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Nonsingular representation of three-body equations

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An alternate representation of the three-body scattering equations of Alt, Grassberger, and Sandhas is suggested. Like the formulation of Karlsson and Zeiger these equations require only two-body bound-state wave functions and half-off-shell transition amplitudes as input and contain three-body energy-independent effective potentials which become real after partial-wave decomposition. It is emphasized that such representations are particularly suitable for writing singularity-free momentum-space integral equations in the scattering region. One scheme for writing such equations is discussed.

I. INTRODUCTION

Recently, Karlsson and Zeiger[1] (KZ) presented Faddeev-type[2] scattering integral equations for the three-body transition amplitudes. The on-the-energy-shell (on-shell) values of the unknowns of these equations give the physical elastic, rearrangement, and breakup amplitudes and hence agree with the corresponding Alt, Grassberger, and Sandhas[3] (AGS) transition amplitudes. But the solutions of KZ involve a different off-the-energy-shell (off-shell) continuation from that of the AGS transition amplitudes and hence do not agree with the AGS amplitudes for off-shell or half-off-the-energy-shell (half-off-shell) values of momentum parameters.

In the case of the two-body problem the outgoing-wave scattering wave function in the plane-wave basis has a scattering pole whose residue is the half-shell t matrix which is less singular and does not possess this pole.[4] It is usual to formulate the two-body problem in terms of the plane-wave representation of this t matrix. The on-shell value of this t matrix gives the physical transition amplitude.

In the case of the three body problem the generalization of the above-mentioned two-body recipe is not unique and there are various possibilities. First, one may consider the Faddeev equations[2] satisfied by the wave-function components, factor out the primary singularities,[1] and formulate the theory in terms of less singular amplitudes. This method of approach was chosen in KZ where the authors used a complete set of channel eigenstates for the intermediate states involved. Osborn and Kowalski[5] also followed this approach essentially but used the complete set of plane-wave states instead. KZ pointed out that the basic advantage of choosing the complete set of channel eigenstates is that the resulting equations have energy-independent effective potentials which can be made real after partial-wave analysis.

In this note we consider a different generalization of the two-body recipe. We consider the full three-body scattering states corresponding to various elastic and rearrangement channels. The projection of these wave functions on a complete set of channel eigenstates contains the elastic, rearrangement, and breakup poles. The residues at these poles are identified with the matrix elements of the AGS transition amplitudes. We work with these amplitudes and using a complete set of channel eigenstates write coupled equations for them, the solution of which gives the AGS transition amplitudes both on-shell and off-shell. The effective potentials are energy independent and can be made real after a partial-wave analysis.

The analytic structure of these equations is remarkably simple compared to the usual Faddeev equations. The three-body parametric energy only appears in the resolvent operators of these equations and not in the off-shell two-body t matrix that appears in the equation. The three-body energy-dependent propagators take the simple form of propagators of two-body Lippmann-Schwinger equations after a change of variables. Hence we can use techniques used in two-body scattering theory to remove the singularity of these equations. Eyre and Osborn[6] recently used this idea to propose a scheme for writing singularity-free momentum-space integral equations. The method they use is essentially a generalization of two-body K-matrix approach to the case of KZ equations.

In this note we propose another method for writing such equations. The present method is essentially a generalization of the method of Ref. 7 to this problem. The present method relies on writing an auxiliary equation with nonsingular kernel. The auxiliary equation we propose has a much weaker kernel and presumably will have a rapidly convergent iterative solution. Then the solution of the original equation is written in terms of the solution of the auxiliary equation. An iterative solution to the auxiliary equation will provide a practical way of solving the original equation. We demonstrate this in the case of the AGS transition amplitudes, but the same idea can be applied to the case of the KZ equations.

In Sec. II we consider the Lippmann-Schwinger equation for the full wave function, show how it naturally leads to the AGS transition amplitudes after a factorization of singularities, and write KZ-type equations for the AGS transition amplitudes. In Sec. III we describe a method for writing a nonsingular representation for the set of equations proposed in Sec. II and finally in Sec. IV we give a brief summary and concluding remarks.

II. THE FORMULATION

Our notation is the usual one and we briefly state it here. The indices α, β, and γ will be used to denote a pair. $G_0 \equiv (z-H_0)^{-1}$, $G_\alpha \equiv (z-H_\alpha)^{-1}$, and $G\equiv (z-H)^{-1}$ where $z=E+i\epsilon$ are the resolvent operators to the free Hamiltonian H_0, the channel Hamiltonian $H_\alpha \equiv (H_0+V_\alpha)$, $(\alpha=1,2,3)$, and the full Hamiltonian $H\equiv(H_0+V)$, where V_α is a pair interaction and $V=\sum_{\alpha=1}^{3} V_\alpha$ is the total interaction. The channel interaction will be denoted by $\overline{V}_\alpha \equiv (V-V_\alpha)$. We also use $\overline{\delta}_{\beta\alpha}=(1-\delta_{\beta\alpha})$. The energy dependence of the energy-dependent operators will not be explicitly shown in this note.

We introduce the following types of momentum variables.[1,2] $\vec{q}_\beta$ denotes the relative momentum of the pair β, and $\vec{p}_\beta$ denotes the relative momentum of the center of mass of the pair β and the remaining particle. The associated reduced masses are μ_β and n_β, respectively, as in Ref. 1. We also use $\tilde{p}_\beta{}^2 = p_\beta{}^2/2n_\beta$ and $\tilde{q}_\beta{}^2 = q_\beta{}^2/2\mu_\beta$. $\kappa_\alpha{}^2$ denotes the binding energy of the pair α.

The channel eigenstates are the products of a two-body bound state ϕ_α and a momentum eigenstate $\vec{p}_\alpha$ of relative motion between the spectator particle and the pair α. We assume that there is only one two-body bound state per channel. The scattering state $\Psi_\alpha^{(+)}$ which refers to the initial channel α is defined by

$$|\Psi_\alpha^{(+)}\rangle = i\epsilon G|\vec{p}_\alpha\phi_\alpha\rangle , \tag{1}$$

in the limit $\epsilon\to 0$. Using the resolvent identity

$$G=G_\beta+G_\beta\overline{V}_\beta G , \tag{2}$$

and taking the limit $\epsilon\to 0$, Eq. (1) becomes

$$|\Psi_\alpha^{(+)}\rangle = \delta_{\beta\alpha}|\vec{p}_\alpha\phi_\alpha\rangle + G_\beta\overline{V}_\beta|\Psi_\alpha^{(+)}\rangle . \tag{3}$$

From now on we shall assume that the channel state $\vec{p}_\alpha\phi_\alpha$ which appears on the right is an on-shell state and satisfies $\kappa_\alpha{}^2+E=\tilde{p}_\alpha{}^2$. We introduce the complete set of channel eigenstates in the channel β by $\vec{p}_\beta\phi_\beta$ and $\vec{p}_\beta\psi^-_{\vec{q}_\beta}$ where $\psi^-_{\vec{q}_\beta}$ is the incoming two-body scattering state. A projection of Eq. (3) on these states gives

$$\langle\vec{p}_\beta\phi_\beta|\Psi_\alpha^{(+)}\rangle = \delta_{\beta\alpha}\delta^3(\vec{p}_\beta-\vec{p}_\alpha)+\frac{Y_{\beta\alpha}(\vec{p}_\beta,\vec{p}_\alpha;z)}{z-\tilde{p}_\beta{}^2+\kappa_\beta{}^2} \tag{4}$$

and

$$\langle\vec{p}_\beta\psi^-_{\vec{q}_\beta}|\Psi_\alpha^{(+)}\rangle = \frac{W_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta,\vec{p}_\alpha;z)}{z-\tilde{p}_\beta{}^2-\tilde{q}_\beta{}^2} , \tag{5}$$

where

$$Y_{\beta\alpha}(\vec{p}_\beta,\vec{p}_\alpha;z)=\langle\vec{p}_\beta\phi_\beta|\overline{V}_\beta|\Psi_\alpha^{(+)}\rangle \tag{6}$$

and

$$W_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta,\vec{p}_\alpha;z)=\langle\vec{p}_\beta\psi^-_{\vec{q}_\beta}|\overline{V}_\beta|\Psi_\alpha^{(+)}\rangle . \tag{7}$$

Using Eqs. (4) and (5) the plane-wave projection of $\Psi_\alpha^{(+)}$ can be written as

$$\begin{aligned}\langle\vec{p}_\beta\vec{q}_\beta|\Psi_\alpha^{(+)}\rangle &= \delta_{\beta\alpha}\delta^3(\vec{p}_\beta-\vec{p}_\alpha)\phi_\alpha(\vec{q}_\alpha)\\ &\quad+\phi_\beta(\vec{q}_\beta)\frac{Y_{\beta\alpha}(\vec{p}_\beta,\vec{p}_\alpha;z)}{z-\tilde{p}_\beta{}^2+\kappa_\beta{}^2}\\ &\quad+\int d\vec{q}_\beta{}'\,\psi^-_{\vec{q}_\beta{}'}(\vec{q}_\beta)\frac{W_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta{}',\vec{p}_\alpha;z)}{z-\tilde{p}_\beta{}^2-\tilde{q}_\beta{}'^2} .\end{aligned} \tag{8}$$

Equation (8) is another possible three-body generalization of Eq. (1.1) of KZ. We have factored out the kinematic singularities of the full wave function and introduced less singular amplitudes Y and W in terms of which we shall formulate our theory. As the residues of the wave function at elastic, rearrangement, and breakup poles give the corresponding transition amplitudes the half-on-shell values of Y and W defined by Eqs. (6) and (7) can be readily related to physical scattering amplitudes. Though known in certain circles we show here explicitly that Y and W are really the matrix elements of the AGS transition operators defined by[3]

$$\begin{aligned}U_{\beta\alpha} &= (1-\delta_{\beta\alpha})(z-H_0)+V-V_\alpha-V_\beta\\ &\quad+\delta_{\beta\alpha}V_\alpha+\overline{V}_\beta G\overline{V}_\alpha\end{aligned} \tag{9}$$

and

$$G=\delta_{\beta\alpha}G_\beta+G_\beta U_{\beta\alpha}G_\alpha . \tag{10}$$

Substituting Eq. (10) in Eq. (1) and taking the limit $\epsilon\to 0$ we get

$$|\Psi_\alpha^{(+)}\rangle = \delta_{\beta\alpha}|\vec{p}_\alpha\phi_\alpha\rangle + G_\beta U_{\beta\alpha}|\vec{p}_\alpha\phi_\alpha\rangle . \tag{11}$$

If we take projection of Eq. (11) on the channel eigenstates for the channel β we get

$$\langle\vec{p}_\beta\phi_\beta|\Psi_\alpha^{(+)}\rangle = \delta_{\beta\alpha}\delta^3(\vec{p}_\beta-\vec{p}_\alpha)+\frac{\langle\vec{p}_\beta\phi_\beta|U_{\beta\alpha}|\vec{p}_\alpha\phi_\alpha\rangle}{z-\tilde{p}_\beta{}^2+\kappa_\beta{}^2} \tag{12}$$

and

$$\langle \vec{p}_\beta \psi_{\vec{q}_\beta} | \Psi_\alpha^{(+)} \rangle = \frac{\langle \vec{p}_\beta \psi_{\vec{q}_\beta} | U_{\beta\alpha} | \vec{p}_\alpha \phi_\alpha \rangle}{z - \tilde{p}_\beta^{\,2} - \tilde{q}_\beta^{\,2}} . \tag{13}$$

Comparing Eqs. (4) and (5) with Eqs. (12) and (13) we identify

$$Y_{\beta\alpha}(\vec{p}_\beta, \vec{p}_\alpha; z) = \langle \vec{p}_\beta \phi_\beta | U_{\beta\alpha} | \vec{p}_\alpha \phi_\alpha \rangle \tag{14}$$

and

$$W_{\beta\alpha}(\vec{p}_\beta \vec{q}_\beta, \vec{p}_\alpha; z) = \langle \vec{p}_\beta \psi_{\vec{q}_\beta} | U_{\beta\alpha} | \vec{p}_\alpha \phi_\alpha \rangle . \tag{15}$$

We recall that the AGS operators U satisfy the AGS equations

$$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_\gamma \bar{\delta}_{\beta\gamma} V_\gamma G_\gamma U_{\gamma\alpha} . \tag{16}$$

In terms of the complete set of channel eigenstates the explicit matrix element of Eq. (16) becomes

$$Y_{\beta\alpha}(\vec{p}_\beta, \vec{p}_\alpha; z) = \bar{\delta}_{\beta\alpha} Y^{(0)}_{\beta\alpha}(\vec{p}_\beta, \vec{p}_\alpha; z) + \sum_\gamma \int d^3p_\gamma' \, \mathcal{V}^{11}_{\beta\gamma}(\vec{p}_\beta, \vec{p}_\gamma') \frac{1}{z - \tilde{p}_\gamma^{\,2} + \kappa_\gamma^2} Y_{\gamma\alpha}(\vec{p}_\gamma', \vec{p}_\alpha; z)$$
$$+ \sum_\gamma \iint d^3p_\gamma' d^3q_\gamma' \, \mathcal{V}^{12}_{\beta\gamma}(\vec{p}_\beta, \vec{p}_\gamma' \vec{q}_\gamma') \frac{1}{z - \tilde{p}_\gamma'^{\,2} - \tilde{q}_\gamma'^{\,2}} W_{\gamma\alpha}(\vec{p}_\gamma' \vec{q}_\gamma', \vec{p}_\alpha; z) , \tag{17}$$

$$W_{\beta\alpha}(\vec{p}_\beta \vec{q}_\beta, \vec{p}_\alpha; z) = \bar{\delta}_{\beta\alpha} W^{(0)}_{\beta\alpha}(\vec{p}_\beta \vec{q}_\beta, \vec{p}_\alpha; z) + \sum_\gamma \int d^3p_\gamma' \, \mathcal{V}^{21}_{\beta\gamma}(\vec{p}_\beta \vec{q}_\beta, \vec{p}_\gamma') \frac{1}{z - \tilde{p}_\gamma'^{\,2} + \kappa_\gamma^2} Y_{\gamma\alpha}(\vec{p}_\gamma', \vec{p}_\alpha; z)$$
$$+ \sum_\gamma \iint d^3p_\gamma' d^3q_\gamma' \, \mathcal{V}^{22}_{\beta\gamma}(\vec{p}_\beta \vec{q}_\beta, \vec{p}_\gamma' \vec{q}_\gamma') \frac{1}{z - \tilde{p}_\gamma'^{\,2} - \tilde{q}_\gamma'^{\,2}} W_{\gamma\alpha}(\vec{p}_\gamma' \vec{q}_\gamma', \vec{p}_\alpha; z) , \tag{18}$$

where

$$Y^{(0)}_{\beta\alpha}(\vec{p}_\beta, \vec{p}_\alpha; z) = -\phi_\beta(\vec{q}_{\beta(2)})(\kappa_\alpha^2 + \tilde{q}_{\alpha(1)}^{\,2})\phi_\alpha(\vec{q}_{\alpha(1)}) , \tag{19}$$

$$W^{(0)}_{\beta\alpha}(\vec{p}_\beta \vec{q}_\beta, \vec{p}_\alpha; z) = -\psi^*_{\vec{q}_\beta}(\vec{q}_{\beta(2)})(\kappa_\alpha^2 + \tilde{q}_{\alpha(1)}^{\,2})\phi_\alpha(\vec{q}_{\alpha(1)}) , \tag{20}$$

$$\mathcal{V}^{11}_{\beta\gamma}(\vec{p}_\beta, \vec{p}_\gamma) = -\phi_\beta(\vec{q}_{\beta(2)})(\kappa_\gamma^2 + \tilde{q}_{\gamma(1)}^{\,2})\phi_\gamma(\vec{q}_{\gamma(1)})\bar{\delta}_{\beta\gamma} , \tag{21}$$

$$\mathcal{V}^{12}_{\beta\gamma}(\vec{p}_\beta, \vec{p}_\gamma \vec{q}_\gamma) = \phi_\beta(\vec{q}_{\beta(2)}) t^*_\gamma(\vec{q}_{\gamma(1)}, \vec{q}_\gamma; \tilde{q}_\gamma^{\,2} + i\epsilon)\bar{\delta}_{\beta\gamma} , \tag{22}$$

$$\mathcal{V}^{21}_{\beta\gamma}(\vec{p}_\beta \vec{q}_\beta, \vec{p}_\gamma) = -\psi^*_{\vec{q}_\beta}(\vec{q}_{\beta(2)})(\kappa_\gamma^2 + \tilde{q}_{\gamma(1)}^{\,2})\phi_\gamma(\vec{q}_{\gamma(1)})\bar{\delta}_{\beta\gamma} , \tag{23}$$

and

$$\mathcal{V}^{22}_{\beta\gamma}(\vec{p}_\beta \vec{q}_\beta, \vec{p}_\gamma \vec{q}_\gamma) = \psi^*_{\vec{q}_\beta}(\vec{q}_{\beta(2)}) t^*_\gamma(\vec{q}_{\gamma(1)}, \vec{q}_\gamma, \tilde{q}_\gamma^{\,2} + i\epsilon)\bar{\delta}_{\beta\gamma} , \tag{24}$$

where t_γ represents two-body half-off-shell transition amplitude. In Eqs. (19) and (20) we have used the fact that $\vec{p}_\alpha \phi_\alpha$ is an on-shell state and we have introduced[1,2]

$$\vec{q}_{\beta(2)} = \frac{\mu_\beta}{m_\gamma} \vec{p}_\beta + \vec{p}_\alpha \tag{25}$$

and

$$\vec{q}_{\alpha(1)} = -\vec{p}_\beta - \frac{\mu_\alpha}{m_\gamma} \vec{p}_\alpha . \tag{26}$$

Here m_γ is the mass of the spectator particle for the pair γ. In Eqs. (21)–(24), $\vec{q}_{\beta(2)}$ and $\vec{q}_{\gamma(1)}$ are defined similarly.

Equations (17) and (18) have all the interesting features of the KZ equations, e.g.,

(1) the input consists of two-body bound-state wave functions and half-off-shell transition amplitudes (at positive energies) and

(2) the effective potentials are energy independent and as we shall see in the following they become real after a partial-wave analysis.

To illustrate the advantages associated with a partial-wave decomposition of Eqs. (17) and (18) we shall be limited to the consideration of a simple case. We take the total three-body angular momemtum to be zero and consider only S-wave two-body interactions.

We recall that S-wave two-body scattering states $\psi^\pm$ can be written as[4]

$$\psi^\pm_{q_\beta}(r) = q_\beta \phi_{q_\beta}(r) \frac{1}{\mathcal{L}_\pm(q_\beta)} , \tag{27}$$

where $\phi_{q_\beta}(r)$ is the S-wave regular solution to the Schrödinger equation and $\mathcal{L}_\pm$ are the two-body Jost functions satisfying[4]

$$\mathcal{L}_\pm(q_\beta) = |\mathcal{L}(q_\beta)| e^{\mp i\delta(q_\beta)} \tag{28}$$

and

$$\mathcal{L}_+(q_\beta) = \mathcal{L}^*_-(q_\beta) . \tag{29}$$

The two-body t matrix has the same phase as the

Jost function and satisfies[4]

$$t_\beta(q_\beta, q'_\beta; \tilde{q}_\beta^{\,2} + i0) = -\frac{1}{q_\beta} e^{i\delta(q_\beta)} r(q_\beta, q'_\beta)\,, \tag{30}$$

where r is a real function. Following KZ we introduce

$$\hat{W}_{\beta\alpha}(p_\beta q_\beta, p_\alpha; z) = \mathcal{L}_+(q_\beta) W_{\beta\alpha}(p_\beta q_\beta, p_\alpha; z)\,. \tag{31}$$

Then an S-wave projection of Eqs. (17) and (18) yields

$$Y_{\beta\alpha}(p_\beta, p_\alpha; z) = \mathcal{V}^{11}_{\beta\alpha}(p_\beta, p_\alpha) + \sum_\gamma \int_0^\infty p'^2_\gamma dp'_\gamma \mathcal{V}^{11}_{\beta\gamma}(p_\beta, p'_\gamma) \frac{1}{z - \tilde{p}'^2_\gamma + \kappa_\gamma^{\,2}} Y_{\gamma\alpha}(p'_\gamma, p_\alpha; z)$$
$$+ \sum_\gamma \int_0^\infty \int_0^\infty p'^2_\gamma dp'_\gamma q'^2_\gamma dq'_\gamma \hat{\mathcal{V}}^{12}_{\beta\gamma}(p_\beta, p'_\gamma q'_\gamma) \frac{|\mathcal{L}(q'_\gamma)|^{-2}}{z - \tilde{p}'^2_\gamma - \tilde{q}'^2_\gamma} \hat{W}_{\gamma\alpha}(p'_\gamma q'_\gamma, p_\alpha; z) \tag{32}$$

and

$$\hat{W}_{\beta\alpha}(p_\beta q_\beta, p_\alpha; z) = \hat{\mathcal{V}}^{21}_{\beta\alpha}(p_\beta q_\beta, p_\alpha) + \sum_\gamma \int_0^\infty p'^2_\gamma dp'_\gamma \hat{\mathcal{V}}^{21}_{\beta\gamma}(p_\beta q_\beta, p'_\gamma) \frac{1}{z - \tilde{p}'^2_\gamma + \kappa_\gamma^{\,2}} Y_{\gamma\alpha}(p'_\gamma, p_\alpha; z)$$
$$+ \sum_\gamma \int_0^\infty \int_0^\infty p'^2_\gamma dp'_\gamma q'^2_\gamma dq'_\gamma \hat{\mathcal{V}}^{22}_{\beta\gamma}(p_\beta q_\beta, p'_\gamma q'_\gamma) \frac{|\mathcal{L}(q'_\gamma)|^{-2}}{z - \tilde{p}'^2_\gamma - \tilde{q}'^2_\gamma} \hat{W}_{\gamma\alpha}(p'_\gamma q'_\gamma, p_\alpha; z)\,. \tag{33}$$

The S-wave components of the effective potential appearing in Eqs. (32) and (33) are defined by

$$\mathcal{V}^{11}_{\beta\gamma}(p_\beta, p_\gamma) = -\tfrac{1}{2} \int_{-1}^{+1} dx\, \phi_\beta(q_{\beta(2)}) (\kappa_\gamma^{\,2} + \tilde{q}_{\gamma(1)}^{\,2}) \phi_\gamma(q_{\gamma(1)}) \bar{\delta}_{\beta\gamma}\,, \tag{34}$$

$$\hat{\mathcal{V}}^{12}_{\beta\gamma}(p_\beta, p_\gamma q_\gamma) = \tfrac{1}{2} \left[\int_{-1}^{+1} dx\, \phi_\beta(q_{\beta(2)}) t^*_\gamma(q_{\gamma(1)}, q_\gamma; \tilde{q}_\gamma^{\,2} + i\epsilon) \right] \mathcal{L}_-(q_\gamma) \bar{\delta}_{\beta\gamma}\,, \tag{35}$$

$$\hat{\mathcal{V}}^{21}_{\beta\gamma}(p_\beta q_\beta, p_\gamma) = -\tfrac{1}{2} \mathcal{L}_+(q_\beta) \int_{-1}^{+1} dx\, \psi^{-*}_{q_\beta}(q_{\beta(2)}) (\kappa_\gamma^{\,2} + \tilde{q}_{\gamma(1)}^{\,2}) \phi_\gamma(q_{\gamma(1)}) \bar{\delta}_{\beta\gamma}\,, \tag{36}$$

and

$$\hat{\mathcal{V}}^{22}_{\beta\gamma}(p_\beta q_\beta, p_\gamma q_\gamma) = \tfrac{1}{2} \mathcal{L}_+(q_\beta) \left[\int_{-1}^{+1} dx\, \psi^{-*}_{q_\beta}(q_{\beta(2)}) t^*_\gamma(q_{\gamma(1)}, q_\gamma, \tilde{q}_\gamma^{\,2} + i\epsilon) \right] \mathcal{L}_-(q_\gamma) \bar{\delta}_{\beta\gamma}\,, \tag{37}$$

where x is the cosine of the angle between $\vec{p}_\beta$ and $\vec{p}_\gamma$.

With this redefinition the effective potentials defined by Eqs. (34)–(37) are real quantities. This can easily be verified by using Eqs. (27)–(30). Hence the effective potentials of Eqs. (32) and (33) are not only independent of three-body energy but also real.

III. NONSINGULAR EQUATIONS

We now study the equations introduced in the last section—Eqs. (32) and (33)—in detail and reduce them to a set of practical nonsingular equations. Such equations will provide a practical way of solving the original Eqs. (32) and (33). The reduction procedure starts with the following change of variables in the double integrals of Eqs. (32) and (33)[6]:

$$\mu_\gamma^{1/4} p'_\gamma = n_\gamma^{1/4} p'_0 \cos\omega'_\gamma\,,$$
$$n_\gamma^{1/4} q'_\gamma = \mu_\gamma^{1/4} p'_0 \sin\omega'_\gamma\,, \tag{38}$$

such that

$$\tilde{p}'^2_\gamma + \tilde{q}'^2_\gamma = p'^2_0 (4 n_\gamma \mu_\gamma)^{-1/2} \equiv \tilde{p}'^2_0\,.$$

Then we have

$$p'^2_\gamma dp'_\gamma q'^2_\gamma dq'_\gamma = p'^5_0 dp'_0 \cos^2\omega'_\gamma \sin^2\omega'_\gamma d\omega'_\gamma \equiv p'^5_0 dp'_0 d\xi'_\gamma\,. \tag{39}$$

In terms of these new variables Eqs. (32) and (33) become

$$Y_{\beta\alpha}(p_\beta,p_\alpha;z)=\mathcal{V}^{11}_{\beta\alpha}(p_\beta,p_\alpha)+\sum_\gamma\int_0^\infty dp'_\gamma p'^2_\gamma\mathcal{V}^{11}_{\beta\gamma}(p_\beta,p'_\gamma)(\tilde{k}_\gamma^2-\tilde{p}'^2_\gamma+i\epsilon)^{-1}Y_{\gamma\alpha}(p'_\gamma,p_\alpha;z)$$
$$+\sum_\gamma\int_0^{\pi/2}d\xi'_\gamma\int_0^\infty dp'_0p'^5_0\mathcal{V}^{12}_{\beta\gamma}(p_\beta,p'_0\omega'_\gamma)(\tilde{k}_0^2-\tilde{p}'^2_0+i\epsilon)^{-1}W_{\gamma\alpha}(p'_0\omega'_\gamma,p_\alpha;z) \tag{40}$$

and

$$W_{\beta\alpha}(p_0\omega_\beta,p_\alpha;z)=\mathcal{V}^{21}_{\beta\alpha}(p_0\omega_\beta,p_\alpha)+\sum_\gamma\int_0^\infty dp'_\gamma p'^2_\gamma\mathcal{V}^{21}_{\beta\gamma}(p_0\omega_\beta,p'_\gamma)(\tilde{k}_\gamma^2-\tilde{p}'^2_\gamma+i\epsilon)^{-1}Y_{\gamma\alpha}(p'_\gamma,p_\alpha;z)$$
$$+\sum_\gamma\int_0^{\pi/2}d\xi'_\gamma\int_0^\infty dp'_0p'^5_0\mathcal{V}^{22}_{\beta\gamma}(p_0\omega_\beta,p'_0\omega'_\gamma)(\tilde{k}_0^2-\tilde{p}'^2_0+i\epsilon)^{-1}W_{\gamma\alpha}(p'_0\omega'_\gamma,p_\alpha;z)\,. \tag{41}$$

Here the carets on various operators have been omitted for simplicity and $|\,(q'_\gamma)|^{-2}$ in the double integral has been absorbed in the potential. [$\mathcal{V}^{12}$ and $\mathcal{V}^{22}$ of Eqs. (40) and (41) are different from those of Eqs. (35) and (37) by a factor of $|\mathcal{L}(q'_\gamma)|^{-2}$.] The unknowns involving $p_\beta q_\beta$ are written in terms of equivalent variables (p_0,ω_β). The on-shell momenta k_γ and k_0 are defined by $\tilde{k}_\gamma^2=z+\kappa_\gamma^2$ and $\tilde{k}_0^2=z$.

Equations (40) and (41) are similar to multichannel scattering equations. In order to complete the analogy with the usual multichannel equations we introduce two other unknowns $S_{\beta\alpha}(p_\beta,p_0\omega_\alpha;z)$ and $T_{\beta\alpha}(p_0\omega_\beta,p'_0\omega_\alpha;z)$. The functions S and T satisfy Eqs. (40) and (41) with $\mathcal{V}^{11}_{\beta\alpha}$, $\mathcal{V}^{12}_{\beta\alpha}$, Y, and W replaced by $\mathcal{V}^{12}_{\beta\alpha}$, $\mathcal{V}^{22}_{\beta\alpha}$, S, and T, respectively. S and T are related to the matrix elements of the AGS transition operators U with $\vec{p}_\alpha\psi^-_{\vec{q}_\alpha}$ in the initial state and we deduce in the Appendix the equations they satisfy after partial-wave projection. The four operators Y, W, S, and T satisfy the following schematic coupled equations (see Appendix for detail):

$$\begin{bmatrix}Y & S\\ W & T\end{bmatrix}=\begin{bmatrix}\mathcal{V}^{11} & \mathcal{V}^{12}\\ \mathcal{V}^{21} & \mathcal{V}^{22}\end{bmatrix}+\begin{bmatrix}\mathcal{V}^{11} & \mathcal{V}^{12}\\ \mathcal{V}^{21} & \mathcal{V}^{22}\end{bmatrix}\begin{bmatrix}G'_0 & 0\\ 0 & G_0\end{bmatrix}\begin{bmatrix}Y & S\\ W & T\end{bmatrix}, \tag{42}$$

where the spectral representations of G'_0 and G_0 are

$$G'_0=\int_0^\infty dp'_\gamma p'^2_\gamma|p'_\gamma\rangle(\tilde{k}_\gamma^2-\tilde{p}'^2_\gamma+i\epsilon)^{-1}\langle p'_\gamma| \tag{43}$$

and

$$G_0=\int_0^{\pi/2}d\xi'_\gamma\int_0^\infty dp'_0p'^5_0|p'_0\omega'_\gamma\rangle(\tilde{k}_0^2-\tilde{p}'^2_0+i\epsilon)^{-1}\langle p'_0\omega'_\gamma|\,. \tag{44}$$

The only singularity of Eq. (42) appears in the resolvent operators of Eqs. (43) and (44). The energy denominators of Eqs. (43) and (44) have the same form as that in the two-body case. Hence we generalize a recently proposed method[7] for writing nonsingular two-body equations in the momentum space to the case of the three-body equations. We rewrite Eq. (42) in such a way that a part of the kernel is explicitly nonsingular. Then we write a nonsingular equation with this piece of the kernel and relate the solution of the original equation to that of the nonsingular equation. We explicitly demonstrate this for one of the equations—Eq. (40)—of Eq. (42). Equation (40) is rewritten as

$$Y_{\beta\alpha}(p_\beta,p_\alpha;z)=\mathcal{V}^{11}_{\beta\alpha}(p_\beta,p_\alpha)+\sum_\gamma\int_0^\infty dp'_\gamma p'^2_\gamma A^{11}_{\beta\gamma}(p_\beta,p'_\gamma,k_\gamma)Y_{\gamma\alpha}(p'_\gamma,p_\alpha;z)$$
$$+\sum_\gamma\int_0^{\pi/2}d\xi'_\gamma\int_0^\infty dp'_0p'^5_0A^{12}_{\beta\gamma}(p_\beta,p'_0\omega'_\gamma,k_0)W_{\gamma\alpha}(p'_0\omega'_\gamma,p_\alpha;z)$$
$$+\sum_\gamma\mathcal{V}^{11}_{\beta\gamma}(p_\beta,k_\gamma)\int_0^\infty dp'_\gamma p'^2_\gamma(\tilde{k}_\gamma^2-\tilde{p}'^2_\gamma+i\epsilon)^{-1}Y_{\gamma\alpha}(p'_\gamma,p_\alpha;z)$$
$$+\sum_\gamma\int_0^{\pi/2}d\xi'_\gamma\mathcal{V}^{12}_{\beta\gamma}(p_\beta,k_0\omega'_\gamma)\int_0^\infty dp'_0p'^5_0(\tilde{k}_0^2-\tilde{p}'^2_0+i\epsilon)^{-1}W_{\gamma\alpha}(p'_0\omega'_\gamma,p_\alpha;z)\,, \tag{45}$$

where

$$A^{11}_{\beta\gamma}(p_\beta, p'_\gamma, k_\gamma) = [\mathcal{V}^{11}_{\beta\gamma}(p_\beta, p'_\gamma) - \mathcal{V}^{11}_{\beta\gamma}(p_\beta, k_\gamma)] \times (\tilde{k}_\gamma{}^2 - \tilde{p}_\gamma{}'^2 + i\epsilon)^{-1} \tag{46}$$

and

$$A^{12}_{\beta\gamma}(p_\beta, p'_0\omega'_\gamma, k_0) = [\mathcal{V}^{12}_{\beta\gamma}(p_\beta, p'_0\omega'_\gamma) - \mathcal{V}^{12}_{\beta\gamma}(p_\beta, k_0\omega'_\gamma)] \times (\tilde{k}_0{}^2 - \tilde{p}_0'^2 + i\epsilon)^{-1} \tag{47}$$

are the nonsingular kernels of Eq. (45). We can write four such equations corresponding to the four equations of Eq. (42). These four equations of type (45) can be combined to yield the following equation in schematic matrix notation

$$\begin{bmatrix} Y & S \\ W & T \end{bmatrix} = \begin{bmatrix} \mathcal{V}^{11} & \mathcal{V}^{12} \\ \mathcal{V}^{21} & \mathcal{V}^{22} \end{bmatrix} + \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix} \begin{bmatrix} Y & S \\ W & T \end{bmatrix} + \begin{bmatrix} \mathcal{W}^{11} & \mathcal{W}^{12} \\ \mathcal{W}^{21} & \mathcal{W}^{22} \end{bmatrix} \begin{bmatrix} G'_0 & 0 \\ 0 & G_0 \end{bmatrix} \begin{bmatrix} Y & S \\ W & T \end{bmatrix}, \tag{48}$$

where

$$A^{21}_{\beta\gamma}(p_0\omega_\beta, p'_\gamma, k_\gamma) = [\mathcal{V}^{21}_{\beta\gamma}(p_0\omega_\beta, p'_\gamma) - \mathcal{V}^{21}_{\beta\gamma}(p_0\omega_\beta, k_\gamma)] \times (\tilde{k}_\gamma{}^2 - \tilde{p}_\gamma'^2 + i\epsilon)^{-1}, \tag{49}$$

$$A^{22}_{\beta\gamma}(p_0\omega_\beta, p'_0\omega'_\gamma, k_0) = [\mathcal{V}^{22}_{\beta\gamma}(p_0\omega_\beta, p'_0\omega'_\gamma) - \mathcal{V}^{22}_{\beta\gamma}(p_0\omega_\beta, k_0\omega'_\gamma)] \times (\tilde{k}_0{}^2 - \tilde{p}_0'^2 + i\epsilon)^{-1}, \tag{50}$$

$$\mathcal{W}^{11}_{\beta\gamma}(p_\beta, p'_\gamma) = \mathcal{V}^{11}_{\beta\gamma}(p_\beta, k_\gamma), \tag{51}$$

$$\mathcal{W}^{12}_{\beta\gamma}(p_\beta, p'_0\omega'_\gamma) = \mathcal{V}^{12}_{\beta\gamma}(p_\beta, k_0\omega'_\gamma), \tag{52}$$

$$\mathcal{W}^{21}_{\beta\gamma}(p_0\omega_\beta, p'_\gamma) = \mathcal{V}^{21}_{\beta\gamma}(p_0\omega_\beta, k_\gamma), \tag{53}$$

and

$$\mathcal{W}^{22}_{\beta\gamma}(p_0\omega_\beta, p'_0\omega'_\gamma) = \mathcal{V}^{22}_{\beta\gamma}(p_0\omega_\beta, k_0\omega'_\gamma). \tag{54}$$

It is to be noted that Eqs. (42) and (48) are essentially the same equations. In Eq. (48) the kernel of Eq. (42) has been broken into two parts. The part containing A obviously does not have the pole of the resolvent operator whereas the singular part is contained in the last term on the right-hand side of Eq. (48).

Now following Ref. 7 we define the following nonsingular equation in schematic matrix notation:

$$\begin{bmatrix} \overline{Y} & \overline{S} \\ \overline{W} & \overline{T} \end{bmatrix} = \begin{bmatrix} \mathcal{V}^{11} & \mathcal{V}^{12} \\ \mathcal{V}^{21} & \mathcal{V}^{22} \end{bmatrix} + \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix} \begin{bmatrix} \overline{Y} & \overline{S} \\ \overline{W} & \overline{T} \end{bmatrix}. \tag{55}$$

Equation (56) contains four equations of which the first one reads

$$\overline{Y}_{\beta\alpha}(p_\beta, p_\alpha; z) = \mathcal{V}^{11}_{\beta\alpha}(p_\beta, p_\alpha) + \sum_\gamma \int_0^\infty dp'_\gamma p'^2_\gamma A^{11}_{\beta\gamma}(p_\beta, p'_\gamma, k_\gamma) \overline{Y}_{\gamma\alpha}(p'_\gamma, p_\alpha; z) + \sum_\gamma \int_0^{\pi/2} d\xi'_\gamma \int_0^\infty dp'_0 p_0'^5 A^{12}_{\beta\gamma}(p_\beta, p'_0\omega'_\gamma, k_0) \overline{W}_{\gamma\alpha}(p'_0\omega'_\gamma, p_\alpha; z). \tag{56}$$

The other equations of (55) can be written down similarly. Now following Ref. 7 we can relate the unknowns of Eqs. (48) and (55) and we have in schematic notation

$$\begin{bmatrix} Y & S \\ W & T \end{bmatrix} = \begin{bmatrix} \overline{Y} & \overline{S} \\ \overline{W} & \overline{T} \end{bmatrix} + \begin{bmatrix} \tilde{Y} & \tilde{S} \\ \tilde{W} & \tilde{T} \end{bmatrix} \begin{bmatrix} G'_0 & 0 \\ 0 & G_0 \end{bmatrix} \begin{bmatrix} Y & S \\ W & T \end{bmatrix}, \tag{57}$$

where $\tilde{Y}$'s are related to $\overline{Y}$'s by

$$\tilde{Y}_{\beta\gamma}(p_\beta, p'_\gamma; z) = \overline{Y}_{\beta\gamma}(p_\beta, k_\gamma; z), \tag{58}$$

$$\tilde{S}_{\beta\gamma}(p_\beta, p'_0\omega'_\gamma; z) = \overline{S}_{\beta\gamma}(p_\beta, k_0\omega'_\gamma; z), \tag{59}$$

$$\tilde{W}_{\beta\gamma}(p_0\omega_\beta, p_\gamma; z) = \overline{W}_{\beta\gamma}(p_0\omega_\beta, k_\gamma; z), \tag{60}$$

and

$$\tilde{T}_{\beta\gamma}(p_0\omega_\beta, p'_0\omega'_\gamma; z) = \overline{T}_{\beta\gamma}(p_0\omega_\beta, k_0\omega'_\gamma; z). \tag{61}$$

Equation (57) contains four equations. The two equations for Y and W now become in explicit notation

$$Y_{\beta\alpha}(p_\beta, p_\alpha; z) = \overline{Y}_{\beta\alpha}(p_\beta, p_\alpha; z) + \sum_\gamma \overline{Y}_{\beta\gamma}(p_\beta, k_\gamma; z) D_{\gamma\alpha}(p_\alpha; z) + \sum_\gamma \int_0^{\pi/2} d\xi'_\gamma \overline{S}_{\beta\gamma}(p_\beta, k_0\omega'_\gamma; z) E_{\gamma\alpha}(\omega'_\gamma, p_\alpha; z), \tag{62}$$

and

$$W_{\beta\alpha}(p_0\omega_\beta, p_\alpha; z) = \overline{W}_{\beta\alpha}(p_0\omega_\beta, p_\alpha; z) + \sum_\gamma \overline{W}_{\beta\gamma}(p_0\omega_\beta, k_\gamma; z) D_{\gamma\alpha}(p_\alpha; z) + \sum_\gamma \int_0^{\pi/2} d\xi'_\gamma \overline{T}_{\beta\gamma}(p_0\omega_\beta, k_0\omega'_\gamma; z) \times E_{\gamma\alpha}(\omega'_\gamma, p_\alpha; z), \tag{63}$$

where

$$D_{\gamma\alpha}(p_\alpha;z)=\int_0^\infty dp'_\gamma p'^2_\gamma(\tilde{k}_\gamma^2-\tilde{p}'^2_\gamma+i\epsilon)^{-1}\times Y_{\gamma\alpha}(p'_\gamma,p_\alpha;z) \tag{64}$$

and

$$E_{\gamma\alpha}(\omega_\gamma,p_\alpha;z)=\int_0^\infty dp'_0 p'^5_0(\tilde{k}_0^2-\tilde{p}'^2_0+i\epsilon)^{-1}\times W_{\gamma\alpha}(p'_0\omega_\gamma,p_\alpha;z)\,. \tag{65}$$

So at the moment the present formulation depends on solving the nonsingular equations (55), (62), and (63).

It is to be noted that the structures of Eqs. (62) and (63) are much simpler than original equations (32) and (33) once the nonsingular equations (55) for $\overline{Y}$'s are solved in advance. To exploit the simple structure of (62) and (63) we write a set of equations for the two unknowns D and E appearing in (62) and (63). Multiplying Eqs. (62) and (63) by obvious appropriate factors from the left and integrating we get

$$D_{\beta\alpha}(p_\alpha;z)=\overline{D}_{\beta\alpha}(p_\alpha;z)+\sum_\gamma\overline{D}_{\beta\gamma}(k_\gamma;z)D_{\gamma\alpha}(p_\alpha;z)+\sum_\gamma\int_0^{\pi/2}d\xi'_\gamma F_{\beta\gamma}(\omega'_\gamma;z)E_{\gamma\alpha}(\omega'_\gamma,p_\alpha;z) \tag{66}$$

and

$$E_{\beta\alpha}(\omega_\beta,p_\alpha;z)=\overline{E}_{\beta\alpha}(\omega_\beta,p_\alpha;z)+\sum_\gamma\overline{E}_{\beta\gamma}(\omega_\beta,k_\gamma;z)D_{\gamma\alpha}(p_\alpha;z)+\sum_\gamma\int_0^{\pi/2}d\xi'_\gamma J_{\beta\gamma}(\omega_\beta,\omega'_\gamma;z)\times E_{\gamma\alpha}(\omega'_\gamma,p_\alpha;z)\,, \tag{67}$$

where $\overline{D}$ and $\overline{E}$ satisfy (64) and (65) with Y and W replaced by $\overline{Y}$ and $\overline{W}$, respectively. The functions F and J are defined by

$$F_{\gamma\alpha}(\omega'_\alpha;z)=\int_0^\infty dp'_\gamma p'^2_\gamma(\tilde{k}_\gamma^2-\tilde{p}'^2_\gamma+i\epsilon)^{-1}\overline{S}_{\gamma\alpha}(p'_\gamma,k_0\omega'_\alpha;z) \tag{68}$$

and

$$J_{\gamma\alpha}(\omega_\gamma,\omega'_\alpha;z)=\int_0^\infty dp'_0p'^5_0(\tilde{k}_0^2-\tilde{p}'^2_0+i\epsilon)^{-1}\times\overline{T}_{\gamma\alpha}(p'_0\omega_\gamma,k_0\omega'_\alpha;z)\,. \tag{69}$$

The basic equations of the present formalism are Eqs. (55), (66), (67), (62), and (63).

Equations (66) and (67) are coupled integral-linear equations in E and D. The kernel and the Born terms of these equations are complex quantities. The kernel is a smooth function of the variables and is also nonsingular in nature. Hence in principle Eqs. (66) and (67) reduce essentially to an integral equation in one variable, with smooth kernel that is easy to solve in practice. We have reduced the original equations—Eqs. (32) and (33)—to two sets of nonsingular equations.

Another important feature of Eq. (55) apart from the fact that its kernel is nonsingular is that the kernel is sufficiently weak compared to the kernel of the original equations. Such kernel has been tested analytically and numerically in two-body problems to give rapid convergence for the iterative series[8] and is one of a wider class of nonsingular kernels. Hence Eq. (55) is expected to give a rapidly convergent iterative solution.

It is not clear at present whether an approximate real solution of Eq. (55) when used in Eqs. (62), (63), (66), and (67) will yield a scattering amplitude satisfying constraints of unitarity. Careful study of the unitarity relation is needed to make unitary approximations.

IV. DISCUSSION AND CONCLUSION

In this note we consider a representation of the AGS equation using the complete set of channel eigenstates as basis functions. The resulting equations have simple formal structure. The input to these equations are the two-body wave functions and the half-off-shell two-body t matrices. The effective potentials are energy independent and can be made real after a partial-wave analysis.

KZ wrote similar equations not for the AGS operators but for the operators K which are related to the AGS operators U by

$$K_{\beta\alpha}=V_\beta G_0 U_{\beta\alpha}G_0V_\alpha\,. \tag{70}$$

The operators K satisfy

$$K_{\beta\alpha}=\overline{\delta}_{\beta\alpha}V_\beta G_0V_\alpha+\sum_\gamma\overline{\delta}_{\beta\gamma}V_\beta G_\gamma K_{\gamma\alpha}\,. \tag{71}$$

On-shell the operator K agrees with the more commonly used AGS operators U but off-shell values of K are not easily related to the corresponding values of U. It was not *a priori* clear that KZ-type equations with real energy-independent potentials can be written with the more commonly used AGS transition amplitudes. We explicitly derive these equations here and show that this is indeed the case.

Next we discuss the utility of such equations in writing a nonsingular representation. The analytic structure of the equations we obtain is very simple in comparison with that of the usual plane-wave representation of the three-body equations. It is well known that the moving logarithmic singularities in the kernel and in the Born term of the usual momentum-space three-body equations, even in a simple separable potential model make a numerical solution difficult in practice. But in the present case after partial-wave decomposition the Born term is three-body energy independent and real whereas the kernel is shown to have a fixed-point singularity as in that of a two-body Lippmann-Schwinger equation. Hence all the two-body techniques for writing nonsingular equations can be easily extended to the present case. We use a recently developed technique for writing nonsingular two-body equations to this case. The method[7] relies on relating the solution of the present set of equations to that of an auxiliary nonsingular set of equations whose kernel is free from the fixed-point singularity of the original equation.

The kernel of the auxiliary nonsingular set of equations, defined by Eqs. (46), (47), (49), and (50), can be made more general by the introduction of an arbitrary function γ as in Ref. 7. But in order to present our point of view in a clear and transparent way we did not introduce the function γ, which will only inhibit this purpose and complicate the method formally and algebraically. In principle it is simple to introduce this function. For example, Eq. (46) should be redefined as

$$A^{11}_{\beta\gamma}(p_\beta,p'_\gamma,k_\gamma)=[\mathcal{V}^{11}_{\beta\gamma}(p_\beta,p'_\gamma)-\mathcal{V}^{11}_{\beta\gamma}(p_\beta,k_\gamma)\gamma^{11}(k_\gamma,p'_\gamma)]\times(\tilde{k}_\gamma^{2}-\tilde{p}'^{2}_\gamma+i\epsilon)^{-1} \qquad (72)$$

with $\gamma^{11}(k_\gamma,k_\gamma)=1$. Other equations of Sec. III should be modified in a way indicated in Ref. 7. It is easy to see that the kernel A defined by Eq. (72), like the one defined by Eq. (46), apart from being nonsingular is also weaker in nature compared to the original kernel. Hence the auxiliary equation will probably have a convergent iterative solution. In an actual implementation of the method the function γ should be varied in order to achieve and/or improve the rate of convergence of the iterative solution of the auxiliary equation. Such an iterative solution, apart from being simple to implement, is also interesting in practice because any approximate real (iterative) solution of the auxiliary nonsingular equation may lead to schemes for making unitary approximations. Various approximation schemes will probably emerge in the future based on the present and related approaches.[6,9]

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APPENDIX

We introduce

$$S_{\beta\alpha}(\vec{p}_\beta,\vec{p}_\alpha\vec{q}_\alpha;z)=\langle\vec{p}_\beta\phi_\beta\,|\,U_{\beta\alpha}\,|\,\vec{p}_\alpha\psi^-_{\vec{q}_\alpha}\rangle \qquad (A1)$$

and

$$T_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta,\vec{p}_\alpha\vec{q}_\alpha;z)=\langle\vec{p}_\beta\psi^-_{\vec{q}_\beta}\,|\,U_{\beta\alpha}\,|\,\vec{p}_\alpha\psi^-_{\vec{q}_\alpha}\rangle, \qquad (A2)$$

and find the equations they satisfy after S-wave projection. From Eq. (16) it is easy to see that S and T satisfy Eqs. (17) and (18) with Y, $Y^{(0)}$, W, and $W^{(0)}$ replaced by S, $S^{(0)}$, T, and $T^{(0)}$, respectively, where $S^{(0)}$ and $T^{(0)}$ are defined by

$$S^{(0)}_{\beta\alpha}(\vec{p}_\beta,\vec{p}_\alpha\vec{q}_\alpha;z)=\phi_\beta(\vec{q}_{\beta(2)})\times t^*_\alpha(\vec{q}_{\alpha(1)},\vec{q}_\alpha;\tilde{q}_\alpha^{\,2}+i\epsilon) \qquad (A3)$$

and

$$T^{(0)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta,\vec{p}_\alpha\vec{q}_\alpha;z)=\psi^{-*}_{\vec{q}_\beta}(\vec{q}_{\beta(2)})\times t^*_\alpha(\vec{q}_{\alpha(1)},\vec{q}_\alpha,\tilde{q}_\alpha^{\,2}+i\epsilon). \qquad (A4)$$

Next we consider as in Sec. II a simple case with zero three-body total angular momentum and S-wave two-body interactions only. As in Eq. (31) we define new quantities S and T by

$$\hat{S}_{\beta\alpha}(p_\beta,p_\alpha q_\alpha;z)=S_{\beta\alpha}(p_\beta,p_\alpha q_\alpha;z)\mathcal{L}_+^{-1}(q_\alpha), \qquad (A5)$$

$$\hat{T}_{\beta\alpha}(p_\beta q_\beta,p_\alpha q_\alpha;z)=\mathcal{L}_+(q_\beta)T_{\beta\alpha}(p_\beta q_\beta,p_\alpha q_\alpha;z)\times\mathcal{L}_+^{-1}(q_\alpha). \qquad (A6)$$

Then as in Sec. II an S-wave projection of the equations for S and T becomes

$$\hat{S}_{\beta\alpha}(p_\beta,p_\alpha q_\alpha;z)=\hat{\mathcal{V}}^{12}_{\beta\alpha}(p_\beta,p_\alpha q_\alpha)\,|\mathcal{L}(q_\alpha)|^{-2}+\sum_\gamma\int_0^\infty p'^2_\gamma dp'_\gamma\mathcal{V}^{11}_{\beta\gamma}(p_\beta,p'_\gamma)\frac{1}{z-\tilde{p}'^2_\gamma+\kappa_\gamma^2}S_{\gamma\alpha}(p'_\gamma,p_\alpha q_\alpha;z)$$
$$+\sum_\gamma\int_0^\infty\int_0^\infty p'^2_\gamma dp'_\gamma q'^2_\gamma dq'_\gamma\hat{\mathcal{V}}^{12}_{\beta\gamma}(p_\beta,p'_\gamma q'_\gamma)\frac{|\mathcal{L}(q'_\gamma)|^{-2}}{z-\tilde{p}'^2_\gamma-\tilde{q}'^2_\gamma}\hat{T}_{\gamma\alpha}(p'_\gamma q'_\gamma,p_\alpha q_\alpha;z) \qquad (A7)$$

and

$$\hat{T}_{\beta\alpha}(p_\beta q_\beta,p_\alpha q_\alpha;z)=\hat{\mathcal{V}}^{22}_{\beta\alpha}(p_\beta q_\beta,p_\alpha q_\alpha)\,|\mathcal{L}(q_\alpha)|^{-2}+\sum_\gamma\int_0^\infty p'^2_\gamma dp'_\gamma\hat{\mathcal{V}}^{21}_{\beta\gamma}(p_\beta q_\beta,p'_\gamma)\frac{1}{z-\tilde{p}'^2_\gamma+\kappa_\gamma^2}S_{\gamma\alpha}(p'_\gamma,p_\alpha q_\alpha;z)$$
$$+\sum_\gamma\int_0^\infty\int_0^\infty p'^2_\gamma dp'_\gamma q'^2_\gamma dq'_\gamma\hat{\mathcal{V}}^{22}_{\beta\gamma}(p_\beta q_\beta,p'_\gamma q'_\gamma)\frac{|\mathcal{L}(q'_\gamma)|^{-2}}{z-\tilde{p}'^2_\gamma-\tilde{q}'^2_\gamma}\hat{T}_{\gamma\alpha}(p'_\gamma q'_\gamma,p_\alpha q_\alpha;z). \qquad (A8)$$

Omitting the carets on various operators and absorbing the factor of $|\mathcal{L}(q)|^{-2}$ in the potential as in Eqs. (40) and (41), Eqs. (A7) and (A8) can be explicitly written [in terms of new variables of Eqs. (38) and (39)] as

$$S_{\beta\alpha}(p_\beta, k_0\omega_\alpha; z) = \mathcal{V}^{12}_{\beta\alpha}(p_\beta, k_0\omega_\alpha) + \sum_\gamma \int_0^\infty dp'_\gamma p'^2_\gamma \mathcal{V}^{11}_{\beta\gamma}(p_\beta, p'_\gamma)(\tilde{k}_\gamma^2 - p'^2_\gamma + i\epsilon)^{-1} S_{\gamma\alpha}(p'_\gamma, k_0\omega_\alpha; z)$$

$$+ \sum_\gamma \int_0^{\pi/2} d\xi'_\gamma \int_0^\infty dp'_0 p'^5_0 \mathcal{V}^{12}_{\beta\gamma}(p_\beta, p'_0\omega'_\gamma)(\tilde{k}_0^2 - \tilde{p}'^2_0 + i\epsilon)^{-1} T_{\gamma\alpha}(p'_0\omega'_\gamma, k_0\omega_\alpha; z) \quad \text{(A9)}$$

and

$$T_{\beta\alpha}(p_0\omega_\beta k_0\omega_\alpha; z) = \mathcal{V}^{22}_{\beta\alpha}(p_0\omega_\beta, k_0\omega_\alpha) + \sum_\gamma \int_0^\infty dp'_\gamma p'^2_\gamma \mathcal{V}^{21}_{\beta\gamma}(p_0\omega_\beta, p'_\gamma)(\tilde{k}_\gamma^2 - \tilde{p}'^2_\gamma + i\epsilon)^{-1} S_{\gamma\alpha}(p'_\gamma, k_0\omega_\alpha; z)$$

$$+ \sum_\gamma \int_0^{\pi/2} d\xi'_\gamma \int_0^\infty dp'_0 p'^5_0 \mathcal{V}^{22}_{\beta\gamma}(p_0\omega_\beta, p'_0\omega'_\gamma)(\tilde{k}_0^2 - \tilde{p}'^2_0 + i\epsilon)^{-1} T_{\gamma\alpha}(p'_0\omega'_\gamma, k_0\omega_\alpha; z)\,. \quad \text{(A10)}$$

Equations (40), (41), (A9), and (A10) are combined to give the compact operator form (42).

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