\left[-i e_0 (x^j - x^i)_\mu A_\mu \left(\frac{x^i + x^j}{2} \right) \right], \tag{3.7}$$

where x^i_μ are the coordinates of the site i. Considering a plaquette in the (μ, ν) plane and Taylor expanding A_μ about the center of this plaquette, we find with a little suppressed algebra

$$\begin{aligned} \delta_{\square} &= 1 - \tfrac{1}{2} \mathrm{Tr}(U_{ij} U_{jk} U_{kl} U_{li}) \\ &= \tfrac{1}{8} e_0^2 a^4 \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu} + F_{\nu\mu} F_{\nu\mu}) + O(a^5), \end{aligned} \tag{3.8}$$

with no implied sum over μ and ν. Here, a is the lattice spacing. Combining the a^4 with the sum over plaquettes gives a four-dimensional integral over space-time; consequently, we conclude that

$$\tfrac{1}{4} \beta e_0^2 = 1 \tag{3.9}$$

reproduces the classical theory when the lattice spacing goes to zero. This justifies Eq. (2.5).

We now turn to the renormalization group. In a conventional perturbation treatment of the quantized theory one defines a renormalized charge e_R in terms of some physical correlation function at a scale of mass μ. The precise definition is a matter of convention; we merely assume it is made such that when an ultraviolet cutoff of length scale a is in effect

$$e_R(e_0, \mu, a) = e_0 + O(e_0^3). \tag{3.10}$$

The continuum limit of the quantum theory follows by taking a to zero while adjusting e_0 to keep the physically defined e_R fixed. Of course, in a theory with more parameters such as bare masses, additional physical quantities need fixing. The Gell-Mann–Low function[18] is defined as

$$\gamma(e_R) = \mu \frac{\partial}{\partial \mu} e_R(e_0, \mu, a). \tag{3.11}$$

If the continuum limit of the theory is physically sensible $\gamma(e_R)$ should remain finite as a is taken to zero. For SU(2) the perturbative expansion of $\gamma(e_R)$ begins[1]

$$\gamma(e_R) = -\frac{11 e_R^3}{24\pi^2} + O(e_R^5). \tag{3.12}$$

Through terms of order e_R^5, the function $\gamma(e_R)$ is independent of the details of the definition of e_R.

Remarkably, if a perturbative analysis is ever valid for SU(2) gauge theory, then Eq. (3.12) tells us how e_0 must vary in the continuum limit. Because e_R is being held fixed, we have

$$0 = a \frac{d}{da} e_R(e_0, \mu, a) = \frac{\partial e_R}{\partial e_0} a \frac{de_0}{da} + a \frac{\partial e_R}{\partial a}. \tag{3.13}$$

Simple dimensional analysis gives

$$a \frac{\partial e_R}{\partial a} = \mu \frac{\partial e_R}{\partial \mu} = \gamma(e_R). \tag{3.14}$$

Combining Eqs. (3.10)–(3.14), we obtain

$$a \frac{de_0}{da} = \frac{11 e_0^3}{24\pi^2} + O(e_0^5). \tag{3.15}$$

We now assume that e_0 does not get hung up on a fixed point away from the origin. If e_0 eventually is small enough that the $O(e_0^5)$ term in Eq. (3.15) is negligible, then we can integrate to obtain

$$e_0^2 = \frac{12\pi^2}{11 \ln(\hat{a}/a)} + O((\ln a)^{-2}). \tag{3.16}$$

This analysis does not determine the integration constant $\hat{a}$. For later comparison with our Monte Carlo results, we rewrite Eq. (3.16) in the form

$$a^2 \underset{e_0 \to 0}{\sim} \hat{a}^2 \exp\left(-\frac{24\pi^2}{11 e_0^2} \right) = \hat{a}^2 \exp\left(-\frac{6\pi^2 \beta}{11} \right). \tag{3.17}$$

This is the prediction of asymptotic freedom.

In the lattice formulation a natural physical quantity to use for renormalization purposes is the string tension.[12] This quantity can be extrac-

ted from a study of Wilson loops. For a closed contour C comprised of links in the lattice, the Wilson loop is defined

$$W(C)=\left\langle \tfrac{1}{2}\mathrm{Tr}\left(\prod_C U\right)_{\mathrm{PO}}\right\rangle . \tag{3.18}$$

Here, PO represents "path ordering"; that is, the U_{ij} are ordered and oriented as they are encountered in circulating around the contour. The simplest Wilson loop arises when C is a single plaquette, in which case

$$W(\square)=1-\langle S_\square\rangle . \tag{3.19}$$

If for large separations the interaction energy of two static sources in the fundamental representation of the gauge group increases linearly with distance, then one expects for large contours[6]

$$\ln W(C)=-KA(C)+O(p(C)), \tag{3.20}$$

where $A(C)$ is the minimum area enclosed by C and $p(C)$ is the contour perimeter. The string tension K is the coefficient of the linear part of the static quark-antiquark potential. Measuring A in physical units, it equals the lattice spacing squared times $N_\square(C)$, the minimum number of plaquettes forming a surface bounded by C. Thus we write

$$\ln W(C)=-(a^2K)N_\square(C)+O(p(C)) . \tag{3.21}$$

By measuring the exponential falloff of $W(C)$ with $N_\square$, we effectively measure the dimensionless quantity a^2K as a function of the bare coupling e_0.

As the string tension is a physical quantity, we can use it to define the continuum limit of the theory. Thus we consider the renormalization scheme of holding K fixed by adjusting e_0 while the lattice spacing a is taken to zero. As K is a nonperturbative quantity, this scheme does not directly use an analog of Eq. (3.10). However, the asymptotic-freedom result in Eq. (3.17) is independent of renormalization method, and thus we expect to find for small e_0, i.e., large β,

$$a^2K\sim\exp\left(\frac{-6\pi^2\beta}{11}\right) . \tag{3.22}$$

Indeed, the left-hand side of Eq. (3.15) represents a definition of a Gell-Mann–Low function for this renormalization prescription.

Although the classical theory contains only dimensionless parameters, the quantum theory requires introduction of a dimensional quantity such as K. This is an example of the phenomenon of dimensional transmutation as discussed by Coleman and Weinberg.[19] Note also the essential singularity at vanishing coupling in Eq. (3.22). Thus the string tension is a strictly nonperturbative quantity.

In the small-β region Wilson's strong-coupling expansion is valid. The first nonvanishing term in this expansion is

$$W(C)\to(\tfrac{1}{4}\beta)^{N_\square} . \tag{3.23}$$

Consequently, we expect at high temperatures

$$a^2K\to-\ln(\tfrac{1}{4}\beta) . \tag{3.24}$$

The main goal of the calculations of the next section is to join the behaviors in Eqs. (3.22) and (3.24).

IV. RESULTS

In Fig. 1, we illustrate the convergence of the Monte Carlo procedure. Working at $\beta=2.3$, we show the average plaquette P, defined in Eq. (2.21), as a function of number of iterations for a total of 30 iterations.[20] We show runs begun both randomly as well as ordered and on lattices of dimensions $4\times4\times4\times4$, $6\times6\times6\times6$, $8\times8\times8\times8$, and $10\times10\times10\times10$. Note that the convergence rate is essentially independent of lattice size; only the fluctuations grow on the smaller lattices. This supports the absence of a phase transition in this region.

In Fig. 2 we study the evolution from the zero-temperature ordered state on an 8^4 lattice for several values of β. Note that the convergence time is not strongly β dependent; a slight increase appears in the range $\beta\approx2$–2.4. At all β, equilibrium is essentially complete after 20 iterations. Convergence becomes extremely good in the high- and low-temperature regions; consequently, the method is not tied to either strong or weak coupling.

In Fig. 3 we show the expectation values of

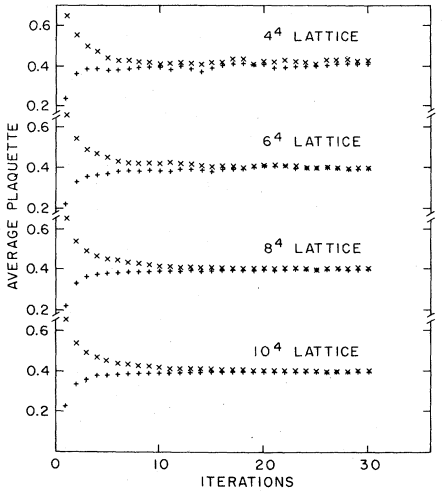

FIG. 1. The average plaquette as a function of number of iterations at $\beta=2.3$.

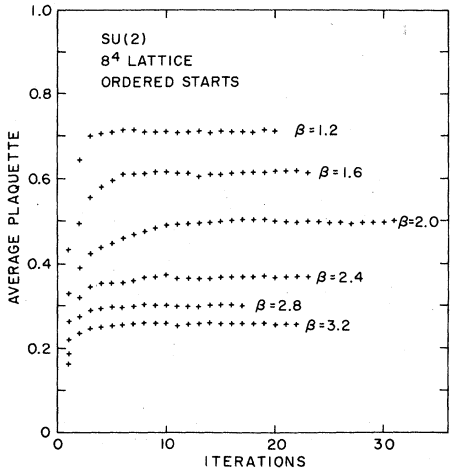

FIG. 2. The evolution of the average plaquette at several values of β.

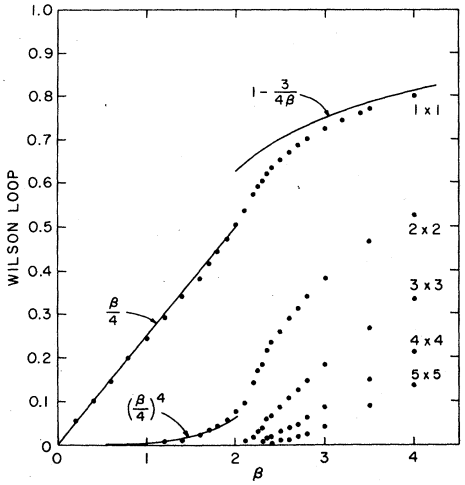

FIG. 4. Wilson loops as a function of β.

square Wilson loops at $\beta=3$ as a function of lattice size. These loops are taken to lie in a fundamental plane of the lattice and are up to six links on a side. Each measurement is an average over all similar loops in the lattice and the error bars represent the standard deviation for the fluctuations over five iterations after attaining equilibrium. As intuitively expected, larger loops show the finite-size effects most strongly. On a 10^4 lattice, loops of up to five sites on a side appear to have stabilized. This represents the largest loop used in the subsequent analysis.

In Fig. 4 we summarize the values for these square Wilson loops as a function of β. In the important region from $\beta=2.1$ to $\beta=3$, these numbers came from a 10^4 lattice whereas elsewhere we used an 8^4 lattice. In Fig. 4 we also plot the strong-coupling result

$$W(\square) \underset{\beta\to 0}{\sim} \tfrac{1}{4}\beta, \qquad W(2\times 2) \underset{\beta\to 0}{\sim} (\tfrac{1}{4}\beta)^4, \tag{4.1}$$

and the weak-coupling limit[10]

$$W(\square) \underset{\beta\to\infty}{\sim} 1-\frac{3}{4\beta}. \tag{4.2}$$

All loops approach unity inversely with β at low temperatures.

To extract a string tension at a given value of β, we fit these loops to the form

$$W(S)=\exp[-(A+BS+CS^2)], \tag{4.3}$$

where S is the loop side. We adjusted the parameters A, B, and C to minimize the mean square deviation of this fit from the "measured" loops. In Fig. 5 we show some representative fits. Be-

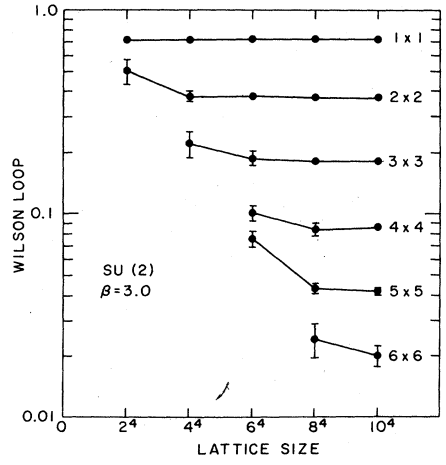

FIG. 3. Wilson loops at $\beta=3$ as a function of lattice size.

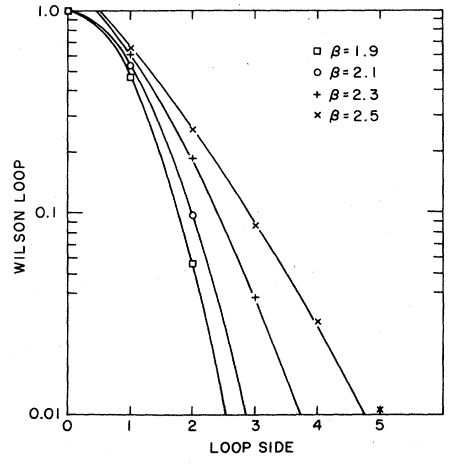

FIG. 5. Representative fits to the loops.

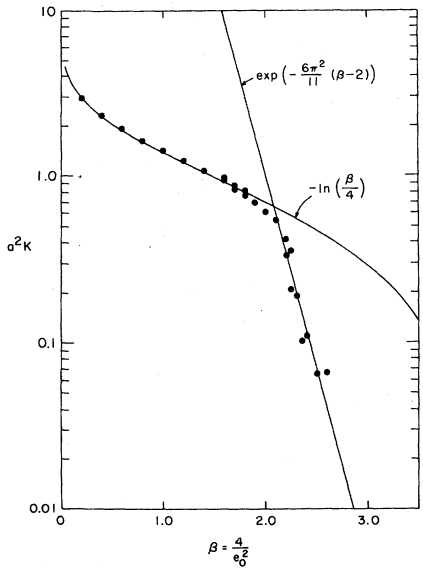

FIG. 6. The cutoff squared times the string tension as a function of β. The solid lines are the strong- and weak-coupling limits.

low $\beta=2.1$ only loops of side 1 and 2 are significantly different from zero so we must include the loop of side 0 in the fit. Below $\beta=1.6$ only the loop of side 1 is significant and we assume the area term C dominates. From Eq. (3.21) we identify

$$C=a^2K. \tag{4.4}$$

In Fig. 6 we summarize these results by plotting a^2K versus β. Here we also plot the strong-coupling result of Eq. (3.24) and the weak-coupling conclusion of Eq. (3.22) with an arbitrarily chosen normalization. From $\beta=1.6$ to 1.8 we plot both the least-square fit and the result of assuming pure area-law behavior. For $\beta=2.2$ and 2.25 we plot fits including and not including the loop of side zero. Above $\beta=2.5$ the area law is too subdominant relative to the perimeter law for accurate determination. As each temperature is treated independently of the others, the fluctuations apparent in this figure represent the statistical error of this analysis.

V. DISCUSSION

Note that the changeover from the strong-coupling behavior of Eq. (3.24) to the weak-coupling behavior of Eq. (3.22) occurs rather sharply over a range of about 10% in β about $\beta=2$. This appearance of the confinement mechanism occurs at

$$\frac{e_0^2}{4\pi}\approx 0.16. \tag{5.1}$$

The rapid evolution out of the perturbative regime may be responsible for the remarkable phenomenological successes of the bag model.[21] High-temperature-series results,[12] as well as semiclassical treatments,[5] have also suggested an abrupt onset of confinement.

Our analysis allows a determination of the renormalization scale of the coupling in terms of the string tension. Using the observed asymptotic normalization

$$a^2K \underset{\beta\to\infty}{\sim} \exp\left(-\frac{6\pi^2}{11}(\beta-2)\right), \tag{5.2}$$

we can solve for e_0^2 to give

$$\frac{e_0^2}{4\pi} \underset{a\to 0}{\sim} \frac{3\pi}{11\ln(1/a\Lambda)}, \tag{5.3}$$

where the renormalization scale is

$$\Lambda\approx\sqrt{K}\exp\left(-\frac{6\pi^2}{11}\right)\approx\frac{1}{200}\sqrt{K}. \tag{5.4}$$

Thus we see the appearance of a rather large dimensionless number. The uncertainty in this coefficient is roughly a factor of two because of the large coefficient in the exponential. The renormalization mass should be strongly dependent on both the gauge group and addition of quarks.

We have shown the onset of asymptotic freedom for the bare coupling constant in a renormalization scheme based on confinement. This is strongly suggestive that SU(2) non-Abelian gauge theory simultaneously exhibits confinement and asymptotic freedom. Furthermore, by reproducing the asymptotic-freedom prediction, we strengthen ties between the lattice formulation and the more conventional perturbative approaches to gauge theory.

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