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Existence proof of a nonconfining phase in four-dimensional U(1) lattice gauge theory

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A rigorous lower bound is obtained for the Wilson loop expectation value $\langle A[C]\rangle$ in the four-dimensional U(1) lattice gauge theory with an action of the Villain form. The bound, which holds for $g^2<0.168$, has the form of a Coulomb interaction and guarantees that electric charges are not confined. Using the strong-coupling results of Osterwalder and Seiler, one concludes that this model has at least one phase transition. Using the work of Elitzur, Pearson, and Shigemitsu, one also concludes that the four-dimensional Villain $Z(N)$ lattice gauge theory has at least three phases if $N>37$. Finally, it is also shown that the infinite-volume limit of $\langle A[C]\rangle$ exists.

I. INTRODUCTION

The U(1) lattice gauge theory was formulated by Wilson[1] in 1974, primarily for the purpose of understanding how a gauge theory can result in the confinement of quarks. Wilson studied the strong-coupling limit of the theory by means of a perturbation expansion, and concluded (nonrigorously) that the Wilson loop expectation value $\langle A[C]\rangle$ behaves for large loops as

$$\langle A[C]\rangle \sim c_1 e^{-c_2\Sigma}\,, \tag{1.1}$$

where Σ is the area of the minimal surface which spans the loop C. This behavior corresponds to the confinement of electric charge, providing an analogy with the confinement of quarks in non-Abelian gauge theories. However, if the lattice theory is to provide a valid approximation to QED, then the confinement property is clearly unrealistic. Wilson therefore conjectured that in four dimensions the model undergoes a phase transition as the coupling constant g^2 is varied, with a nonconfining phase in weak coupling. Wilson's conclusions about the strong-coupling behavior of the theory have since been rigorously established by Osterwalder and Seiler.[2]

Various authors have studied the weak-coupling behavior of the theory, but to my knowledge the conjectured phase transition remains unproven. The space-time lattice formulation has been studied using a modified form of the action which was proposed for the two-dimensional XY model by Villain.[3] The Villain action resembles qualitatively the cosine function proposed by Wilson, and in fact approaches the Wilson action for both large and small values of g^2. Using the Villain action, Savit[4] and Banks, Myerson, and Kogut[5] have shown that the U(1) theory can be rewritten as a theory involving only closed loops of magnetic current (in four-dimensional space-time) with Coulomb interactions. The action associated with a loop is proportional to $1/g^2$, so they become very sparse as $g^2\to 0$. Banks *et al.* argue that they can therefore be ignored in weak coupling, and this argument leads to a Coulomb force law between charges. This argument has been refined slightly by Ukawa, Windey, and myself,[6] who have derived a perturbation expansion to describe the effect of a low density of magnetic loops. An explicit calculation of the leading term indicates that the Coulomb form of the long-range force is unchanged, but the effective coupling is increased slightly:

$$g_{\rm eff}{}^2 = g^2 + 8\pi^2 e^{-\pi^2/g^2} + \cdots\,. \tag{1.2}$$

Also using the Villain form, Peskin[7] has shown that the model can be rewritten as a singular limit of a (noncompact) Abelian Higgs model. For strong coupling the Higgs field acquires an expectation value, and Peskin argues that this implies confinement. For weak coupling one expects this expectation value to vanish, and hence the cause of confinement is removed.

Drell, Quinn, Svetitsky, and Weinstein[8] have examined a Hamiltonian formulation of the theory (i.e., discrete space with continuous time). Using variational techniques, they reduce the problem to a form very similar to that of Banks *et al.* and then make similar arguments.

The U(1) lattice gauge theory has also been studied rigorously by Glimm and Jaffe,[9] who derive the leading term for the free energy in asymptotically weak coupling. (Their approach is equivalent to using the Villain form of the theory, although they describe it in different terms.)

This paper will also concern the Villain form of the theory. I will prove that for g^2 sufficiently small (numerically, $g^2<0.168$), the Wilson loop expectation value obeys a bound

$$\langle A[C]\rangle \geq \exp\left(-\tfrac{1}{2}g'^2 \sum_{b,b'} j_{b'}\Box^{-1}_{b'b} j_b\right)\,, \tag{1.3}$$

where g'^2 is a function of g^2, b and b' are summed over the bonds of the lattice, $\Box^{-1}$ is the inverse lattice Laplacian, and j_b is the "current" associated with the Wilson loop C (i.e., $j_b=1$ for bonds contained in C, and $j_b=0$ otherwise). The bound is shown to hold on finite lattices with specified boundary conditions, denoted by Λ_R^* and described in Sec. II. It is later shown that the infinite-volume limit of $\langle A[C]\rangle$ exists and therefore also obeys this bound.

The electrostatic potential is defined in terms of rectangular loops $C(r,T)$ with sides of length r and T:

$$\langle A[C(r,T)]\rangle \underset{T\to\infty}{\sim} \text{const}\times e^{-V(r)T}. \tag{1.4}$$

Assuming that this limit exists, it follows from (1.3) that $V(r)$ is bounded by a Coulomb potential

$$V(r)\le g'^2[\Box_3^{-1}(0)-\Box_3^{-1}(r)], \tag{1.5}$$

where $\Box_3^{-1}$ denotes the inverse three-dimensional lattice Laplacian. Thus, the bound implies that electric charges are not confined. When this result is combined with the strong-coupling bound of Osterwalder and Seiler,[2] one sees that the model has at least two phases, distinguished by the qualitatively different behavior of $\langle A[C]\rangle$.

The result also has implications for the Villain form of the four-dimensional $Z(N)$ gauge theory. Elitzur, Pearson, and Shigemitsu[10] have shown that if the Villain U(1) theory has a phase transition between a confining and nonconfining phase at $g^2=g_{\text{cr}}^2$, then the $Z(N)$ theory has at least three phases for $N>2\pi/g_{\text{cr}}^2$.

The paper is organized as follows. The formalism is described in Sec. II. Boundary and coboundary operators are defined, and their relevant properties are explained. Some technical details are relegated to Appendix A. The gauge theory is also defined. In Sec. III the gauge theory is transformed by the method of Banks, Myerson, and Kogut. In Sec. IV an inequality is established which bounds $\langle A[C]\rangle$ from below by an expectation value in the XY model computed in the high-temperature region. In Sec. V a cluster expansion technique is used to prove the final result. The proofs of several technical lemmas are given in Sec. VI and Appendix B. In Appendix C it is shown that the infinite-volume limit of $\langle A[C]\rangle$ exists.

The proofs in the paper are elementary in the sense that no sophisticated mathematical theorems are used, at least not without detailed explanation. The paper should therefore be reasonably easy for any physicist to follow, though it will perhaps seem cumbersome to an experienced mathematical physicist.

II. FORMULATION OF THE THEORY

Before describing the gauge theory, it will be useful to summarize the formalism for describing and manipulating functions on a lattice.[11] Let Λ^∞ denote an infinite D-dimensional Euclidean cubic lattice. Let c_r denote an oriented r-dimensional cell (r-cell) of Λ^∞. (For $D=4$, the cells are the sites, bonds, plaquettes, cubes, and hypercubes of the lattice.) For each c_r, there exists a corresponding r-cell which differs only in its orientation and is denoted by $-c_r$.[12] Let $I(c_r, c_{r+1})$ denote the incidence function. [That is, $I(c_r, c_{r+1})$ takes on the value $+1$ if c_r is contained in c_{r+1} in the proper orientation, -1 if it is contained in c_{r+1} in the reverse orientation, and 0 otherwise.] The incidence function satisfies

$$\sum_{c_r} I(c_{r-1}, c_r)I(c_r, c_{r+1})=0. \tag{2.1}$$

In this paper we will be concerned mainly with finite sublattices $\Lambda\subset\Lambda^\infty$. The incidence function for Λ is defined to be the incidence function for Λ^∞ restricted to those r-cells which are contained in Λ. A sublattice will be called closed if

$$c_{r+1}\in\Lambda \text{ and } I(c_r,c_{r+1})\ne 0 \Rightarrow c_r\in\Lambda.$$

A sublattice is open if

$$c_r\in\Lambda \text{ and } I(c_r,c_{r+1})\ne 0 \Rightarrow c_{r+1}\in\Lambda.$$

To aid visualization, two-dimensional examples of closed and open sublattices of Λ^∞ are shown in Figs. 1(a) and 1(b), respectively. Note that either the open or closed conditions imply that Eq. (2.1) holds for the sublattice.

On a finite lattice there is no need to distinguish between chains and cochains. Thus, for this paper I define an r-chain f_r to be a function of r-cells which takes values in an Abelian group G, with the property that $f(-c_r)=-f(c_r)$. In this paper G will be either the real numbers, the integers, or U(1), where the latter will be represented as the real numbers modulo 2π. (The group operation will always be addition.) The boundary (∂) and coboundary (∇) operators are then defined in the usual way. Thus, if f_r is an r-chain, then $g_{r-1}\equiv\partial f_r$ is an $(r-1)$-chain defined by

$$g(c_{r-1})\equiv\sum_{c_r} I(c_{r-1},c_r)f(c_r). \tag{2.2}$$

The sum is defined to include only one of the two orientations of each r-cell. Similarly, $h_{r+1}\equiv\nabla f_r$ is defined by

$$h(c_{r+1})\equiv\sum_{c_r} I(c_r,c_{r+1})f(c_r). \tag{2.3}$$

(a) (b)

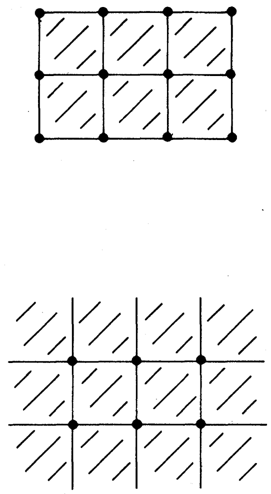

FIG. 1. (a) and (b) are two-dimensional examples of closed and open sublattices of Λ^∞, respectively. These lattices are dual to each other and serve as examples of lattices Λ_R and Λ_R^*.

From (2.1), one has $\partial^2 = \nabla^2 = 0$. One can also define an inner product on r-chains:

$$\langle f'_r, f_r \rangle \equiv \sum_{c_r} f'(c_r) f(c_r) . \tag{2.4}$$

The multiplication on the right will be well defined whenever both chains are real, or whenever one chain is integral. With respect to this inner product, ∇ is the adjoint of ∂.

The Laplace-Beltrami operator is defined by

$$\Box \equiv \nabla\partial + \partial\nabla . \tag{2.5}$$

It is a positive-semidefinite symmetric linear operator which commutes with ∂ or ∇. It agrees with the usual finite difference approximation to the Laplacian, except that the latter is ill-defined at the boundaries of a finite lattice.

It will be useful to consider also the lattice Λ^* which is dual to a given lattice Λ. Each cell $c_r^* \in \Lambda^*$ is dual to a cell $c_{D-r} \in \Lambda$. The incidence function I^* is given by

$$I^*(c_r^*, c_{r+1}^*) = I(c_{D-(r+1)}, c_{D-r}) , \tag{2.6}$$

where the cells which are arguments of I are dual to the cells which are arguments of I^* (but in the opposite order). If Λ is a closed sublattice of Λ^∞, then Λ^* is an open sublattice of $\Lambda^{\infty *}$, and vice versa. (The lattices Λ^∞ and $\Lambda^{\infty *}$ are of course isomorphic, so there is no need to distinguish them.) In Fig. 1, the lattices (a) and (b) are dual to each other.

One can of course define r-chains on the lattice Λ^*, and the boundary and coboundary operators are defined in terms of the incidence functions I^*. There is also a natural mapping $g_r \equiv {}^*f_{D-r}$ which maps a $(D-r)$-chain on Λ to an r-chain on Λ^* (or vice versa), using the simple prescription

$$g(c_r^*) = f(c_{D-r}) , \tag{2.7}$$

where c_r^* is dual to c_{D-r}. One then has

$$\begin{aligned} &** = 1 , \\ &*\nabla = \partial * , \\ &*\partial = \nabla * . \end{aligned} \tag{2.8}$$

The bulk of this paper will concern a particularly simple sublattice, which will be called Λ_R. Its sites are chosen to be a finite rectangular array of the sites of Λ^∞. The r-cells of Λ are then selected recursively on r, using the criterion that $c_r \in \Lambda$ if and only if $c_{r-1} \in \Lambda$ for all c_{r-1} with $I(c_{r-1}, c_r) \neq 0$. Thus, Λ_R is closed. Figure 1(a) shows a two-dimensional version of Λ_R and Fig. 1(b) shows its dual.

There are several homology properties of the lattices Λ_R and Λ_R^* which will be used later. For the lattice Λ_R (if $r \neq 0$) or for the lattice Λ_R^* (if $r \neq 4$) we have the following:

(i) If $\partial f_r = 0$, then there exists a g_{r+1} such that $f_r = \partial g_{r+1}$. (For $r = 4$, this means $f = 0$.)

(ii) If $\nabla f_r = 0$, then there exists a g_{r-1} such that $f_r = \nabla g_{r-1}$. (For $r = 0$, this means $f = 0$.)

(iii) The operator $\Box$ is invertible when operating on real-valued r-chains.

For completeness, an elementary proof of these properties is given in Appendix A. (Note that for the excluded values of r, these properties are violated by the constant function.)

The gauge theory can now be formulated on an arbitrary finite sublattice Λ. The basic variables will be $\theta_b \in \mathrm{U}(1)$, defined on each bond $b \in \Lambda$. Then $\nabla\theta$ is a 2-chain, defined on plaquettes $p \in \Lambda$. The gauge theory partition function has the form

$$Z = \int_0^{2\pi} \{d\theta_b\} \exp\left\{ - \sum_p S[(\nabla\theta)_p] \right\} , \tag{2.9}$$

where

$$\{d\theta_b\} \equiv \prod_{b \in \Lambda} d\theta_b , \tag{2.10}$$

and $S(\phi)$ is some action function. The form originally proposed by Wilson is

$$S_W(\phi) = -(1/g^2)\cos\phi . \tag{2.11}$$

This paper will concern the Villain[3] form of the action, defined by

$$\exp[-S_V(\phi)] = \sum_{l=-\infty}^{\infty} \exp[-(1/2g^2)(\phi - 2\pi l)^2] \tag{2.12a}$$

$$= \frac{g}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp[-\tfrac{1}{2}g^2 n^2]\, e^{in\phi} . \tag{2.12b}$$

The expectation value to be studied is that of the Wilson loop operator $A[C]$. The closed loop C can be described by an integer-valued 1-chain j_b, which is equal to the number of times the loop C crosses the bond b, each counted with the appropriate sign. Thus,

$$\partial j = 0\,. \tag{2.13}$$

Then

$$\langle A[C]\rangle \equiv \frac{1}{Z}\int_0^{2\pi}\{d\theta_b\}\sum_{\{l_p\}=-\infty}^{\infty}\exp[-(1/2g^2)\|\nabla\theta - 2\pi l\|^2]\,e^{i\langle j,\theta\rangle}\,, \tag{2.14}$$

where I have used the notation $\|\ \|^2 \equiv \langle\,,\rangle$.

In the proof that will follow, most of the work will be carried out in terms of the dual lattice Λ^*. Since the lattice Λ_R is simpler than its dual, it will prove convenient to formulate the gauge theory on the lattice $\Lambda \equiv \Lambda_R^*$. The Wilson-loop expectation value $\langle A[C]\rangle$ can then be rewritten in terms of quantities associated with the lattice Λ_R. The loop C, which was described by a 1-chain on Λ_R^*, will be described by a 3-chain on Λ_R:

$$\tilde{j}_c \equiv {}^*j_b\,, \tag{2.15}$$

where the subscript c denotes a function of cubes. Then

$$\nabla\tilde{j}_c = 0 \tag{2.16}$$

and

$$\langle A[C]\rangle = \frac{1}{Z}\int_0^{2\pi}\{d\theta_c\}\sum_{\{l_p\}=-\infty}^{\infty}\exp[-(1/2g^2)\|\partial\theta - 2\pi l\|^2]e^{i\langle\tilde{j},\theta\rangle}\,, \tag{2.17}$$

where now all of the r-chains are defined on the lattice Λ_R.

III. THE BANKS-MYERSON-KOGUT TRANSFORMATION

The first step in proving a lower bound for $\langle A[C]\rangle$ is to carry out the transformation introduced by Savit[4] and Banks, Myerson, and Kogut.[5] (This transformation is similar to one which has been used to discuss a class of two-dimensional models.[13])

The starting point is Eq. (2.17). The transformation can be carried out in several steps. To begin, one must remove the gauge symmetry

$$\theta_c \to \theta_c' = \theta_c + \partial\psi_h\,, \tag{3.1}$$

where ψ_h is an arbitrary function of hypercubes. This is done by choosing a set $\overline{C}$ of cubes and requiring

$$\theta_c = 0 \quad \text{if } c \in \overline{C}\,. \tag{3.2}$$

There are many choices of $\overline{C}$ which will work, but the simplest choice is constructed as follows. Let $h_1, \ldots, h_N$ denote a sequence of all the hypercubes of Λ_R, with the property that h_i and h_{i+1} share one cube. Let $\overline{C}$ be the set of these shared cubes, along with one additional cube which is contained in only one hypercube. It is then easily seen that there exists a unique gauge transformation ψ_h such that θ_c' obeys the gauge condition (3.2). Thus, (2.17) can be rewritten with the substitution

$$\int_0^{2\pi}\{d\theta_c\} \to \int_0^{2\pi}\{d\theta_c\}_{c\in C}\,,$$

where C denotes the complement of $\overline{C}$.

For any fixed choice of $\{l_p\}$, one can define

$$m_b \equiv \partial l_p\,. \tag{3.3}$$

Then

$$\partial m_b = 0\,. \tag{3.4}$$

One then defines an integer-valued 2-chain $G_p[m]$ satisfying

$$\partial G_p[m] = m_b\,. \tag{3.5}$$

Then $\partial(l_p - G_p) = 0$, which implies that there exists an integer-valued 3-chain χ_c satisfying

$$l_p = G_p[m] + \partial\chi_c\,. \tag{3.6}$$

χ_c can be fixed uniquely by requiring

$$\chi_c = 0 \quad \text{if } c \in \overline{C}\,.$$

Equation (2.17) can then be modified by

$$\sum_{\{l_p\}=-\infty}^{\infty} \to \sum_{\substack{\{m_b\}=-\infty \\ \partial m=0}}^{\infty}\ \sum_{\substack{\{\chi_c\}=-\infty \\ c\in C}}^{\infty}\,.$$

Then, setting $a_c \equiv \theta_c - 2\pi\chi_c$, one has

$$\langle A[C]\rangle = \frac{1}{Z}\int_{-\infty}^{\infty}\{da_c\}_{c\in C} \sum_{\substack{\{m_b\}=-\infty \\ \partial m=0}}^{\infty} \exp[-(1/2g^2)\|\partial a - 2\pi G[m]\|^2] e^{i\langle \tilde{j}, a\rangle}. \tag{3.7}$$

In the above equation, and in all subsequent expressions in this paper which are normalized by $1/Z$, the symbol Z refers to the same quantity which appears in the numerator except that the source j is set equal to zero. The symbol Z which appears in one equation will not necessarily be equal to the same symbol in a different equation.

Finally, one can integrate out $\{a_c\}$. One shifts the variables of integration so that the integrand undergoes a shift $a_c \to a_c + \hat{a}_c$. The a_c integration is then seen to decouple provided that

$$\nabla\partial\hat{a} = 2\pi\nabla G + i g^2 \tilde{j}. \tag{3.8}$$

The general solution to (3.8) is then

$$\hat{a} = \Box^{-1}(2\pi\nabla G + i g^2\tilde{j}) + \partial\psi_h, \tag{3.9}$$

for any 4-chain ψ_h. In principle one can choose ψ_h so that $\hat{a}_c$ vanishes for $c \in \overline{C}$, but this choice need not be carried out explicitly. After some manipulations one has

$$\langle A[C]\rangle = A_0[C]E[C], \tag{3.10}$$

where

$$A_0[C] \equiv \exp(-\tfrac{1}{2}g^2\langle \tilde{j}, \Box^{-1}\tilde{j}\rangle) \tag{3.11}$$

and

$$E[C] \equiv \frac{1}{Z}\sum_{\substack{\{m_b\}=-\infty \\ \partial m=0}}^{\infty} \exp\{-\tfrac{1}{2}(2\pi/g)^2\langle m, \Box^{-1}m\rangle + 2\pi i\langle \tilde{j}, \Box^{-1}\nabla G[m]\rangle\}. \tag{3.12}$$

The variables m_b are interpreted physically as a magnetic current density.

Equation (3.12) can be rewritten by using the property $\nabla\tilde{j} = 0$ to introduce an integer-valued 2-chain H_p which satisfies

$$\nabla H_p = \tilde{j}_c. \tag{3.13}$$

Then note that

$$\exp\{2\pi i\langle \tilde{j}, \Box^{-1}\nabla G[m]\rangle\} = \exp(i\langle m, a\rangle), \tag{3.14}$$

where

$$a_b \equiv -2\pi\Box^{-1}\partial H_p. \tag{3.15}$$

Then

$$E[C] = \frac{1}{Z}\sum_{\substack{\{m_b\}=-\infty \\ \partial m=0}}^{\infty} \exp[-\tfrac{1}{2}(2\pi/g)^2\langle m, \Box^{-1}m\rangle + i\langle m, a\rangle]. \tag{3.16}$$

Since $E[C]$ is the expectation value of a complex quantity with modulus one, it follows that $E[C] \leq 1$. Thus, one immediately has an upper bound on $\langle A[C]\rangle$:

Theorem I:

$$\langle A[C]\rangle \leq \exp(-\tfrac{1}{2}g^2\langle \tilde{j}, \Box^{-1}\tilde{j}\rangle). \tag{3.17}$$

IV. RELATION TO THE XY MODEL

The remaining problem is to analyze the behavior of the quantity $E[C]$ defined by (3.16). $E[C]$ is an expectation value computed in a statistical ensemble of magnetic current loops with Coulomb interactions. This is the Coulomb dipole gas discussed by Glimm and Jaffe.[9] Because the Coulomb interaction is difficult to treat by standard mathematical methods, the strategy here is to eliminate it from the problem as soon as possible. This is accomplished by means of an inequality of the Griffiths-Kelly-Sherman[14] type. The particular technique used here follows very closely that of Elitzur, Shigemitsu, and Pearson.[10] The end result will be to bound $E[C]$ from below by an expectation value computed in the XY model. The weak-coupling region of the gauge theory will correspond to the high-temperature region of the XY model.

Lemma 4.1. Let $K_{bb'}$ be a real symmetric positive-semidefinite matrix, with indices $b, b' \in \Lambda_R$. Furthermore, let λ be a real parameter such that the matrix $(\Box^{-1} + \lambda K)$ is positive definite. Then define

$$\hat{E}[C;K;\lambda] \equiv \frac{1}{Z}\sum_{\substack{\{m_b\}=-\infty \\ \partial m=0}}^{\infty} \exp[-\tfrac{1}{2}(2\pi/g)^2\langle m, (\Box^{-1}+\lambda K)m\rangle + i\langle m, a\rangle]. \tag{4.1}$$

(Here Z is defined as the value of the numerator when the source a_b is set equal to zero.) Then

$$\frac{\partial \hat{E}}{\partial \lambda} \geq 0 . \tag{4.2}$$

Proof. Differentiating (4.1),

$$\frac{\partial \hat{E}}{\partial \lambda} = -\frac{1}{2}\left(\frac{2\pi}{g}\right)^2 \frac{1}{Z^2} \sum_{\substack{\{m_b\}=-\infty \\ \partial m=0}}^{\infty} \sum_{\substack{\{m_b'\}=-\infty \\ \partial m'=0}}^{\infty} [\langle m, Km\rangle - \langle m', Km'\rangle]$$
$$\times \exp\{-\tfrac{1}{2}(2\pi/g)^2[\langle m, (\Box^{-1}+\lambda K)m\rangle + \langle m', (\Box^{-1}+\lambda K)m'\rangle] + i\langle m, a\rangle\} .$$

In the above equation and for the rest of this proof, the symbol Z denotes the value it was assigned in Eq. (4.1). Now introduce the new variables

$$\mu_b \equiv m_b + m_b' ,$$
$$\mu_b' \equiv m_b - m_b' .$$

Note that for each b, μ_b and μ_b' are either both even or both odd. This constraint can be accommodated by defining a parity function

$$\pi(\mu) = \begin{cases} 0 & \text{if } \mu \text{ is even}, \\ 1 & \text{if } \mu \text{ is odd}, \end{cases}$$

and requiring $\pi(\mu_b) = \pi(\mu_b') \equiv \pi_b$. Then

$$\frac{\partial \hat{E}}{\partial \lambda} = -\frac{1}{2}\left(\frac{2\pi}{g}\right)^2 \frac{1}{Z^2}$$
$$\times \sum_{\{\pi_b\}=0,1} \sum_{\substack{\{\mu_b\}=-\infty \\ \partial\mu=0 \\ \pi(\mu_b)=\pi_b}}^{\infty} \sum_{\substack{\{\mu_b'\}=-\infty \\ \partial\mu'=0 \\ \pi(\mu_b')=\pi_b}}^{\infty} \langle \mu', K\mu\rangle \exp\{-\tfrac{1}{4}(2\pi/g)^2[\langle \mu', (\Box^{-1}+\lambda K)\mu'\rangle + \langle \mu, (\Box^{-1}+\lambda K)\mu\rangle] + \tfrac{1}{2}i\langle \mu'+\mu, a\rangle\} .$$

[In the above sum, the only nonzero contributions will come from parity assignments $\{\pi_b\}$ which satisfy $\partial\pi = 0$ (mod 2).] Rearranging,

$$\frac{\partial \hat{E}}{\partial \lambda} = \frac{1}{2}\left(\frac{2\pi}{g}\right)^2 \frac{1}{Z^2} \sum_{\{\pi_b\}} \langle V_b[\pi], KV_b[\pi]\rangle , \tag{4.3}$$

where

$$V_b[\pi] = i \sum_{\substack{\{\mu_b\}=-\infty \\ \partial\mu=0 \\ \pi(\mu_b)=\pi_b}}^{\infty} \mu_b \exp[-\tfrac{1}{4}(2\pi/g)^2\langle \mu, (\Box^{-1}+\lambda K)\mu\rangle + \tfrac{1}{2}i\langle \mu, a\rangle] . \tag{4.4}$$

The symmetry $\mu_b \to -\mu_b$ guarantees that $V_b[\pi]$ is real. Since K is positive semidefinite, the lemma is proven.

Lemma 4.2:

$$\langle m, \Box m\rangle \leq 16\|m\|^2 .$$

The proof of this lemma will be given at the end of Appendix A.

Lemma 4.3. Let $E[C]$ be the quantity defined by (3.16). Then

$$E[C] \geq F[C] \equiv \frac{1}{Z} \sum_{\substack{\{m_b\}=-\infty \\ \partial m=0}}^{\infty} \exp(-\tfrac{1}{2}\gamma\|m\|^2 + i\langle m, a\rangle) , \tag{4.5}$$

where

$$\gamma = \tfrac{1}{16}(2\pi/g)^2 . \tag{4.6}$$

Proof. By Lemma 4.2, the matrix

$$K_{bb'} = \Box^{-1}_{bb'} - \tfrac{1}{16}\delta_{bb'}$$

is positive semidefinite. Furthermore, the matrix $(\Box^{-1}+\lambda K)$ is positive definite for $-1 \leq \lambda \leq 0$. Then by Lemma 4.1, the quantity $\hat{E}[C;K;\lambda]$ is monotonically nondecreasing in λ. But $\hat{E}[C;K;0]=E[C]$ and $E[C;K;-1]=F[C]$, so the lemma is proven.

The quantity $F[C]$ is actually an expectation value in the XY model. This equivalence can be seen by introducing an integration over variables $\{\theta_s\}$ to enforce the constraint $\partial m=0$:

$$F[C]=\frac{1}{Z}\int_0^{2\pi}\{d\theta_s\}\sum_{\{m_b\}=-\infty}^{\infty}\exp[-\tfrac{1}{2}\gamma\|m\|^2+i\langle\partial m,\theta\rangle+i\langle m,a\rangle]. \tag{4.7}$$

Now define

$$\exp[-V(\phi)]\equiv\sum_{m=-\infty}^{\infty}\exp[-\tfrac{1}{2}\gamma m^2+im\phi], \tag{4.8}$$

which is again the form of a Villain action [see (2.12b)]. Then

$$F[C]=\frac{1}{Z}\int_0^{2\pi}\{d\theta_s\}\exp\left[-\sum_b V((\nabla\theta)_b+a_b)\right]. \tag{4.9}$$

In this form one clearly recognizes the XY model with Villain action. As $g^2\to 0$, $\gamma\to\infty$, and thus the problem has been reduced to that of the high-temperature behavior of the XY model.

V. APPLICATION OF THE CLUSTER EXPANSION

In this section it will be shown that $F[C]$ can be bounded from below by the use of a cluster expansion. The cluster expansion technique was introduced in constructive quantum field theory by Glimm, Jaffe, and Spencer,[15] and was later used in the context of lattice gauge theory by Osterwalder and Seiler.[2]

To begin, one defines

$$W(\phi)\equiv\exp[-V(\phi)]=\sum_{m=-\infty}^{\infty}\exp(-\tfrac{1}{2}\gamma m^2+im\phi) \tag{5.1a}$$

$$=\left(\frac{2\pi}{\gamma}\right)^{1/2}\sum_{l=-\infty}^{\infty}\exp[-(1/2\gamma)(\phi-2\pi l)^2]. \tag{5.1b}$$

Equation (4.9) can then be rewritten as

$$F[C]=\frac{1}{Z}\int_0^{2\pi}\{d\theta_s\}\prod_b W((\nabla\theta)_b+a_b). \tag{5.2}$$

Now let $W_{\min}$ denote the minimum value of $W(\phi)$. Positivity of $W_{\min}$ is assured by (5.1b). The quantity $\rho(\phi)$ is then defined by

$$W(\phi)\equiv W_{\min}[1+\rho(\phi)]. \tag{5.3}$$

It follows immediately that

$$0\le\rho(\varphi)\le\frac{W(0)-W_{\min}}{W_{\min}}\equiv\epsilon(\gamma). \tag{5.4}$$

It is easily seen that $\epsilon(\gamma)\to 0$ as $\gamma\to\infty$. The cluster expansion is obtained by inserting (5.3) into (5.2) and then expanding the product:

$$F[C]=\frac{1}{Z}\sum_Q G_Q[a], \tag{5.5}$$

where

$$G_Q[a]\equiv\int_0^{2\pi}\left\{\frac{d\theta_s}{2\pi}\right\}\prod_{b\in Q}\rho((\nabla\theta)_b+a_b) \tag{5.6}$$

and Q is summed over all subsets of the bonds in Λ_R. From (5.4) one has

$$0\le G_Q[a]\le[\epsilon(\gamma)]^{L(Q)}, \tag{5.7}$$

where $L(Q)$ is the number of bonds contained in the cluster Q.

Now note that any cluster Q can be uniquely partitioned into connected clusters $R_1,\ldots,R_n$. (Two bonds are *connected* if they share a site, and a cluster is connected if any two of its bonds can be joined by a sequence of connected bonds in the cluster.) Then

$$G_Q[a]=\prod_{i=1}^{n}G_{R_i}[a] \tag{5.8}$$

and

$$F[C]=\frac{1}{Z}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{R_1\cdots R_n\\ \text{disjoint}}}\prod_{i=1}^{n}G_{R_i}[a]. \tag{5.9}$$

In the above sum, each R_i is summed over all connected clusters, subject only to the constraint that $\{R_1\cdots R_n\}$ is disjoint (e.g., no bond in one cluster R_i is connected to any bond in another cluster R_j).

The next problem is to bound each $G_R[a]$ from below. This bound is stated below as Lemma 5.1, and it will be proven in Sec. VI. In the analysis of $G_R[a]$, it is convenient to describe the effect of the Wilson loop by the quantity

$$f_p\equiv\partial\Box^{-1}\vec{j}_c. \tag{5.10}$$

One can think of $\Box^{-1}\vec{j}_c$ as the electromagnetic potential arising from the current $\vec{j}_c$, and then f_p is the electromagnetic field strength. The bound on $G_R[a]$ can then be stated as follows.

Lemma 5.1. Let R be a connected cluster of L bonds. Furthermore, let $\bar{f}_R$ denote the maximum value of $|f_p|$ on any plaquette which is within a distance $\frac{1}{2}L$ of the cluster R. (The distance between a plaquette p and the cluster R is defined as the minimum number of plaquettes in a connected sequence which starts with p and ends with a plaquette which contains a bond of R.) Then

$$G_R[a] \geq (1-\Delta_R)G_R[0], \tag{5.11}$$

where

$$\Delta_R = \kappa(\gamma)\bar{f}_R{}^2 L^6[\tau(\gamma)]^L. \tag{5.12}$$

As $\gamma\to\infty$, $\kappa(\gamma)$ and $\tau(\gamma)$ each approach a constant value. [Explicit expressions for κ and τ will be given below as (6.16) and (6.15).]

It should be pointed out that the bound stated in the above lemma is a rather poor one if either L or $\bar{f}_R$ is large. However, it will be good enough to prove our result, as can be expected from the following reasoning. The contributions from large L will be suppressed by the cluster expansion as $\gamma\to\infty$. The contributions from large $\bar{f}_R$ will arise only from clusters in the vicinity of the Wilson loop. These contributions will modify the length dependent term in the answer (the self-energy term), but they will not be confused with an area law.

Given the above lemma, it is now possible to prove the following.

Lemma 5.2:

$$F[C] \geq \exp\left(-2\sum_R \Delta_R[\epsilon(\gamma)]^{L(R)}\right), \tag{5.13}$$

where R is to be summed over all connected clusters.

Proof. The connected clusters R_i can be divided into two classes, depending on whether Δ_{R_i} is less than or greater than one-half. Let V_i be a label which is summed over all connected clusters with $\Delta_{V_i}\leq\frac{1}{2}$, and let W_i be summed over all connected clusters with $\Delta_{W_i}>\frac{1}{2}$. The partition function of Eq. (5.9) can be expanded as

$$Z=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!\,m!}\sum_{\substack{V_1\cdots V_n\;W_1\cdots W_m\\ \text{disjoint}}}\left(\prod_{i=1}^{n}G_{V_i}[0]\right)\times\left(\prod_{i=1}^{m}G_{W_i}[0]\right).$$

In understanding the motivation of what follows, the reader should bear in mind that there are many clusters V_i and few clusters W_i. Thus, the contribution of the W_i can be bounded rather poorly without significantly weakening the result. The first step is to ignore the disjointness requirement on W_i:

$$Z\leq K_1\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{V_1\cdots V_n\\ \text{disjoint}}}\prod_{i=1}^{n}G_{V_i}[0],$$

where

$$K_1\equiv\exp\left(\sum_W G_W[0]\right).$$

Using (5.11) and the requirement $\Delta_{V_i}\leq\frac{1}{2}$,

$$Z\leq K_1\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{V_1\cdots V_n\\ \text{disjoint}}}\prod_{i=1}^{n}\{(1+2\Delta_{V_i})G_{V_i}[a]\}=K_1\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!\,m!}\sum_{\substack{V_1\cdots V_n\;V_1'\cdots V_m'\\ \text{disjoint}}}\left[\prod_{i=1}^{n}G_{V_i}[a]\right]\left[\prod_{i=1}^{n}2\Delta_{V_i'}G_{V_i'}[a]\right].$$

Now remove the disjointness requirement on the V_i'. Then

$$Z\leq K_1K_2\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{V_1\cdots V_n\\ \text{disjoint}}}\prod_{i=1}^{n}G_{V_i}[a],$$

where

$$K_2\equiv\exp\left(\sum_V 2\Delta_V G_V[a]\right).$$

The sum over $V_1\cdots V_n$ can now be extended to all connected clusters, so

$$Z\leq K_1K_2\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{R_1\cdots R_n\\ \text{disjoint}}}\prod_{i=1}^{n}G_{R_i}[a].$$

Comparing with (5.9), one sees

$$F[C]\geq 1/K_1K_2.$$

Noting that

$$K_1\leq\exp\left(\sum_W 2\Delta_W G_W[0]\right)$$

and recalling the bound (5.7) on $G_Q[a]$, the lemma is proven.

Given Lemma 5.2, it is now just a matter of counting clusters to obtain the final bound, which is the main result of this paper.

Theorem II. Let $\langle A[C]\rangle$ denote the expectation value of the Wilson loop operator on the finite lattice Λ_R^*. If g^2 is sufficiently small so that $\epsilon(\gamma)\tau(\gamma)<\frac{1}{196}$, then

$$\langle A[C]\rangle\geq\exp[-\tfrac{1}{2}(g^2+\delta g^2)\langle j,\Box^{-1}j\rangle], \tag{5.14}$$

where

$$\delta g^2=C_1\kappa(\gamma)\sum_{L=0}^{\infty}L^{10}[196\epsilon(\gamma)\tau(\gamma)]^L. \tag{5.15}$$

[Here C_1 is a constant, ϵ and γ were defined by (5.4) and (4.6), and κ and τ were introduced in

Lemma 5.1.]

The condition on g^2 in Theorem II corresponds numerically to

$$g^2 < 0.168 . \tag{5.16}$$

Proof of Theorem II.

Note that Lemmas 5.1 and 5.2 imply

$$F[C] \geq \exp[-2\kappa(\gamma)I],$$

where

$$I = \sum_R \bar{f}_R{}^2 [L(R)]^6 [\epsilon(\gamma)\tau(\gamma)]^{L(R)} .$$

For each cluster R there exists a plaquette $p(R)$ with the properties (i) $p(R)$ lies within a distance $\frac{1}{2}L$ of the cluster R, and (ii) $|f_{p(R)}| = \bar{f}_R$. [If this plaquette is not unique, one can choose one plaquette $p(R)$ which has these properties.] Then

$$I = \sum_{p_0} f_{p_0}{}^2 \alpha(p_0) ,$$

where

$$\alpha(p_0) = \sum_{\substack{R \\ p(R)=p_0}} [L(R)]^6 [\epsilon\tau]^{L(R)} .$$

Clearly,

$$\alpha(p_0) \leq \sum_{L=0}^{\infty} n(L) L^6 [\epsilon\tau]^L ,$$

where $n(L)$ is the maximal number of connected clusters of length L which lie within a distance $\frac{1}{2}L$ of a given plaquette p_0. To bound $n(L)$, note that the cluster must contain one bond within a distance $\frac{1}{2}L$ of p_0. This bond must therefore be contained within a cubic volume of L^4 sites, and so the number of such bonds is less than $4L^4$. Having specified the first bond, the number of possible connected clusters can be shown by standard techniques[16] to be bounded by $14^{2(L-1)}$. Thus,

$$n(L) \leq 4L^4 (14)^{2(L-1)} .$$

Next, note that

$$\sum_{p_0} f_{p_0}{}^2 = \langle \partial\Box^{-1}\bar{j}, \partial\Box^{-1}\bar{j} \rangle = \langle j, \Box^{-1} j \rangle ,$$

where one has used the fact that $\nabla \bar{j} = 0$. Combining these bounds with (3.10), (3.11), and (4.5), the theorem is proven.

VI. THE BOUND ON $G_R[a]$

The bound on $G_R[a]$ which was stated as Lemma 5.1 will be proven in this section. The first step is to examine the Fourier expansion of $\rho(\phi)$:

$$\rho(\phi) = \sum_{m=-\infty}^{\infty} r(m) e^{im\phi} , \tag{6.1}$$

where

$$r(m) = \frac{1}{W_{\min}} (e^{-\gamma m^2/2} - W_{\min}\,\delta_{m,0}) . \tag{6.2}$$

In terms of these Fourier coefficients, the expression (5.6) for $G_R[a]$ becomes

$$G_R[a] = \sum_{\substack{\{m_b\}=-\infty \\ b\in R \\ \partial m=0}}^{\infty} e^{i\langle m,a\rangle} \prod_{b\in R} r(m_b) . \tag{6.3}$$

To bound this expression, one must look at the properties of the functions m_b which are being summed over.

Lemma 6.1. Let R be a connected cluster of L bonds. It is always possible to choose a set of integer-valued 1-chains $\{M_b^{(i)}\}$, where $i = 1, \ldots, n$, which vanish for $b \notin R$ and which have the following properties:

(a) For each i,

$$\partial M^{(i)} = 0 . \tag{6.4}$$

(b) The most general integer 1-chain m_b which vanishes for $b \notin R$ and which satisfies $\partial m = 0$ can be written as

$$m_b = \sum_{i=1}^{n} k_i M_b^{(i)} , \tag{6.5}$$

where the $\{k_i\}$ are arbitrary integers.

(c) For each i there exists a special bond $b_0(i)$, with the property that

$$M_{b_0(i)}^{(j)} = \delta_{ij} . \tag{6.6}$$

Thus, $k_i = m_{b_0(i)}$.

Furthermore, it is possible to choose integer-valued 2-chains $\{S_p^{(i)}\}$, where $i = 1, \ldots, n$, which have the following properties:

(d) We have

$$\partial S_p^{(i)} = M_b^{(i)} . \tag{6.7}$$

(e) There exists a constant C_2 such that

$$\sum_i \sum_p |S_p^i| \leq C_2 L^3 . \tag{6.8}$$

(f) Any plaquette for which $S_p^i \neq 0$ for some i lies within a distance $\frac{1}{2}L$ of the cluster R. (The definition of distance was given in Lemma 5.1.)

The proof of Lemma 6.1 will be given in Appendix B. Using this lemma and the definition of a_b given by (3.15) and (3.13), $G_R[a]$ can be rewritten as

$$G_R[a] = \sum_{\{k_i\}=-\infty}^{\infty} \mu[k] \cos\left(\sum_{i=1}^{n} k_i \Phi_i\right) , \tag{6.9}$$

where

$$\mu[k]=\prod_{b\in R} r\left(\sum_{i=1}^{n} k_i M_b^{(i)}\right) \qquad (6.10)$$

and

$$\Phi_i = 2\pi\langle S^{(i)}, f\rangle\,, \qquad (6.11)$$

where f_p is the electromagnetic field strength defined by (5.10). Using the fact that $\cos\theta \geq 1 - \frac{1}{2}\theta^2$ for all θ, one has

$$G_R[a] \geq \sum_{\{k_i\}=-\infty}^{\infty} \mu[k]\left[1-\frac{1}{2}\left(\sum_{i=1}^{n} k_i\Phi_i\right)^2\right]. \qquad (6.12)$$

This expression can be bounded by the use of the following lemma.

Lemma 6.2. There exist functions $\sigma(\gamma)$ and $\tau(\gamma)$ such that

$$\frac{\sum_{\{k_i\}=-\infty}^{\infty} \mu[k]|k_j k_l|}{\sum_{\{k_i\}=-\infty}^{\infty} \mu[k]} \leq \sigma(\gamma)[\tau(\gamma)]^L. \qquad (6.13)$$

As $\gamma\to\infty$, $\sigma(\gamma)$ and $\tau(\gamma)$ each approach constants.

Proof. Consider first $\langle k_j^2\rangle$ for some fixed value of j, and let $b_0=b_0(j)$ as defined in Lemma 6.1. Then

$$\langle k_j^2\rangle \equiv \sum_{\{k_i\}=-\infty}^{\infty} \mu[k]k_j^2 \Big/ \sum_{\{k_i\}=-\infty}^{\infty} \mu[k]$$

$$= \sum_{\substack{\{m_b\}=-\infty \\ b\in R \\ \partial m=0}}^{\infty} m_{b_0}^2 \prod_{b\in R} r(m_b) \Big/ \sum_{\substack{\{m_b\}=-\infty \\ b\in R \\ \partial m=0}}^{\infty} \prod_{b\in R} r(m_b)\,.$$

$\langle k_j^2\rangle$ is bounded from above by deleting the restriction $\partial m=0$ from the numerator and restricting the denominator to the contribution from $m_b\equiv 0$. Then

$$\langle k_j^2\rangle \leq \sigma(\gamma)[\tau(\gamma)]^L\,,$$

where

$$\sigma(\gamma)=\sum_{m=-\infty}^{\infty} m^2 r(m) \Big/ \sum_{m=-\infty}^{\infty} r(m) \qquad (6.14)$$

and

$$\tau(\gamma)=\frac{1}{r(0)}\sum_{m=-\infty}^{\infty}{}' r(m)\,. \qquad (6.15)$$

The limit of $\gamma\to\infty$ can be extracted by using (5.1a) to yield

$$W_{\min}=1-2e^{-\gamma/2}+O(e^{-2\gamma})\,,$$

and then using (6.2) for $r(m)$. One finds $\sigma(\gamma)\to\frac{1}{2}$, and $\tau(\gamma)\to 2$. The bound is extended to $\langle|k_j k_l|\rangle$ by a simple Schwarz inequaltiy.

Using the lemma above with (6.12), one has

$$\frac{G_R[a]}{G_R[0]} \geq 1-\frac{1}{2}\left[\sum_{i=1}^{n}|\Phi_i|\right]^2 \sigma(\gamma)[\tau(\gamma)]^L\,,$$

where Φ_i was defined by (6.11). Using Lemma 6.1(f) and the definition of $\bar{f}_R$ given in Lemma 5.1, one has

$$|\Phi_i| \leq 2\pi\bar{f}_R \sum_p |S_p^{(i)}|\,.$$

The proof of Lemma 5.1 is completed by using Lemma 6.1(e). One has

$$\kappa(\gamma)=2\pi^2 C_2^2\sigma(\gamma)\,. \qquad (6.16)$$

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APPENDIX A: HOMOLOGY PROPERTIES

In this appendix, I will give an elementary proof of the homology properties of the lattices Λ_R and Λ_R^* which were stated in Sec. II. In addition, the proof of Lemma 4.2 will appear at the end of this appendix.

The first step is to give an explicit construction of the lattice Λ^∞. The sites can be labeled by the integer-valued D-vectors $\underline{n}\equiv(n_1,\ldots,n_D)$, with $-\infty<n_\mu<\infty$ for each μ. Now introduce D unit vectors $\hat{\mu}$, with $\hat{\mu}_\nu\equiv\delta_{\mu\nu}$. Bonds can then be labeled by pairs $(\underline{n};\mu)$, denoting the bond which contains $\underline{n}$ positively and $\underline{n}+\hat{\mu}$ negatively. The general r-cell can be labeled by $(\underline{n};\,\mu_1,\ldots,\mu_r)$, where $\mu_1<\mu_2<\cdots<\mu_r$. The incidence function is given by

$$I[(\underline{n}_a;\mu_1,\ldots,\mu_r),(\underline{n}_b;\nu_1,\ldots,\nu_{r+1})]=\sum_{i=1}^{r+1}(-1)^{r+1-i}\left(\delta_{\underline{n}_a,\underline{n}_b}-\delta_{\underline{n}_a,\underline{n}_b+\hat{\nu}_i}\right)\delta_{\mu_1}^{\nu_1}\cdots\delta_{\mu_{i-1}}^{\nu_{i-1}}\delta_{\mu_i}^{\nu_{i+1}}\cdots\delta_{\mu_r}^{\nu_{r+1}}. \qquad (A1)$$

[Oppositely oriented r-cells may be denoted by $-(\underline{n};\mu_1,\ldots,\mu_r)$, and the incidence function can easily be generalized to allow these new arguments. However, it is simpler to construct the formalism so that each r-cell has a fixed standard orientation corresponding to the incidence function (A1).] It is straightforward to verify that the incidence function obeys Eq. (2.1).

The finite sublattice Λ_R is obtained by restricting the sites to the region

$$0 < n_\mu \leq L_\mu \quad \text{for each } \mu\,, \tag{A2}$$

where $\{L_\mu\}$ is a set of positive integers. The criterion for r-cells becomes

$$(\underline{n};\mu_1,\ldots,\mu_r)\in\Lambda_R \Leftrightarrow \begin{cases} 0<n_\nu\leq L_\nu-1 & \text{if } \nu=\mu_i \\ 0<n_\nu\leq L_\nu & \text{otherwise}\,. \end{cases} \tag{A3}$$

An r-chain can then be denoted by

$$f(\underline{n};\mu_1,\ldots,\mu_r)\equiv f_{\mu_1\cdots\mu_r}(\underline{n})\,. \tag{A4}$$

Using the conventions above, these tensors are defined only for $\mu_1<\cdots<\mu_r$. However, it is convenient to extend the definition to all r-tuples $(\mu_1,\ldots,\mu_r)$ by defining the tensor to be antisymmetric.

The boundary and coboundary operators can then be worked out explicitly. The only complication is to correctly treat the boundaries of Λ_R. For this purpose it is useful to extend the definition of $f_{\mu_1\cdots\mu_r}(\underline{n})$ to Λ^∞ by requiring it to vanish outside Λ_R. Then if $g_{r-1}=\partial f_r$,

$$g_{\mu_1\cdots\mu_{r-1}}(\underline{n})=-\nabla_{-\lambda}f_{\mu_1\cdots\mu_{r-1}\lambda}(\underline{n})\,, \tag{A5}$$

where repeated indices are summed and

$$\nabla_{-\lambda}f(\underline{n})\equiv f(\underline{n}-\hat{\lambda})-f(\underline{n})\,. \tag{A6}$$

If $h_{r+1}=\nabla f_r$, then

$$h_{\mu_1\cdots\mu_{r+1}}(\underline{n})=\sum_{i=1}^{r+1}(-1)^{r-i}\nabla_{\mu_i}f_{\mu_1\cdots[\mu_i]\cdots\mu_{r+1}}(\underline{n})\,, \tag{A7}$$

where the square brackets denote an omitted index and

$$\nabla_\mu f(\underline{n})\equiv f(\underline{n}+\hat{\mu})-f(\underline{n})\,. \tag{A8}$$

It is worth noting that expression (A5) for the boundary chain g will vanish identically outside Λ_R, while expression (A7) for the coboundary chain h will take on spurious values on cells one unit outside Λ_R.

I will first prove property (i) for the lattice Λ_R: for $r\neq 0$, if $\partial f_r=0$, then there exists a g_{r+1} such that $f_r=\partial g_{r+1}$. g_{r+1} will be constructed explicitly, using induction on the dimension D of the lattice.

First consider the case $D=r\geq 1$. Then

$$f_{\mu_1\cdots\mu_D}(\underline{n})=\epsilon_{\mu_1\cdots\mu_D}\tilde{f}(\underline{n})\,, \tag{A9}$$

where $\tilde{f}(\underline{n})$ will vanish unless $0<n_\mu\leq L_\mu-1$ for each μ. Then $\partial f_r=0$ implies

$$\sum_\lambda \epsilon_{\mu_1\cdots\mu_{D-1}\lambda}[\tilde{f}(\underline{n})-\tilde{f}(\underline{n}-\hat{\lambda})]=0 \tag{A10}$$

for all $\underline{n}$ and for all $\mu_1\cdots\mu_{D-1}$. Thus

$$\tilde{f}(\underline{n})-\tilde{f}(\underline{n}-\hat{\lambda})=0$$

for all $\underline{n}$ and for all λ, and with the boundary conditions this implies $\tilde{f}(\underline{n})=0$.

Now suppose that property (i) has been shown for dimension $D-1$. To show that it holds for dimension D, consider the construction

$$h_{\mu_1\cdots\mu_{r+1}}(\underline{n})=\theta(n_D)\tilde{h}_{\mu_1\cdots\mu_{r+1}}(\underline{n})\,, \tag{A11}$$

where

$$\tilde{h}_{\mu_1\cdots\mu_{r+1}}(\underline{n})=\sum_{i=1}^{r+1}(-1)^{r-i}\delta_{\mu_i,D}\times\sum_{k=1}^{\infty}f_{\mu_1\cdots[\mu_i]\cdots\mu_{r+1}}(\underline{n}+k\hat{D}) \tag{A12}$$

and

$$\theta(n_D)\equiv\begin{cases}1 & \text{if } n_D>0\,,\\ 0 & \text{otherwise}\,.\end{cases} \tag{A13}$$

$\tilde{h}$ is to be considered an $(r+1)$-chain on Λ^∞, while h has been constructed so that it vanishes identically outside Λ_R. An illustration of such a construction is shown in Fig. 2. Using $\partial f=0$, one finds

$$-\nabla_{-\lambda}\tilde{h}_{\mu_1\cdots\mu_r\lambda}(\underline{n})=f_{\mu_1\cdots\mu_r}(\underline{n})\,. \tag{A14}$$

Then

$$-\nabla_{-\lambda}h_{\mu_1\cdots\mu_r\lambda}(\underline{n})=f_{\mu_1\cdots\mu_r}(\underline{n})+\tilde{f}_{\mu_1\cdots\mu_r}(\underline{n})\,, \tag{A15}$$

where

$$\tilde{f}_{\mu_1\cdots\mu_r}(\underline{n})=\delta_{n_D,1}\tilde{h}_{\mu_1\cdots\mu_r D}(\underline{n}-\hat{D})\,. \tag{A16}$$

Note that $\tilde{f}_{\mu_1\cdots\mu_r}(\underline{n})$ vanishes if $n_D\neq 1$ or if any $\mu_i=D$. Thus, $\tilde{f}$ is an r-chain in dimension $D-1$. Furthermore,

$$\partial\tilde{f}=\partial(\partial h-f)=0\,. \tag{A17}$$

Thus, by the induction hypothesis there exists a $\tilde{g}_{r+1}$ with $\tilde{f}_r=\partial\tilde{g}_{r+1}$, and then

$$g_{r+1}=h_{r+1}-\tilde{g}_{r+1} \tag{A18}$$

obeys the desired relation $f=\partial g$.

Using standard homology and cohomology theory,[18] it is possible to use property (i) to prove properties (ii) and (iii). However, for the benefit of the reader who may not be familiar with this theory, I will complete the proof in an elementary way.

The next step is to prove property (ii) for the lattice Λ_R: For $r\neq 0$, if $\nabla f_r=0$, then there exists a g_{r-1} such that $f_r=\nabla g_{r-1}$. The proof will again use induction on D. Consider the construction

$$h_{\mu_1\cdots\mu_{r-1}}(\underline{n})=\sum_{k=0}^{L_D-n_D-1}f_{\mu_1\cdots\mu_{r-1}D}(\underline{n}+k\hat{D})\,. \tag{A19}$$

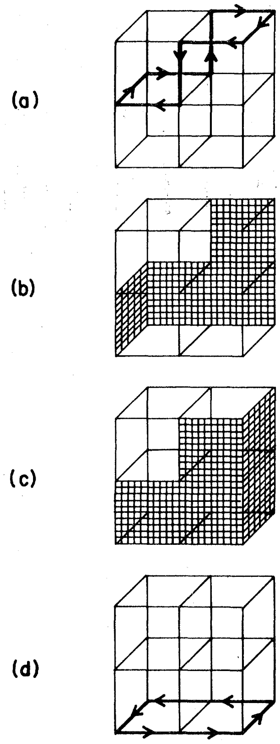

FIG. 2. An illustration of the homology construction. Here $D=3$ and $r=1$, and the 3-axis is vertical. In (a), darkened bonds with arrows indicate unit values of $f_\mu(\underline{n})$, with $f_\mu(\underline{n})=0$ elsewhere. Unit values of $h_{\mu\nu}(\underline{n})$ are indicated by shaded plaquettes in (b) or (c). In (d), darkened bonds illustrate $\tilde{f}=\partial h-f$.

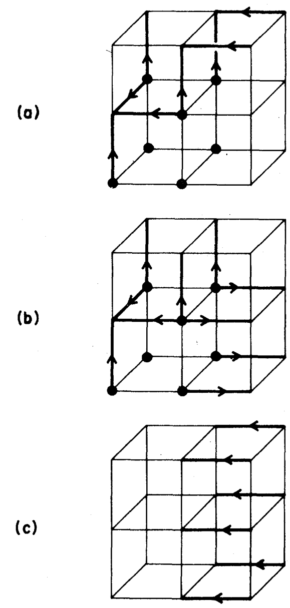

FIG. 3. An illustration of the cohomology construction. Here $D=3$ and $r=1$. In (a), darkened bonds with arrows indicate unit values of $f_\mu(\underline{n})$, with $f_\mu(\underline{n})=0$ elsewhere. Unit values of $h(n)$ are indicated by darkened sites. In (b), the 1-chain ∇h is indicated by darkened bonds and (c) illustrates the 1-chain $\tilde{f}=f-\nabla h$.

An illustration of this construction is shown in Fig. 3. Explicit calculation then gives

$$\nabla h = f - \tilde{f}, \tag{A20}$$

where

$$\tilde{f}_{\mu_1\cdots\mu_r}(\underline{n}) = f_{\mu_1\cdots\mu_r}(\underline{n}_\perp, n_D = L_D). \tag{A21}$$

[Here $\underline{n}_\perp$ denotes the $(D-1)$-vector consisting of the first $D-1$ components of $\underline{n}$.] As in the previous case, it follows immediately that $\nabla\tilde{f}=0$. Note that $\tilde{f}_{\mu_1\cdots\mu_r}$ vanishes if any $\mu_i=D$, and that it depends only on $\underline{n}_\perp$. Thus, $\tilde{f}$ is an r-chain in dimension $D-1$. If $D=r$, f vanishes and the problem is solved. For $D>r$, the proof is completed by induction.

Properties (i) and (ii) for the lattice Λ_R^* are then established using the duality properties expressed by Eqs. (2.7) and (2.8).

To prove property (iii), suppose $\Box f=0$ for some real-valued r-chain f_r. The goal is to show that $f=0$. Note that

$$0 = \langle f, \Box f\rangle = \|\partial f\|^2 + \|\nabla f\|^2, \tag{A22}$$

so $\partial f=0$ and $\nabla f=0$. For the values of r for which (i) applies, $\nabla f=0$ implies $f=\partial g$ from some g. Then

$$\|f\|^2 = \langle f, \partial g\rangle = \langle \nabla f, g\rangle = 0, \tag{A23}$$

so the result is proven.

Finally, we turn to the proof of Lemma 4.2, which for arbitrary D becomes

$$\langle f, \Box f\rangle \le 4D\|f\|^2. \tag{A24}$$

On Λ^∞, it is easily shown that

$$\Box f_{\mu_1\cdots\mu_r}(\underline{n}) = \nabla_{-\lambda}\nabla_\lambda f_{\mu_1\cdots\mu_r}(\underline{n}). \tag{A25}$$

To compute $\Box f$ on Λ_R, it is convenient to extend f to Λ^∞ as before. The only complication is that ∇f can then take on nonvanishing values outside Λ_R. Thus,

$$\Box_{\Lambda_R} f = (\nabla\partial + \partial\nabla)_{\Lambda_R} f = \Box_{\Lambda^\infty} f - g, \tag{A26}$$

where $g=\partial h$ and

$$h = \nabla f\,|_{\overline{\Lambda}_R}.$$

(The notation above means that h equals ∇f restricted to those r-cells contained in the complement of Λ_R.) By examining the incidence function (A1) and the Λ_R boundary conditions (A3), one notes that if $c_{r+1}\in\overline{\Lambda}_R$, then there is at most one cell $c_r\in\Lambda_R$ with $I(c_r, c_{r+1})\ne 0$. Thus, for $c_r\in\Lambda_R$,

$$g(c_r)=\sum_{c_{r+1}\in\bar{\Lambda}_R} I(c_r,c_{r+1})\sum_{c_r'} I(c_r',c_{r+1})f(c_r')$$

$$=f(c_r)\sum_{c_{r+1}\in\bar{\Lambda}_R}[I(c_r,c_{r+1})]^2 . \quad \text{(A27)}$$

Then clearly $\langle f,g\rangle \geq 0$, so

$$\langle f,\Box_{\Lambda_R} f\rangle \leq \langle f,\Box_{\Lambda^\infty} f\rangle = \sum_{\underline{n},\lambda} f(\underline{n})[2f(\underline{n})-f(\underline{n}+\hat{\lambda}) - f(\underline{n}-\hat{\lambda})],$$

where the indices $\mu_1\cdots\mu_r$ on f have been suppressed. The Schwarz inequality implies

$$\left|\sum_{\underline{n}} f(\underline{n})f(\underline{n}\pm\hat{\lambda})\right| \leq \|f\|^2,$$

and then (A24) follows immediately.

APPENDIX B: PROOF OF LEMMA 6.1

This appendix will contain the proof of Lemma 6.1. I will first prove parts (a)–(c).

Several definitions will be useful. A 1-cycle is defined to be an integer-valued 1-chain m_b which satisfies $\partial m=0$. A 1-cycle on R has the additional property that $m_b=0$ if $b\notin R$. A closed path is defined as a sequence of oriented bonds with the following properties:

(i) Each successive bond begins at the site where the previous bond ended.

(ii) The final bond in the sequence ends at the site where the first bond began.

Two sequences which differ only by a cyclic permutation are considered equivalent, defining the same closed path. Given any closed path, a corresponding 1-cycle is defined by setting m_b equal to the number of times the bond b occurs in the closed path, each counted with the appropriate sign.

To prove parts (a)–(c), one proceeds by induction on L. Given any cluster R containing L bonds, one can always find a bond b_1 such that R is the union of b_1 and a connected cluster R' of $L-1$ bonds. Assume that n functions $\{M_b^{(i)}\}$ have been chosen which satisfy properties (a)–(c) for the cluster R'. There are now two cases to consider. (i) Suppose b_1 is connected to R' through only one of its sites. Then any 1-cycle on R must have $m_{b_1}=0$, and thus the functions $\{M_b^{(i)}\}$ will satisfy (a)–(c) also for the cluster R.[19] (ii) Suppose b_1 is connected to R' through both of its sites. Since R' is connected, there must be a sequence of bonds through R' which joins the two end points of b_1. With b_1, this sequence forms a closed path. Let μ_b denote the corresponding 1-cycle, so $\mu_{b_1}=1$. Then let

$$M_b^{(n+1)}=\mu_b-\sum_{i=1}^{n}\mu_{b_0(i)}M_b^{(i)} \quad \text{(B1)}$$

and let $b_0(n+1)=b_1$. It is then easily verified that this enlarged set of functions satisfies (a)–(c) for the cluster R.

I will now go on to prove parts (d)–(f). Again one proceeds by induction, assuming that the Lemma has already been established for the cluster R'. Case (i) will again be trivial, and for case (ii) one must analyze the properties of $M_b\equiv M_b^{(n+1)}$.

Lemma B1. There exists a closed path P for which the corresponding 1-cycle is the function $M_b^{(n+1)}$ defined by (B1). The length of P is less than or equal to the number L of bonds in the cluster.

Proof. This construction is very similar to the famous Konigsberg bridge problem of Euler,[17] so I will translate it into that language. Each site is to be considered an island, and each bond for which $M_b\neq 0$ is to be considered a set of $|M_b|$ bridges. On each bridge is a one-way sign which corresponds to the sign of M_b. Starting on the bridge b_1, one chooses a closed path P which crosses each bridge no more than once and only in the legal direction. The condition $\partial M=0$ guarantees that each island has the same number of bridges leaving it as entering it, so such a path can always be continued until it returns to b_1. To show that P crosses each bridge, let $\tilde{M}$ denote the 1-cycle which corresponds to P. Then $M_b'\equiv\tilde{M}_b-M_b$ is a 1-cycle on R', and $M'_{b_0(i)}=0$ for $i=1,\ldots,n$. By the induction hypothesis, $M_b'=0$. Finally, the bound on the length of P is obtained by showing that no bond is crossed more than once. This statement can be proven by contradiction. If P crosses the bond b_2 twice, then one can form the subsequence $\tilde{P}$ which starts and ends at b_2 and which does not include the bond b_1. P is a closed path, and there is a corresponding 1-cycle $\tilde{M}$ on R', with $\tilde{M}_{b_0(i)}=0$ for $i=1,\ldots n$. This, however, contradicts the induction hypotheses.

Lemma B2. Given a closed path P of length L with a corresponding 1-cycle M_b, there exists an integer-valued 2-chain S_p with the following properties:

(a) $\partial S_p=M_b$.

(b) $\sum_p|S_p|\leq\frac{1}{8}[(D-1)/D]L^2$, where D is the dimension of the lattice. ($D=4$ in our case.)

(c) Any plaquette for which $S_p\neq 0$ lies within a distance $\frac{1}{2}L$ of P.

Proof. This lemma is proven by induction on L. Given that P returns to its starting point, one knows that the number of bonds in the μ direction must equal the number in the $-\mu$ direction

$(\mu = 1, \ldots, D)$. Call this number l_μ, so

$$L = 2 \sum_{\mu=1}^{D} l_\mu .$$

For purposes of the proof, replace property (b) by the property (b′):

$$(\mathrm{b}') \ \sum_p |S_p| \leq \sum_{\mu<\nu} l_\mu l_\nu .$$

It will be shown later that (b′) implies (b).

Given P, choose a direction μ with $l_\mu \neq 0$. Now let B_α and B_β denote two oriented bonds in P, one in the μ direction and one in the $-\mu$ direction (in either order). Let $k(B_\alpha, B_\beta)$ denote the number of bonds which occur after B_α and before B_β in the cyclic sequence P. Of all possible pairs, choose B_α and B_β to minimize $k(B_\alpha, B_\beta)$. Cyclically permuting, the sequence P can be written

$$P = (B_\alpha, B_1, \ldots, B_k, B_\beta, B_{k+1}, \ldots, B_{L-2}),$$

where the oriented bonds $B_1, \ldots, B_k$ are all perpendicular to the μ direction. Note that

$$k \leq \sum_{\nu \neq \mu} l_\nu .$$

A three-dimensional example of such a closed path is shown in Fig. 4(a). Now let $\overline{B}_1, \ldots, \overline{B}_k$ denote the bonds which are obtained by translating $B_1, \ldots, B_k$ by one unit in the direction of B_β. Then note that one can form a closed path $\tilde{P}$ defined by

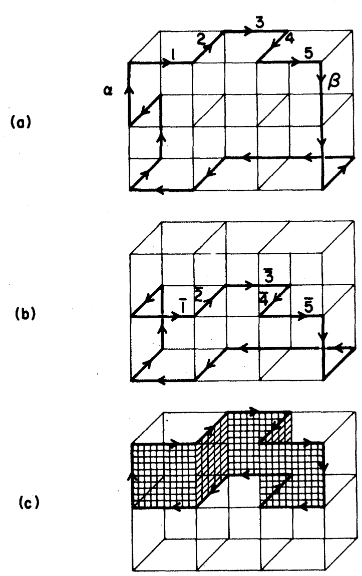

FIG. 4. An illustration of the construction used in the proof of Lemma B2. (a) shows the closed path P, with the bonds B_i labeled by i. The closed path $\tilde{P}$ is shown in (b). The path P' and the 2-chain S' are shown in (c).

$$\tilde{P} = (\overline{B}_1, \ldots, \overline{B}_k, B_{k+1}, \ldots, B_{L-2}),$$

characterized by

$$\tilde{l}_\nu = \begin{cases} l_\mu - 1 & \text{if } \nu = \mu \\ l_\nu & \text{if } \nu \neq \mu . \end{cases}$$

The path $\tilde{P}$ is shown in Fig. 4(b). If $\tilde{M}$ is the corresponding 1-cycle, by the induction hypothesis there is an $\tilde{S}_p$ which satisfies (a), (b′), and (c). Now note that $M' \equiv M - \tilde{M}$ corresponds to the closed path

$$P' = (B_\alpha, B_1, \ldots, B_k, B_\beta, -\overline{B}_k, \ldots, -\overline{B}_1),$$

where the minus sign denotes the opposite orientation. One can then define the 2-chain

$$S'_p \equiv \begin{cases} 1 & \text{if } B_i \in p \text{ and } -\overline{B}_i \in p \text{ for any } i = 1, \ldots, k \\ -1 & \text{if } -B_i \in p \text{ and } \overline{B}_i \in p \text{ for any } i = 1, \ldots, k \\ 0 & \text{otherwise.} \end{cases}$$

The path P' and the surface S' are shown in Fig. 4(c). One then has

$$M' = \partial S'$$

and

$$\sum_p |S'_p| = k .$$

Then let

$$S_p = \tilde{S}_p + S'$$

and property (a) is easily verified. To verify (b′),

$$\sum_p |S_p| \leq \sum_p |\tilde{S}_p| + k$$

$$\leq \sum_{\substack{\nu<\lambda \\ \nu\neq\mu \\ \lambda\neq\mu}} l_\nu l_\lambda + (l_\mu - 1) \sum_{\nu\neq\mu} l_\nu + \sum_{\nu\neq\mu} l_\nu$$

$$= \sum_{\nu<\lambda} l_\nu l_\lambda .$$

To verify (c), note that any plaquette for which $\tilde{S}_p \neq 0$ lies within a distance $\frac{1}{2}(L-2)$ of $\tilde{P}$. It therefore lies within a distance $\frac{1}{2}L$ of P. Furthermore, any plaquette for which $S'_p \neq 0$ lies within a unit distance of P. Thus, (a), (b′), and (c) are established.

To show that (b′) implies (b), define

$$\delta_\mu = l_\mu - L/2D .$$

Then, after some simple algebra,

$$\sum_{\mu<\nu} l_\mu l_\nu = \frac{1}{8} \frac{D-1}{D} L^2 - \tfrac{1}{2} \sum_\mu \delta_\mu{}^2 .$$

It should be mentioned that the bounds (b) and (b′) are both strong bounds in the sense that one can construct closed paths which saturate them.[20]

The bound (c), on the other hand, is probably quite weak.

Lemmas 6.1(d) and 6.1(f) are now proven. To complete the inductive proof of 6.1(e), let

$$C_2=\tfrac{1}{8}(D-1)/D=\tfrac{3}{32}.$$

Then

$$\sum_i\sum_p |S_p^i| \le C_2[(L-1)^3+L^2]$$
$$=C_2[L^3-2(L-1)^2-(L-1)] \le C_2L^3 .$$

APPENDIX C: THE INFINITE-VOLUME LIMIT

In this appendix, I will show that the infinite-volume limit of $\langle A[C]\rangle$ exists. More precisely, the following two theorems will be proven.

Theorem III. Let Λ_i, $i=1,2,\ldots,\infty$, denote an infinite sequence of finite closed sublattices of Λ^∞, with $\Lambda_i\subset\Lambda_{i+1}$. Further, assume that the sequence approaches infinite volume in the sense that any finite lattice Λ is contained in Λ_i for some i. Then let $\langle A[C]\rangle_i$ denote the expectation value of the Wilson loop C on Λ_i. Then $\langle A[C]\rangle_i$ is monotonically nondecreasing in i and approaches a finite limit as $i\to\infty$. This limit is the same for all sequences Λ_i with the above properties.

Theorem IV. Let Λ_i, $i=1,2,\ldots,\infty$, denote an infinite sequence of finite open sublattices of Λ^∞, with $\Lambda_i\subset\Lambda_{i+1}$. Further, assume that each $\overline{\Lambda}_i$ (the complement of Λ_i) has a trivial 1-cohomology (i.e., given any 1-chain f_b with finite support, $\nabla f=0$ implies $f=\nabla g$ for some g). Again assume that the sequence approaches infinite volume. Then $\langle A[C]\rangle_i$ is monotonically nonincreasing in i and approaches a finite limit as $i\to\infty$. This limit is the same for all sequences Λ_i with the above properties.

Before proving these theorems, several remarks are in order. (i) One would expect that the limits of Theorems III and IV would be equal, but this conjecture has not been proven. (ii) The cohomology condition of Theorem IV is satisfied, for example, by the lattice Λ_R^*. Thus, the lower bound of Theorem II is obeyed by the infinite-volume limit in Theorem IV. (iii) For the case of the Wilson action and closed boundary conditions, the infinite-volume limit has been established by Osterwalder and Seiler.[2]

The proof of Osterwalder and Seiler made use of an inequality of the Griffiths-Kelley-Sherman[14] type which was proven by Ginibre. The proof here is based on an extension of this inequality to the Villain form, using the methods of Elitzur, Pearson, and Shigemitsu.[10] The proof is similar to that of Lemma 4.4.

Lemma C1. Let $\langle A[C;g_p]\rangle$ denote the expectation value of the Wilson-loop operator in a theory which contains an arbitrary coupling constant g_p for each plaquette of the lattice Λ:

$$\langle A[C;g_p]\rangle \equiv \frac{1}{Z}\int_0^{2\pi}\{d\theta_b\}\sum_{\{l_p\}=-\infty}^{\infty}\exp\left[-\sum_p\frac{1}{2g_p{}^2}(\nabla\theta-2\pi l)^2+i\langle j,\theta\rangle\right]. \tag{C1}$$

Then $\langle A[C;g_r]\rangle$ is positive and is a monotonically nonincreasing function of each $g_p{}^2$.

Proof. Begin by rewriting (C1) using the form (2.12b) of the Villain action:

$$\langle A[C;g_p]\rangle=\frac{1}{Z}\int_0^{2\pi}\{d\theta_b\}\sum_{\{n_p\}=-\infty}^{\infty}\exp\left(-\tfrac{1}{2}\sum_p g_p{}^2n_p{}^2+i\langle n,\nabla\theta\rangle+i\langle j,\theta\rangle\right).$$

Using $\langle n,\nabla\theta\rangle=\langle\partial n,\theta\rangle$, one has

$$\langle A[C;g_p]\rangle=\frac{1}{Z}\sum_{\substack{\{n_p\}=-\infty\\ \partial n=j}}^{\infty}\exp\left(-\frac{1}{2}\sum_p g_p{}^2n_p{}^2\right). \tag{C2}$$

It follows immediately that $\langle A[C;g_p]\rangle\ge 0$. Differentiating,

$$\frac{\partial\langle A[C;g_p]\rangle}{\partial g_{p_0}{}^2}=-\frac{1}{2Z^2}\sum_{\substack{\{n_p\}=-\infty\\ \partial n=j}}^{\infty}\sum_{\substack{\{n_p'\}=-\infty\\ \partial n'=0}}^{\infty}(n_{p_0}{}^2-n'_{p_0}{}^2)\exp\left(-\tfrac{1}{2}\sum_p g_p{}^2(n_p{}^2+n'_p{}^2)\right).$$

[In the above equation and for the rest of this proof, the symbol Z maintains the value it had in (C2).] Now set

$$\mu_p\equiv n_p+n_p',\quad \mu_p'\equiv n_p-n_p'. \tag{C3}$$

The sums over n_p and n_p' can then be replaced by sums over μ_p and μ_p'. To account for the constraint that μ_p and μ_p' must be both even or both odd, one introduces a parity function π_p as in the proof of Lemma 4.1. Then

$$\frac{\partial\langle A[C;g_p]\rangle}{\partial g_{p_0}{}^2} = -\frac{1}{2Z^2}\sum_{\{\pi_p\}}\sum_{\substack{\{\mu_p\}=-\infty\\ \partial\mu=j\\ \pi(\mu_p)=\pi_p}}^{\infty}\sum_{\substack{\{\mu_p'\}=-\infty\\ \partial\mu'=j\\ \pi(\mu_p')=\pi_p}}^{\infty}(\mu_{p_0}\mu_{p_0}')\exp\left(-\tfrac{1}{4}\sum_p g_p{}^2(\mu_p{}^2+\mu_p'{}^2)\right).$$

Rewriting,

$$\frac{\partial\langle A[C;g_p]\rangle}{\partial g_{p_0}{}^2} = -\frac{1}{2Z^2}\sum_{\{\pi_p\}}\{V_{p_0}[\pi]\}^2, \tag{C4}$$

where

$$V_{p_0}[\pi] = \sum_{\substack{\{\mu_p\}=-\infty\\ \partial\mu=j\\ \pi(\mu_p)=\pi_p}}^{\infty}\mu_{p_0}\exp\left(-\frac{1}{4}\sum_p g_p{}^2\mu_p{}^2\right). \tag{C5}$$

The monotonicity follows from (C4).

Proof of Theorem III. Note that if $\Lambda_i \subset \Lambda_{i+1}$ is closed, then

$$\langle A[C;g_p]\rangle_{\Lambda_i} = \langle A[C;g_p']\rangle_{\Lambda_{i+1}}, \tag{C6a}$$

where

$$g_p' = \begin{cases} g_p, & \text{if } p\in\Lambda_i \\ \infty, & \text{otherwise}. \end{cases} \tag{C6b}$$

The monotonicity of $\langle A[C]\rangle_i$ then follows from Lemma (C1). Since $\langle A[C]\rangle_i$ is bounded from above by one, the limit $i\to\infty$ must exist. Finally, let $\Lambda_i^{(1)}$ and $\Lambda_i^{(2)}$ denote two sequences with the required properties. Let $A_i^{(k)} \equiv \langle A[C]\rangle_i^{(k)}$, $(k=1,2)$, and let $A_\infty^{(k)} = \lim_{i\to\infty}A_i^{(k)}$. For any value of i,

$$A_\infty^{(1)} \geq A_i^{(1)}. \tag{C7a}$$

For any $\epsilon>0$, there exists a value of j such that

$$-A_\infty^{(2)} \geq -A_j^{(2)} - \epsilon. \tag{C7b}$$

Since the sequence $\Lambda_i^{(1)}$ approaches infinite volume, one can then choose i sufficiently large so that $\Lambda_i^{(1)} \supset \Lambda_j^{(2)}$, so

$$A_i^{(1)} \geq A_j^{(2)}. \tag{C7c}$$

Combining (C7a)–(C7c), one has

$$A_\infty^{(1)} - A_\infty^{(2)} \geq -\epsilon. \tag{C8}$$

For (C8) to hold for all ϵ, $A_\infty^{(1)} - A_\infty^{(2)} \geq 0$. The argument can then be reversed, showing that $A_\infty^{(1)} = A_\infty^{(2)}$.

Proof of Theorem IV. The proof is of course nearly identical to that of Theorem III. One must show that

$$\langle A[C;g_p]\rangle_{\Lambda_i} = \langle A[C;g_p']\rangle_{\Lambda_{i+1}}, \tag{C9a}$$

where

$$g_p' = \begin{cases} g_p, & \text{if } p\in\Lambda_i, \\ 0, & \text{otherwise}. \end{cases} \tag{C9b}$$

Equations (C9), however, are somewhat less obvious than their counterparts (C6). To verify (C9), look at the right-hand side of (C9a):

$$\langle A[C;g_p']\rangle_{\Lambda_{i+1}} = \frac{1}{Z}\int_0^{2\pi}\{d\theta_b\}e^{i\langle j,\theta\rangle}\prod_{p\notin\Lambda_i}\{\delta[(\nabla\theta)_p]\}\prod_{p\in\Lambda_i}e^{-S_V[(\nabla\theta)_p,g_p]}, \tag{C10}$$

where the δ function is defined on the argument modulo 2π and S_V denotes the Villain action of (2.12). Now let $\overline{\Lambda}_i$ denote the complement of Λ_i in Λ^∞ and let Λ_δ be the intersection of $\overline{\Lambda}_i$ with Λ_{i+1}. The δ-function constraint requires that $(\nabla\theta)|_{\Lambda_\delta}=0$. (The symbols $f|_\Lambda$ denote the chain f restricted to the sublattice Λ.) Λ_δ is a closed sublattice of Λ_{i+1}, so $(\nabla\theta)|_{\Lambda_\delta} = \nabla\theta^{(\delta)}$, where $\theta^{(\delta)} \equiv \theta|_{\Lambda_\delta}$. (When the coboundary operator acts on a restricted 1-chain, it is defined to be the coboundary operator on the restricted sublattice.) Letting $\theta^{(i)} \equiv \theta|_{\Lambda_i}$, (C10) can be rewritten

$$\langle A[C;g_p']\rangle_{\Lambda_{i+1}} = \frac{1}{Z}\int_0^{2\pi}\{d\theta_b^{(\delta)}\} \times \prod_{p\in\Lambda_\delta}\{\delta[(\nabla\theta^{(\delta)})_p]\}F[\theta^{(\delta)},j], \tag{C11}$$

where

$$F[\theta^\delta,j] = \int_0^{2\pi}\{d\theta_b^{(i)}\}e^{i\langle j,\theta\rangle}\prod_{p\in\Lambda_i}e^{-S_V[(\nabla\theta)_p,g_p]}. \tag{C12}$$

The next step is to use gauge invariance to show that $F[\theta^\delta,j]$ is independent of θ^δ. Then the numerator and denominator Z of (C11) will each factorize, proving Eqs. (C9). To examine $F[\theta^\delta,j]$, it is convenient to extend θ_b to be a 1-chain on Λ^∞ by the prescription $\theta_b=0$ for $b\notin\Lambda_{i+1}$. Since Λ_{i+1} is an open sublattice of Λ^∞, $\nabla\theta$ will vanish outside Λ_{i+1}. Now let $\theta^{\bar{i}} \equiv \theta|_{\overline{\Lambda}_i}$. Since $(\nabla\theta)|_{\overline{\Lambda}_i}=0$, and $\overline{\Lambda}_i$ is a closed sublattice of Λ^∞, it follows that $\nabla\theta^{\bar{i}}=0$. But $\overline{\Lambda}_i$ has a trivial 1-cohomology, so $\theta_b^{\bar{i}} = \nabla\phi_s$, for some 0-chain ϕ_s (s for site). Thus, one can define a change of variables

$$\theta_b' = \theta_b - \nabla\phi_s \tag{C13}$$

for all b, where ϕ_s has been extended to Λ^∞ by the prescription $\phi_s = 0$ for $s \in \Lambda_i$. Then θ_b' will vanish on $\overline{\Lambda}_i$. (C13) is of course a gauge transformation with $\nabla\theta' = \nabla\theta$ and $\langle j, \theta'\rangle = \langle j, \theta\rangle$. Using (C13) as a change of variables in (C12), one finds $F[\theta^6, j] = F[0, j]$, and (C9) follows.

The rest of the proof of Theorem IV is identical to that of Theorem III, except that this time $\langle A[C]\rangle_i$ is monotonically nonincreasing and is bounded from below by zero.

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[12]It may seem strange to introduce an orientation for sites, but it is convenient to do so. A site of one orientation is contained positively in bonds which point toward the site and negatively in bonds which point away from the site, while these signs are reversed for the corresponding site of the opposite orientation. The advantage of using oriented sites is that the duality properties, to be discussed later in this section, can be stated more simply.

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[15]J. Glimm, A. Jaffe, and T. Spencer in *Constructive Quantum Field Theory*, edited by G. Velo and A. Wightman, Springer Lecture Notes in Physics (Springer, Berlin/New York, 1973), Vol. 25.

[16]This technique was used by Glimm *et al.* in Ref. 15. One regards each bond in the cluster as an island. A network of $L-1$ bridges is then constructed as follows: (i) Choose one bond b_0 at random, (ii) choose a connected bond b_1 and imagine a bridge which joins them, (iii) choose any bond b_2 which is connected to a bond b_i which has already been chosen, and imagine a bridge joining b_2 to b_i, (iv) repeat until all bonds are included. According to the famous Konigsberg bridge problem of Euler (see Ref. 17), it is always possible to choose a path which crosses each island and crosses no bridge more than twice. Thus, the path crosses at most $2(L-1)$ bridges. Since each bond is connected to 14 others, the number of such paths is bounded by $14^{2(L-1)}$.

[17]D. B. Blackett, *Elementary Topology* (Academic, New York, 1967), p. 159. A translation from Euler's memoirs appears in Sci. Am. 189, 66 (1953).

[18]See, for example, p. 319 of Hocking and Young (Ref. 11).

[19]Note that here one has used the fact that R is contained in a closed sublattice, and thus each bond contains two sites.

[20]For example, for $L = 2D$ one has the closed path consisting of bonds in the directions $1, 2, \ldots, D, -1, -2, \ldots, -D$.