Gauge field configurations in curved space-times. V. Regularity constraints and quantized actions

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We study the regularity conditions for certain classes of self-dual, finite-action solutions obtained previously in de Sitter, Eguchi-Hanson, and multicenter space-times. These constraints are found to reduce the continuous spectra of the actions obtained before to discrete ones.

I. INTRODUCTION

In the preceding papers of this series (the papers quoted in Ref. 1, hereafter called I, II, III, and IV, respectively) we have obtained various solutions of the equations of motion for SU(2) gauge fields in different metrics. In particular in II, III, and IV self-dual solutions have been obtained for SU(2) fields for de Sitter, Eguchi-Hanson, and multicenter metrics, respectively (Euclidean signature being understood in all cases). The criterion employed for constructing these solutions were self-duality and *finite action*.

This permitted us to obtain certain classes of solutions with a *continous* spectrum for the action. The action is defined as the volume integral of $(\frac{1}{2}\,\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}\sqrt{g}\,)$, which can be converted to suitable surface integrals according to the case considered. Here we will study certain regularity constraints on these solutions, which will be shown to lead to *discrete* spectra for the respective actions, though not always to strictly integral ones. But we would like to emphasize the following remarkable feature arising in the curved spaces to be considered. The class of self-dual, finite-action solutions (even for genuinely non-Abelian ones) is a larger one and includes the class which satisfies supplementary regularity constraints.[2] The types of regularity constraints to be considered depend on the special features of the metrics in question, excluding such a phenomenon, by definition, for flat space.

In our opinion, the broader class of finite-action solutions as a whole should be considered to be of potential interest. In fact, even singular solutions with divergent actions should not be discarded *a priori* (flat-space meron solutions, for example). We intend to study elsewhere adequately the consequences of the existence of self-dual, finite-action solutions violating the types of regularity constraints to be discussed below.

This, however, need not prevent us from studying the regularity conditions in detail, which we now proceed to do. The contents of the foregoing remarks will be clearer after we discuss the particular cases briefly but in a reasonably self-contained fashion. For each case we recapitulate the essential points.

II. DE SITTER METRIC

We study now the self-dual solution of II. The metric is (on the Euclidean section)

$$ds^2=Nd\tau^2+N^{-1}dR^2+R^2(d\theta^2+\sin^2\theta d\varphi^2)\,, \tag{2.1}$$

where

$$N=(1-\tfrac{1}{3}\Lambda R^2)\quad\text{with }\Lambda>0\,.$$

The Euclidean time is periodic with a period

$$T=2\pi\sqrt{3/\Lambda}$$

and

$$0\leq R<\sqrt{3/\Lambda}\,.$$

The limit $R=\sqrt{3/\Lambda}$ can be made free of singularities by using Kruskal-type coordinates. (For details see II and the references quoted therein.) The ansatz for the SU(2) gauge potential is

$$\begin{aligned}&A_0=\chi_*\hat{\Phi}\,,\\&\vec{A}=(e^{\chi}-1)i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]\,,\end{aligned} \tag{2.2}$$

where

$$\hat{\Phi}\equiv\frac{\vec{x}}{R}\cdot\frac{\vec{\sigma}}{2}=\hat{R}\cdot\vec{\sigma}/2$$

and

$$\chi_*\equiv\frac{d}{dR_*}\chi(R_*)\quad\text{with }R_*=\int\frac{dR}{N}\,.$$

It can be shown (II) that one gets self-dual $F_{\mu\nu}$ for

$$\chi=\ln\frac{\alpha\sinh(\sqrt{\Lambda/3}R_*)}{\sinh[\alpha(\sqrt{\Lambda/3}R_*+\beta)]}\;. \tag{2.3}$$

Setting $\beta=0$ and $\alpha>1$ one gets *finite-action* solutions with the action S

$$S=8\pi^2(\alpha-1)\quad(\alpha>1)\,. \tag{2.4}$$

We thus get a continuous spectrum for $S/8\pi^2$.

Now to obtain the regularity constraints one must examine more closely the situation as $R\to\sqrt{3/\Lambda}$, or since

$$\sqrt{\Lambda/3}R=\tanh(\sqrt{\Lambda/3}R_*)\,,$$

the limit $R_*\to\infty$. The action density itself has no singularity there (for $\alpha>1$) as can be checked directly from

$$S=8\pi^2\sqrt{\Lambda/3}\int_0^\infty\left(\frac{d}{dR_*}[(e^{2\chi}-1)\chi_*]\right)dR_*\,. \tag{2.5}$$

So there is no problem in that respect. To check the regularity of the A_μ's one must, however, do so in a coordinate system *which regularizes the metric at* $R=\sqrt{3/\Lambda}$. So let us introduce the Kruskal-type coordinates (ξ,η) such that

$$e^{-2\sqrt{\Lambda/3}R_*}=\eta^2+\xi^2\equiv\zeta^2$$

and (2.6)

$$e^{i\sqrt{\Lambda/3}\tau}=\left(\frac{\eta-i\xi}{\eta+i\xi}\right)^{1/2}=e^{i\psi},\quad\text{with }\tan\psi=\left(\frac{-\xi}{\eta}\right).$$

The line element is now

$$ds^2=\frac{3}{\Lambda}\frac{4}{(1+\zeta^2)^2}\left[d\xi^2+d\eta^2+\left(\frac{1-\zeta^2}{2}\right)^2(d\theta^2+\sin^2\theta d\varphi^2)\right]. \tag{2.7}$$

In (2.2) $\vec{A}$ has only components A_θ and A_φ ($A_R=0$). So in the new coordinates (apart from A_θ and A_φ which remain untouched) one gets (since $\psi=\sqrt{\Lambda/3}\,\tau$ with a period 2π)

$$A_\xi=0\,,\quad A_\psi=\sqrt{3/\Lambda}\,\chi_*\hat{\Phi}\,. \tag{2.8}$$

As $R_*\to\infty$, $\zeta\to0$ and (for $\alpha>1$)

$$\chi_*=\sqrt{\Lambda/3}\left[\frac{1+\zeta^2}{1-\zeta^2}-\alpha\left(\frac{1+\zeta^{2\alpha}}{1-\zeta^{2\alpha}}\right)\right]$$
$$\to\sqrt{\Lambda/3}[(1-\alpha)+2\zeta^2+(\text{higher powers of }\zeta)]\,. \tag{2.9}$$

Thus

$$A_\psi\to-(\alpha-1)\hat{\Phi}\,, \tag{2.10}$$

which is *finite* but not *vanishing* and hence

$$A_\xi=\frac{-\eta}{\xi^2+\eta^2}A_\psi\text{ and }A_\eta=\frac{\xi}{\xi^2+\eta^2}A_\psi \tag{2.11}$$

diverge as ξ and $\eta\to0$. To remove this divergence we now introduce the gauge transformation

$$U_\alpha=e^{-i(\alpha-1)\psi\hat{\Phi}}\,. \tag{2.12}$$

The transformed potentials are (in this "α gauge")

$$A'_\xi=0\,,$$
$$A'_\psi=[\sqrt{3/\Lambda}\chi_*+(\alpha-1)]\hat{\Phi}\,, \tag{2.13}$$
$$\vec{A}'=+e^\chi\sin[(\alpha-1)\psi]\vec{\nabla}\hat{\Phi}$$
$$+\{e^\chi\cos[(\alpha-1)\psi]-1\}i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]\,.$$

[For convenience of notation we continue to use $\vec{A}$ or $\vec{A}'$ to group only the two components A_θ and A_φ with components $(\partial_\theta\hat{\Phi},\partial_\varphi\hat{\Phi})$ of $\vec{\nabla}\hat{\Phi}$.]

Using (2.9), (2.10), and (2.13) one sees that

$$A'_\psi\approx(\xi^2+\eta^2)\hat{\Phi} \tag{2.14}$$

and hence from (2.11) that A'_ξ, A'_η no longer diverge. For U_α to be *single-valued* (since we are using a 2×2 matrix representation) as $\psi\to(\psi+2\pi)$, $(\alpha-1)$ must be an *even* integer if we want to exclude an overall change of sign of U. But such a change of sign has no effect on A'_μ and from (2.13) they are found to be single-valued for *all integer values* of $(\alpha-1)$. (As $R\to0$, to assure regular Cartesian components, we have to go back to the original gauge. In contrast, in Sec. III, we will find a gauge regular everywhere.[3]) Thus, the supplementary regularity constraints lead to a *discrete integer spectrum* for $S/8\pi^2$ in (2.4).

In the foregoing discussion we have used a Kruskal-type coordinate transformation which is useful also in other cases (Schwarzschild metric, for example). But for the particular case of the de Sitter space one can also use a transformation giving an explicitly conformally flat metric. In fact, defining

$$\bar{\xi}=\frac{-2\xi}{(1+\eta)^2+\xi^2},\quad\bar{\eta}=\frac{1-\xi^2-\eta^2}{(1+\eta)^2+\xi^2},$$

one gets

$$ds^2=\frac{3}{\Lambda}\frac{4}{(1+\bar{\xi}^2+\bar{\eta}^2)}[d\bar{\xi}^2+d\bar{\eta}^2+\bar{\eta}^2(d\theta^2+\sin^2\theta d\varphi^2)]\,. \tag{2.15}$$

It can be shown that one is led to the same conclusion as before, namely to a discrete spectrum for the regular fields.

Let us note finally that in formula (2.21) of II we used the same transformation as (2.12) of this paper. But in II we noted the periodicity constraint only on the limiting form (as $R\to\sqrt{3/\Lambda}$). We also examined the potentials in the Kruskal coordinates, but separately. The proper regularity criterion is obtained by combining the gauge transformation with the coordinate transformation as done here.

III. EGUCHI-HANSON METRIC[5,6]

We consider now the self-dual solution of III. The procedure adopted will be very similar to that of Sec. II. The metric is given by

$$ds^2=f^2dr^2+\frac{r^2}{4}(d\theta^2+\sin^2\theta d\varphi^2)$$
$$+\frac{r^2h^2}{4}(d\psi+\cos\theta d\varphi)^2, \qquad (3.1)$$

where

$$f^{-1}=h=[1-(a/r)^4]^{1/2}$$

and

$$a\leqslant r<\infty,\quad 0\leqslant\theta\leqslant\pi,\quad 0\leqslant\varphi\leqslant 2\pi,\quad 0\leqslant\psi\leqslant 2\pi.$$

The ansatz is

$$A_r=0,\quad A_\psi=G(r)\frac{\sigma_3}{2},\quad A_\theta=F(r)\left(\sin\psi\frac{\sigma_1}{2}-\cos\psi\frac{\sigma_2}{2}\right), \qquad (3.2)$$

$$A_\varphi=G(r)\cos\theta\frac{\sigma_3}{2}-F(r)\sin\theta\left(\cos\psi\frac{\sigma_1}{2}+\sin\psi\frac{\sigma_2}{2}\right).$$

Self-dual solutions are obtained for

$$F(r)=\frac{\alpha\sinh\rho}{\sinh[\alpha(\rho+\beta)]}$$

and (3.3)

$$G(r)=\alpha\tanh\rho\coth[\alpha(\rho+\beta)],$$

where α and β are continuous parameters and

$$r^2/a^2=\coth\rho. \qquad (3.4)$$

The action turns out to be (III)

$$S=8\pi^2\int_{\rho=0}^{\infty}\left[\frac{d}{d\rho}(\tfrac{1}{2}G^2+F^2-GF^2)\right]d\rho \qquad (3.5)$$

$$=(8\pi^2)\left(\frac{\alpha^2-1}{2}\right)\ \text{ for }\alpha>1,\beta=0 \qquad (3.6)$$

and

$$S=(8\pi^2)\left(\frac{\alpha^2}{2}\right)\ \text{ for }\alpha>1\text{ and }0<\beta<\infty. \qquad (3.7)$$

The continuous parameter α enters in the limit $\rho\to\infty$ (i.e., $r\to a$) when $F\to 0$, $G\to\alpha$. Hence the regularity of the A_μ's should be checked as $r\to a$. Using the variable[5]

$$u^2=r^2[1-(a/r)^4],$$
$$ds^2=[1+(a/r)^4]^{-2}du^2+\frac{u^2}{4}(d\psi+\cos\theta d\varphi)^2 \qquad (3.8)$$
$$+\frac{r^2}{4}(d\theta^2+\sin^2\theta d\varphi^2),$$

and the metric can be regularized at $r=a$ if $0\leqslant\psi\leqslant 2\pi$ (see the discussion in Ref. 5). As $r\to a$

$$A_u=A_r=0$$

and

$$A_\psi=G\frac{\sigma_3}{2}\to\alpha\frac{\sigma_3}{2}. \qquad (3.9)$$

Holding θ and φ fixed as $r\to a$ $(u\to 0)$

$$ds^2\approx\tfrac{1}{4}(du^2+u^2d\psi^2), \qquad (3.10)$$

hence the regularity should be checked using the Cartesian components

$$\xi=u\cos\psi,\quad \eta=u\sin\psi, \qquad (3.11)$$

when

$$A_\xi=\frac{-\eta}{u^2}A_\psi\text{ and }A_\eta=\frac{\xi}{u^2}A_\psi \qquad (3.12)$$

are seen to diverge as $u\to 0$. To avoid this we introduce, as before, a gauge transformation

$$U_\alpha=e^{i\alpha\psi\sigma_3/2} \qquad (3.13)$$

which generalizes the operator (6) of III. The transformed potentials are

$$A_u'=0,\quad A_\psi'=(G-\alpha)\frac{\sigma_3}{2},$$

$$A_\theta'=-F\left\{\cos[(\alpha-1)\psi]\frac{\sigma_2}{2}+\sin[(\alpha-1)\psi]\frac{\sigma_1}{2}\right\}, \qquad (3.14)$$

$$A_\varphi'=G\cos\theta\frac{\sigma_3}{2}-F\sin\theta\left\{\cos[(\alpha-1)\psi]\frac{\sigma_1}{2}-\sin[(\alpha-1)\psi]\frac{\sigma_2}{2}\right\}.$$

As $u\to 0$,

$$G\to\alpha(1-2u^2/4a^2+\text{higher powers of }u). \qquad (3.15)$$

Hence, A_ξ' and A_η' are regular in this "α gauge." As before, the single-valuedness of A_θ' and A_φ' as $\psi\to\psi+2\pi$ now imposes *integer values* of $(\alpha-1)$. Hence, from (3.6) and (3.7), we have the following results:

(1) For $\beta=0$, $S/8\pi^2$ is an *integer for odd* α and *half-integer for even* α. (In particular, the spin-connection case, with $\alpha=2$, gives $\frac{3}{2}$.)

(2) For $\beta>0$, $S/8\pi^2$ is an *integer for even* α and *half-integer for odd* α. (We will add more remarks at the end, concerning the half-integral values after having discussed the multicenter case.)

To complete the picture we note that while the action integrand in (3.5)

$$\approx e^{-2(\alpha-1)\rho}\text{ as }\rho\to\infty\ (u\to 0), \qquad (3.16)$$

considering separately the derivative of F as $u\to 0$, one has

$$\frac{dF}{du}\approx\frac{1}{2a}\alpha(\alpha-1)e^{-(\alpha-2)\rho}\text{ as }u\to 0. \qquad (3.17)$$

But this does not add any supplementary restriction on α.

IV. MULTICENTER METRICS[4,5,7]

Among the cases studied in IV, only one led to a continuous spectrum [see Eq. (46) of IV]. Here we will consider the regularity constraints for this case in particular. Discrete but fractional values (a term $\approx 1/n$ for n centers) appear more generally whenever the integration on the large sphere (in the sense described in IV) contributes to the action. This feature will be reconsidered at the end. The metric is given by

$$ds^2 = V^{-1}(d\tau + \vec{\omega}\cdot d\vec{x})^2 + V\, d\vec{x}\cdot d\vec{x}, \tag{4.1}$$

where

$$V = V_0 + \sum_{i=1}^{n}\left(\frac{2M}{R_i}\right) \quad (R_i \equiv [(\vec{x}-\vec{x}_i)^2]^{1/2})$$

and

$$\vec{\nabla}\times\vec{\omega} = \vec{\nabla}V.$$

The period of τ is $8\pi M$. The ansatz leading to a continuous spectrum is (with unknown $\vec{G}$ and a *constant* K)

$$A_0 = \vec{G}\cdot\frac{\vec{\sigma}}{2},$$
$$\vec{A} = \vec{\omega}\left(\vec{G}\cdot\frac{\vec{\sigma}}{2}\right) - V\left(\vec{G}\times\frac{\vec{\sigma}}{2}\right) + K\frac{\vec{\sigma}}{2}. \tag{4.2}$$

Self-dual solutions are obtained for

$$\vec{G} = -\left(\frac{\vec{\nabla}H}{VH}\right), \tag{4.3}$$

where

$$\Delta H + K^2 H = 0.$$

It can be shown (IV) that except at singular points the action density is

$$\sqrt{g}\tfrac{1}{2}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) = \vec{\nabla}\cdot\vec{\xi},$$

where

$$\vec{\xi} = \vec{\nabla}\left[\frac{1}{2V}\left(\frac{\vec{\nabla}H}{H}\right)^2\right]. \tag{4.4}$$

Let us briefly recapitulate that for $K=0$,

$$H = H_0 + \sum_{j=1}^{\tilde{n}} \frac{2\mu_j}{R_j}, \tag{4.5}$$

where a finite action is obtained when the R_j's are a *subset* of the R_i's giving the centers of V in (4.1) ($\tilde{n}\le n$). The action is defined by surrounding the centers by small spheres and taking the limiting form as they are contracted to zero. One obtains (for details, see IV), converting S to surface integrals over a large and the small spheres,

$$S = 8\pi^2\left(\tilde{n} - \frac{(1-\epsilon)}{n}\right), \tag{4.6}$$

where $\epsilon = 0$ for $V_0 = 0 = H_0$ and $\epsilon = 1$ otherwise. (For the trivial case $\tilde{n} = 0$, $\epsilon = 1$.) The term $\tilde{n}$ comes from the common centers of H and V and for $H_0 = 0 = V_0$ there is a nonvanishing contribution from the large sphere ($\approx 1/n$). For the spin-connection case ($H = V$ and $V_0 = 0$) one thus has

$$S = 8\pi^2\left(n - \frac{1}{n}\right). \tag{4.7}$$

For $K\ne 0$ (and $V_0 \ne 0$) interesting complex solutions are obtained by setting $K = i\zeta$ (ζ real) and (with constants a_j and b_j)

$$H = \sum_j \frac{a_j \sinh[\zeta(R_j + b_j)]}{\zeta R_j}. \tag{4.8}$$

(We consider only real $b_j \ge 0$. For $b_j \ne 0$ the center j of H must coincide with one of the centers of V.)

One finally obtains for this case (IV) a finite *real* action,

$$S = 8\pi^2\left[\tilde{n} + \frac{1}{n}(2\chi + \chi^2)\right]$$
$$= 8\pi^2\left[\tilde{n} - \frac{1}{n} + \frac{1}{n}(\chi+1)^2\right], \tag{4.9}$$

where

$$\chi \equiv \left(\frac{2nM\zeta}{V_0}\right) \tag{4.10}$$

and $\tilde{n}$ represents the number of centers of H with $b_j \ne 0$. (ζ and hence χ can be taken to be non-negative without loss of generality.) The terms involving the factor $1/n$ again come from integration on the large sphere. This is the case where we obtain finite, continuous action, which will now be studied more closely to obtain the regularity constraints on the parameter χ. Here the constraint arises in trying to eliminate the string singularity due to $\vec{\omega}$. While a more general investigation is possible, it is sufficient for our purpose to study the simple case

$$H = \frac{\sinh[\zeta(R+b)]}{R} \quad (R = \sqrt{\vec{x}^2}) \tag{4.11}$$

with one center of V at the origin for $b \ne 0$.

A *complex* gauge transformation given by

$$U = e^{-\zeta(R+b)(\hat{x}\cdot\vec{\sigma}/2)} \tag{4.12}$$

leads (for real b) to the *Hermitian* potentials (IV)

$$A_0 = \frac{1}{V}\left(\frac{1}{R} - \zeta\coth[\zeta(R+b)]\right)\hat{x}\cdot\frac{\vec{\sigma}}{2},$$
$$\vec{A} = \frac{\vec{\omega}}{V}\left(\frac{1}{R} - \zeta\coth[\zeta(R+b)]\right)\hat{x}\cdot\frac{\vec{\sigma}}{2} + \left(\frac{\zeta R}{\sinh[\zeta(R+b)]} - 1\right)\frac{1}{R}\hat{x}\times\frac{\vec{\sigma}}{2}. \tag{4.13}$$

For V given by (4.1) one can choose

$$\vec{\omega} = 2M\left(\sum_{i=1}^{n} \cos\theta_i \vec{\nabla}\varphi_i\right), \tag{4.14}$$

where θ_i and φ_i are the spherical angles measured with the origin at the ith center. Hence as $R \to \infty$,

$$V \to V_0$$

and (4.15)

$$\vec{\omega} \to 2nM \cos\theta \vec{\nabla}\varphi ,$$

where [where $\vec{\nabla}\varphi = (x^2+y^2)^{-1}(-y, x, 0)$] all the centers being supposed to be at a finite distance from the chosen fixed origin. Keeping only the leading terms, we see that as $R \to \infty$

$$A_0 \to -\frac{\zeta}{V_0}\left(\hat{x}\cdot\frac{\vec{\sigma}}{2}\right)$$

and (4.16)

$$\vec{A} \to -(\chi \cos\theta \vec{\nabla}\varphi)\left(\hat{x}\cdot\frac{\vec{\sigma}}{2}\right) - \frac{1}{R}\hat{x}\times\frac{\vec{\sigma}}{2}.$$

Thus for $\zeta > 0$ (i.e., $\chi > 0$) a string singularity survives asymptotically even as a leading term.

We now consider the gauge transformations

$$U_{\pm} = e^{(\mp)i\chi\varphi(\hat{x}\cdot\vec{\sigma}/2)}. \tag{4.17}$$

The transformed potentials are

$$A_{0(\pm)} = A_0 ,$$
$$\vec{A}_{(\pm)} = \left[\frac{\vec{\omega}}{V}\left(\frac{1}{R} - \zeta \coth[\zeta(R+b)]\right) \pm \chi\vec{\nabla}\varphi\right]\left(\hat{x}\cdot\frac{\vec{\sigma}}{2}\right) + \left\{\frac{\zeta R}{\sinh[\zeta(R+b)]}\cos(\chi\varphi) - 1\right\}\frac{1}{R}\hat{x}\times\frac{\vec{\sigma}}{2} \pm \frac{\zeta R}{\sinh[\zeta(R+b)]}\sin(\chi\varphi)\vec{\nabla}\left(\hat{x}\cdot\frac{\vec{\sigma}}{2}\right). \tag{4.18}$$

As $R \to \infty$,

$$\vec{A}_{(\pm)} \to -\chi(\cos\theta \mp 1)\vec{\nabla}\varphi\left(\hat{x}\cdot\frac{\vec{\sigma}}{2}\right) - \frac{1}{R}\hat{x}\times\frac{\vec{\sigma}}{2}. \tag{4.19}$$

Thus the string now survives for $\theta = \pi$ and $\theta = 0$, respectively. This aspect of the situation is similar to the well-known one arising in the study of Dirac monopoles using two coordinate patches.[2,8] The gauge transformation converting $A_{\mu(+)}$ to $A_{\mu(-)}$ is

$$U = e^{i2\chi\varphi(\hat{x}\cdot\vec{\sigma}/2)}. \tag{4.20}$$

This is single-valued (as $\varphi \to \varphi + 2\pi$) for integer χ, and so are the potentials (4.18). Thus, it is sufficient to consider the regularity of the leading terms to obtain a *discrete* spectrum corresponding to (4.9) or (4.10). Moreover, if instead of (4.11) we use the more general solution (4.8), the asymptotic behavior of $\vec{\nabla}(\ln H)$ remains the same. Hence, although in the general case we cannot obtain a Hermitian gauge like (4.13), the $\vec{\omega}$-dependent term behaves asymptotically in the same fashion. Hence, the foregoing considerations remain essentially unchanged. For completeness we note that using (4.11) as $R \to 0$, the leading term in the coefficient of $(\vec{\nabla}\varphi)(\hat{x}\cdot\vec{\sigma}/2)$ is $(-\frac{1}{3}\zeta^2 R^2 \cos\theta)$ for $b = 0$ and simply $\cos\theta$ for $b \neq 0$. Hence, for $b = 0$ the string is avoided as $R \to 0$. For $b \neq 0$ one has indeed a limiting term $\approx \cos\theta\vec{\nabla}\varphi$. But a treatment of this term analogous to that for the $R \to \infty$ limit evidently involves no further restriction on ζ, the term in question being independent of ζ. Similarly, one sees that the μ_j's in (4.5) are not restricted.

V. REMARKS

In Secs. III and IV we have thus obtained discrete spectra for $S/8\pi^2$ but not one of integers. In fact, (4.9) is now restricted to

$$S/8\pi^2 = \text{integer} + \frac{n_0}{n}, \tag{5.1}$$

where $n_0 = 0, 1, 2, \ldots, (n-1)$. For $V_0 = 0 = H_0$ again one has

$$S/8\pi^2 = \bar{n} - \frac{1}{n}. \tag{5.2}$$

Even the spin-connection ($\bar{n} = n$) case gives a nonintegral value. In Sec. III half-integer values appeared, which is not surprising since the Eguchi-Hanson (EH) metric is equivalent to the $n = 2$ case ($V_0 = 0$). But some differences should also be noted. To start with, the ansatz used in Sec. III belongs to a different class from that used in Sec. IV (comparing, of course, in particular $n = 2$) although both classes include the spin-connection case (see the discussion in IV). In the multicenter formalism the nonintegral values arise only when the large-sphere integral contributes. In the EH coordinates a factor $\frac{1}{2}$ arises in both limits ($\alpha^2/2$ as $r \to a$ and $-\frac{1}{2}$ as $r \to \infty$). This factor $\frac{1}{2}$ is a direct consequence of restricting the period of ψ (of Sec. III to 2π (from the usual 4π). The variable τ of Sec. IV has a period 4π (for the normalization $2M = 1$) for any n. (Hence also for $n = 2$, a fact that appears particularly explicitly in the relevant coordinate transformation linking in the two formalisms.[9,10]) This is why the integrals over the small spheres (for details see IV) contribute integers to (5.1) and (5.2). The asymptotic properties of V that formally lead to the term with a factor $1/n$ are the ones that imply the asymptotic properties of the metric (identification of n points under a discrete group).

For flat-space solutions different types of compactifications of the manifold have been considered[11] which, avoiding the usual asymptotic mapping $S_3 \to S_3$, can give a topological significance to half-integer values associated to Gribov vacuums. Half-integral meron charges are not unrelated to Gribov vacuums. Nonorientable manifolds have been proposed to provide a geometric interpretation of merons. When one starts with a flat-space solution such postulates have to be introduced as additional ingredients. Multicenter metrics introduce from the outset asymptotic features leading to fractional values of $S/8\pi^2$.

Let us note finally some further special features of our solutions. One can start with the de Sitter case, but our point is best illustrated by the Eguchi-Hanson one where our solution includes the flat-space one-instanton solution as a limit. In flat-space many-instanton solutions the centers can be chosen arbitrarily and, in fact, to obtain higher indices one has to increase the numbers of centers and hence to introduce new parameters. The translational symmetry of the flat Euclidean space being absent, such a simple prescription is not possible for the EH metric. But we have been able to construct solutions for all values of the action, in a "form-invariant" fashion so to say, the value of α (for the two cases $\beta = 0$ and $\beta > 0$) fixes the action and (apart from the unrestricted positive β) there are no more free parameters. It is remarkable that such a simple, single solution can be obtained to cover all possible values of the action without adding terms. On the other hand, it does not seem to be directly generalizable to include more parameters. In the multicenter case, for a given action, there is room for more parameters. The permissible number of singular centers of H [those appearing in (4.5) or those in (4.8) with $b_j \neq 0$] is limited by the number of centers of V ($\tilde{n} \leq n$), with which they must coincide if the action is to remain finite. With each, one can associate a parameter μ_j. Apart from this fact, a very special situation arises for the *complex* fields corresponding to (4.8). The number of regular centers ($b_j = 0$) is arbitrary. They do not contribute to the action. In this respect the situation remains analogous to that for flat-space complex monopoles discussed by Manton[12] where such centers can be added without altering the monopole number.[13]

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[3]In Secs. II and III the irregularity studied is associated with a "bolt" singularity while that studied in Sec. IV corresponds to a string associated with a "nut" singularity. Both the analogies and the differences between the two cases should be noted. In Sec. III the α gauge entirely removes the irregularity in question. In Sec. IV, the more familiar one, one obtains at best two gauges each with its own string (on positive and negative z axis, respectively) and then one considers the single-valuedness of the gauge transformation connecting the two. The definitions of nut and bolt singularities can be found in Refs. 4 and 5.

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