PHYSICAL REVIEW D VOLUME 21, NUMBER 8 15 APRIL 1980

Thickening the string. I. The string perfect dust*

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(Received 10 July 1979)

The classical theory of the geometrical string is developed as the theory of a simple, surface-forming timelike bivector field in an arbitrary background space-time. The stress-energy tensor for a perfect dust of such strings is written down, and the conservation laws for such a dust, as well as the equations of motion of the string, are derived from the vanishing of the divergence of the stress-energy tensor. (The boundary conditions for the open string are also derived from the junction conditions for the stress-energy tensor in Appendix A.) The generalization of this model to null strings, and to a perfect fluid of strings, are discussed, and will form the subject of the second and third papers in this series. The problem of a fully general-relativistic string theory, and an alternate approach to the string, based upon defining an acceleration tensor for two- (and higher) dimensional subspaces, are also discussed.

I. INTRODUCTION

In recent years there has been extensive discussion of the theory of the geometrical string, primarily because of its application as a possible interpretation of the dual resonance model of elementary particles.[1] However, the theory has also been discussed on the classical level[2]; and, indeed, from this point of view it provides a remarkably natural generalization of the relativistic theory of a structureless point particle. This generalization could have been made at any time after 1905, and it is rather surprising that such a natural structure should not have been investigated much earlier. This paper will be exclusively concerned with the classical theory of the string; the hope of generalizing special-relativistic string theory into a fully general-relativistic one provides its ultimate motivation.[3]

The geometrical string will be treated as the theory of a surface-forming simple bivector field, subject to field equations which determine the surface. This bivector field is usually treated mathematically by introducing a parametrization of the surface, and a pair of linearly independent vectors, derived from the parametrization, which span the surface. While there is nothing wrong with this procedure, it may tend to make one lose sight of the fact that the resulting theory must be invariant under the family of possible reparametrizations of the surface which leave the bivector unaltered. However, it is equally simple to treat the bivector field intrinsically, as we shall demonstrate, and perhaps more natural to do so, in the sense that it is more natural to discuss any geometrical figure intrinsically rather than via its components with respect to some basis.

The next section will review some well-known mathematical results on simple surface-forming bivector fields that will be needed in the following work.[4] In Sec. III, we shall review the "thickening" of a point particle provided by the incoherent dust model of matter, and recall how the equations of motion and conservation law for the dust may be derived from the vanishing of the divergence of its stress-energy tensor. We shall do the calculations in a way that brings out the parallel to our treatment of the string case.

Section IV treats the thickened string by introducing an incoherent dust model; again, the vanishing of the divergence of the stress-energy tensor leads to the correct equations of motion for the string, as well as a conservation law.

In conclusion, we shall discuss why the thickened string seems a good starting point for developing a properly general-relativistic string theory. The possibility of extending the approach of this paper to null strings, and of developing a theory of elastically interacting strings—in particular of a perfect fluid of strings—will also be mentioned. Finally, an alternative treatment of string as subspaces with a vanishing trace of the acceleration three-tensor will be indicated.

Two appendices treat the boundary conditions for an open string, and the reduction of the bivector conservation law to two vectorial conservation laws.

II. SIMPLE, SURFACE-FORMING BIVECTOR FIELDS

We shall carry out our discussion using ordinary tensor analysis,[4] although the discussion often could be done more elegantly by using the language of differential forms.[5]

A bivector is, of course, an antisymmetric tensor of second rank. Since we shall always work in a four-dimensional space-time with metric of Lorentz signature $(+---)$, we shall not always emphasize the properly covariant or contravariant nature of the entities we consider; but we should think of our bivectors as fundamentally contravariant, since we want them to span surface elements. Thus, if the bivector is simple—that is, may be written as the alternating product of two (contra) vectors:

$$S^{\mu\nu}=\xi^\mu\eta^\nu-\xi^\nu\eta^\mu, \quad (2.1)$$

—it defines a surface element spanned by the two vectors $\vec{\xi}$ and $\vec{\eta}$.[6] It is easily proved[4] that a bivector is simple if and only if it obeys the algebraic condition

$$S^{[\mu\nu}S^{\kappa]\lambda}=0, \quad (2.2)$$

where square brackets indicate total antisymmetrization over all indices included (we shall occasionally indicate that an index is to be omitted from antisymmetrization by enclosing it between upright lines). This is clearly equivalent to

$$S^{\mu\nu}S^{\kappa\lambda}\epsilon_{\mu\nu\kappa\delta}=0, \quad (2.2a)$$

where $\epsilon_{\mu\nu\kappa\delta}$ is the four-dimensional Levi-Civita totally antisymmetric tensor density. Thus, if we define the dual (properly covariant) bivector density $*S_{\mu\nu}$ by

$$*S_{\mu\nu}=\tfrac{1}{2}S^{\kappa\lambda}\epsilon_{\mu\nu\kappa\lambda}, \quad (2.3)$$

we may also rewrite (2.2) as

$$*S_{\mu\nu}S^{\nu\kappa}=0. \quad (2.2b)$$

Now suppose that, instead of a simple bivector at a point of space-time, we are given a simple bivector field over some region of space-time. The surface elements so defined at each point of the region may or may not mesh together smoothly into families of two-surfaces. Again, it can be shown easily[4] that the condition for a simple bivector field to be surface forming is

$$*S_{\mu\nu}\,\partial_\kappa S^{\nu\kappa}=0. \quad (2.4)$$

It follows as a consequence of (2.2b) and (2.4) (as indeed it must), that any multiple of a field of surface-forming simple bivectors is also such a field. In particular, we may multiply the bivector by a scalar density field $\tilde{\rho}$, in which case it will follow that

$$*S_{\mu\nu}\partial_\kappa(\tilde{\rho}S^{\nu\kappa})=0. \quad (2.5)$$

Since the ordinary divergence of a contravariant bivector density is a vector density, each term entering the product in (2.5) is a tensorial entity, while this is not true of the second term in the product (2.4).

The major result of this section that we shall need is that a simple surface-forming bivector field can be intrinsically characterized by Eqs. (2.2b) and (2.5).

We shall also later need the following lemma: If a vector has a vanishing contraction with a timelike simple bivector and its dual, then the vector itself must be the zero vector.

By a timelike simple bivector, we mean one which spans a timelike two-surface element. This means it can be written as the alternating product of a timelike and spacelike vector. Its dual will then be a simple spacelike bivector. This means that the dual can be written as the alternating product of two spacelike vectors, which will be linearly independent of the two vectors spanning the original bivector. Thus, the four vectors spanning the bivector and its dual form a basis for the four-dimensional tangent vector space at the point of space-time in question. Thus, the vanishing of the contraction of any vector with both the bivector and its dual means that the vector has vanishing projections on all four vectors of a basis, and is thus the zero vector.

III. THICKENING OF A POINT PARTICLE

The equation of motion of a free monopole point particle is that of a timelike geodesic of the space-time metric. The geodesic equation can be put into two equivalent forms, one depending explicitly on a parametrization of the world line: $x^\mu=x^\mu(\lambda)$ with

$$\frac{d^2x^\mu}{d\lambda^2}+\left\{{\mu \atop \kappa\nu}\right\}\frac{dx^\kappa}{d\lambda}\frac{dx^\nu}{d\lambda}=0,$$

if we used the preferred (affine) parametrization of the world line. The other form is independent of parametrization, and uses the unit tangent vector V^λ to the geodesic:

$$V^\kappa\nabla_\kappa V_\nu=0, \quad V^\kappa V_\kappa=1.$$

Of course, a link between the two is provided by the explicit parametrization of the $\vec{V}$ field: $V^\mu=dx^\mu/d\lambda$; but the fact remains that the theory of geodesic vector fields can be developed without explicit introduction of parametrization. When dealing with a single vector field, the result is trivial. However, in the string case, the choice of different parametrizations of the surface will correspond to spanning the tangent space at each point by different pairs of vector fields; and the fact that one can develop the theory without any parametrization is slightly less trivial.

Together with this equation of motion, we have the (trivial) conservation law that m_0, the mass associated with the particle, is constant along its world line. The equation of motion can be derived from a variational principle by demanding that the proper time along the (timelike) world line be an extremal with respect to small variations of the world line, if we treat the mass m_0 as a constant along the world line. However, the result may be derived from the vanishing of the divergence of the stress-energy tensor for such a particle, treated as a limiting case of a perfect-dust stress-energy tensor in which the density of the dust is taken as a δ function centered on the world line. In this case, the constancy of the rest mass also follows from the vanishing of the divergence of the stress tensor for the dust.[7] But we need not go to the point limit: One can obtain these results for a "thickened" point particle. By this, we mean a perfect dust, for which the stress-energy tensor is not a δ function, but is nonvanishing only inside of some timelike world tube of finite cross section. Indeed, in so far as we look upon a classical macroscopic treatment of a "point particle" as some sort of approximation to the behavior of an extended body (either in the limit when we are very far away from the body compared to its characteristic extension, or in the limit in which we shrink the size of the body to zero while staying at a fixed distance from it), this seems a natural way of treating such a particle. In any case (as we shall discuss in more detail in the last section), we are forced to some such considerations if we wish to consider the body as a source of gravitational field in general relativity, because the structure of the field near a source simply does not allow interpretation as emanating from a point singularity.[8]

We shall review the theory of such a "thickened" point particle in a form which brings out the similarity to the treatment of the "thickened string." We take the stress-energy tensor of the perfect dust forming the thickened particle to be

$$T^{\mu\nu}=v^\mu v^\nu , \tag{3.1}$$

where v^μ is a future-pointing timelike vector field which is nonvanishing inside some timelike world tube (we restrict ourselves to a single tube, the generalization to several tubes is trivial). Clearly, we can always write v^μ as

$$v^\mu=(\rho)^{1/2}V^\mu, \quad V^\mu V_\mu=1 \tag{3.2}$$

(we use a signature -2), so that

$$T^{\mu\nu}=\rho V^\mu V^\nu, \quad \rho\geqslant 0 . \tag{3.3}$$

Actually, it will prove better to use the mixed form of the stress-energy tensor density (basically, this is because its divergence will then be a force density, which is properly a covariant quantity):

$$\tilde{T}^\nu_\mu=\tilde{\rho}V_\mu V^\nu , \tag{3.3a}$$

where we have put the tensorial density factor into $\tilde{\rho}$. Now, we take the divergence of (3.3a) and set it equal to zero:

$$\nabla_\nu\tilde{T}^\nu_\mu=\partial_\nu(\tilde{\rho}V^\nu)V_\mu+\tilde{\rho}V^\nu\nabla_\nu V_\mu=0 . \tag{3.4}$$

We have here taken advantage of the fact that the covariant divergence of a contravariant vector density coincides with its ordinary divergence. We can also remove the covariant derivative from the second term of (3.4) by noting that the normalization of V^μ, Eq. (3.2), implies that $V^\nu\nabla_\mu V_\nu=0$, so that

$$V^\nu\nabla_\nu V_\mu=V^\nu(\partial_\nu V_\mu-\partial_\mu V_\nu) \tag{3.5}$$

(again taking advantage of the fact that the antisymmetrized convariant derivative of a covariant vector equals its ordinary curl). Thus, we may rewrite (3.4) in a form free of covariant derivatives[9]:

$$\nabla_\nu\tilde{T}^\nu_\mu=\partial_\nu(\tilde{\rho}V^\nu)V_\mu+V^\nu(\partial_\nu V_\mu-\partial_\mu V_\nu)=0 . \tag{3.4a}$$

Now, contracting (3.4) with V^μ, we see that the second term vanishes, while $V^\mu V_\mu=1$, so that

$$V^\mu\nabla_\nu\tilde{T}^\nu_\mu=\partial_\nu(\tilde{\rho}V^\nu)=0 . \tag{3.6}$$

This is the differential form of the conservation law for the world tube, to which we shall return in a moment. Turning back to (3.4), we see that the first term vanishes; the vanishing of the second term is just the condition that the V^μ field be the unit tangent to a family of timelike geodesics which must thus fill up, and form the boundary of, the world tube. While the fact that $V^\nu\nabla_\nu V^\mu=0$ is equivalent to the geodesic equation is well known, we shall sketch the derivation here, just for comparison with the string case in the next section. Since V^μ is tangent to a timelike world line, we can write it as

$$V^\mu=\frac{dx^\mu}{d\lambda} , \tag{3.6a}$$

where λ is a parameter along the world line which, from the normalization of V^μ, Eq. (3.2), is seen to be the proper time. Thus, we can expand $V^\nu\nabla_\nu V^\mu=0$, getting

$$\frac{dx^\nu}{d\lambda}\left[\frac{\partial}{\partial x^\nu}\left(\frac{dx^\mu}{d\lambda}\right)+\left\{{\mu \atop \nu\kappa}\right\}\frac{dx^\kappa}{d\lambda}\right]=0$$

or

$$\frac{d^2x^\mu}{d\lambda^2}+\left\{{\mu \atop \nu\kappa}\right\}\frac{dx^\nu}{d\lambda}\frac{dx^\kappa}{d\lambda}=0 , \tag{3.6b}$$

the usual form of the equation for a geodesic in terms of the preferred (affine) parameter, in this case the proper time along the world line.

The differential form of the conservation law (3.6) can be changed into the integral form in the well-known way using the Gauss-Stokes formula. We integrate it over a four-dimensional volume and transform to an integral over the closed three-surface bounding the four-volume.[10] Choosing the closed three-surface to be the surface of the timelike tube bounding the dust, plus two spacelike cross sections of it, and noting that no matter can flow across the timelike bounding tube, we finally get the law of conservation of mass for the tube:

$$\int_{s_1} \tilde{\rho} V^\mu ds_\mu = \int_{s_2} \tilde{\rho} V^\mu ds_\mu . \tag{3.7}$$

Thus, we may take a slice of any shape across the world tube to define the total mass of our "thickened" particle. In the limit of going far from the world tube compared to its extension, or letting its size shrink to zero while remaining a fixed distance from the tube, the motion of the dust can be roughly characterized by a single "representative" geodesic and by this mass.

A more detailed characterization of the dust would lead to a discussion of source multipole moments.[11]

Notice that the form of the conservation law, which relates a four-volume integral to a three-surface boundary integral, forces us, in a sense, to thicken a point particle to an entire world tube. We shall see that the situation is somewhat different in the case of the string.

Finally, we note that we can formally pass to the limit of the point particle by taking $\tilde{\rho}(x)$ as proportional to a δ function along the world line of the particle

$$\tilde{\rho}(x) = \int_{-\infty}^{\infty} m(\lambda)\delta^4[x - z(\lambda)]\,d\lambda , \tag{3.7a}$$

where $z^\mu(\lambda)$ is the parametrization of the world line. Integrating over a narrow tube just surrounding the world line and bounded by two spacelike disks, we see that the integral conservation law gives us

$$m(\lambda_1) = m(\lambda_2) = m_0 ,$$

where λ_1 and λ_2 are the values of the parameter at any two points along the world line. Hence, the constancy of m_0 along the world line is the content of the conservation law in this case.

IV. THE THICKENED STRING

We can parallel the treatment of the "thickened" particle in the last section, replacing the timelike vector field by a timelike simple bivector field. We can go a certain distance without even having to demand that the field be hypersurface forming. But eventually we shall need that assumption, and thus be led to a theory which will be shown to be the "thickened" version of the classical string.

We start from a simple timelike[12] bivector field $S^{\mu\nu}$, which therefore obeys Eq. (2.2) and any of its equivalent forms. We form the stress-energy tensor

$$T^\nu_\mu = S_\mu^{\cdot\kappa} S_\kappa^{\cdot\nu} , \tag{4.1}$$

and again impose a normalization condition, so that we can write

$$S^{\mu\nu} = (\rho)^{1/2}\Sigma^{\mu\nu}, \quad \Sigma^{\mu\nu}\Sigma_{\mu\nu} = -2 , \quad \rho \geq 0 . \tag{4.2}$$

Again, we introduce a tensor density factor into ρ, so that

$$\tilde{T}^\nu_\mu = \tilde{\rho}\Sigma_\mu^{\cdot\kappa}\Sigma_\kappa^{\cdot\nu} \tag{4.3}$$

Now, we take the divergence of (4.3) and set it equal to zero:

$$\nabla_\nu \tilde{T}^\nu_\mu = \partial_\nu(\tilde{\rho}\Sigma^{\kappa\nu})\Sigma_{\mu\kappa} + \tilde{\rho}\Sigma^{\kappa\nu}\nabla_\nu\Sigma_{\mu\kappa} = 0 . \tag{4.4}$$

We have again taken advantage of the fact that the covariant divergence of a contravariant bivector density equals its ordinary divergence to get rid of the covariant derivative in the first term. We can also remove the convariant derivative from the second term, by noting that the normalization of $\Sigma^{\mu\kappa}$, Eq. (4.2) implies that $\Sigma^{\kappa\nu}\nabla_\mu\Sigma_{\kappa\nu} = 0$, so that

$$\Sigma^{\kappa\nu}\nabla_\nu\Sigma_{\mu\kappa} = \tfrac{1}{2}\Sigma^{\kappa\nu}(\nabla_\nu\Sigma_{\mu\kappa} + \nabla_\kappa\Sigma_{\nu\mu} + \nabla_\mu\Sigma_{\kappa\nu}) , \tag{4.5}$$

(using the antisymmetry of $\Sigma^{\mu\nu}$ shows that the first two terms on the right are equal, while the last one vanishes). But the terms in parentheses on the right-hand side of (4.5) are just equal to the ordinary curl of an antisymmetric tensor, itself always a tensor. Thus, using square brackets to denote total antisymmetrization divided by 3!, we can finally eliminate the covariant derivative from the second term of (4.4), getting

$$\nabla_\nu \tilde{T}^\nu_\mu = \partial_\nu(\tilde{\rho}\Sigma^{\kappa\nu})\Sigma_{\mu\kappa} + \tfrac{3}{2}\tilde{\rho}\Sigma^{\kappa\nu}\partial_{[\nu}\Sigma_{\mu\kappa]} . \tag{4.4a}$$

Now, contracting (4.4a) with $\Sigma^{\mu\lambda}$, we see that the last term vanishes, since $\Sigma^{\kappa\nu}\Sigma^{\mu\lambda}\partial_{[\nu}\Sigma_{\mu\kappa]} = \Sigma^{[\kappa\nu}\Sigma^{\mu]\lambda}\partial_{[\nu}\Sigma_{\mu\kappa]} = 0$, by (2.2) applied to $\Sigma^{\mu\nu}$. So

$$\Sigma^{\mu\lambda}\nabla_\nu\tilde{T}^\nu_\mu = \partial_\nu(\tilde{\rho}\Sigma^{\kappa\nu})\Sigma_{\mu\kappa}\Sigma^{\mu\lambda} = 0 . \tag{4.6}$$

But, it is easily seen that if a vector has vanishing contraction with $\Sigma_{\mu\kappa}\Sigma^{\mu\lambda}$, then it has vanishing contraction with $\Sigma_{\mu\kappa}$ alone (write $\Sigma^{\mu\nu}$ as the alternating product of two vectors to prove this). So

$$\partial_\nu(\tilde{\rho}\Sigma^{\kappa\nu})\Sigma_{\mu\kappa} = 0 . \tag{4.6a}$$

Thus far we have not had to assume that $\Sigma^{\mu\nu}$ was surface forming. We now use that assumption, in the form of Eq. (2.5):

$$\partial_\nu(\tilde{\rho}\Sigma^{\kappa\nu})^*\Sigma_{\mu\kappa} = 0 . \tag{2.5$'$}$$

The lemma of Sec. II now shows that the divergence of $\tilde{\rho}\Sigma^{\kappa\nu}$ must vanish, by virtue of (4.6a) and (2.5′):

$$\partial_\nu(\tilde{\rho}\Sigma^{\kappa\nu}) = 0 . \tag{4.6b}$$

Note, incidentally, that Eq. (4.6b) itself implies, for a simple bivector, that it is surface forming. If we expand (4.6b) and multiply it by $*\Sigma_{\kappa\mu}$, we get

$$\tilde{\rho}\, *\Sigma_{\kappa\mu}\partial_\nu(\Sigma^{\kappa\nu}) + *\Sigma_{\kappa\mu}\Sigma^{\kappa\nu}\partial_\nu\tilde{\rho} = 0\,.$$

The second term vanishes by (2.2), so the first term must vanish. But this is just the condition for a surface-forming bivector, Eq. (2.4).

Thus, Eq. (4.6b), together with the equation of motion gotten by noting that the first term of (4.4a) vanishes by virtue of (4.6b)

$$\Sigma^{\kappa\nu}\partial_{[\nu}\Sigma_{\mu\kappa]} = 0\,, \tag{4.7a}$$

or the equivalent form gotten by going back to (4.4)

$$\Sigma^{\kappa\nu}\nabla_\nu\Sigma_{\mu\kappa} = 0\,, \tag{4.7b}$$

form a complete set of equations for the simple bivector field—of course, remembering the algebraic condition (2.2), which assures us that $\Sigma^{\mu\nu}$ is really a simple bivector. It only remains to show that the equations of motion are equivalent to the usual parametrized form of the classical string equations of motion.

To do this, we note that if the simple bivector field $S^{\mu\nu}$ is surface forming, there must exist two parameters τ^A (A, B, etc. $=1,2$) such that the equations of each two-sheet take the form

$$x^\mu = x^\mu(\tau^A)\,. \tag{4.8}$$

Then, the tangent surface element at each point of the surface is spanned by the two vectors $\xi^\mu_A = \partial x^\mu/\partial\tau^A$:

$$S^{\mu\nu} = \epsilon^{AB}\frac{\partial x^\mu}{\partial\tau^A}\frac{\partial x^\nu}{\partial\tau^B}\,, \tag{4.9}$$

where ϵ^{AB} is the two-dimensional Levi-Civita alternating tensor density. So it must be possible to write $\Sigma^{\mu\nu}$ as proportional to the alternating product of the two ξ_A vectors. If we call the factor of proportionality $(\beta)^{-1/2}$, we have

$$\Sigma^{\mu\nu} = \beta^{-1/2}\epsilon^{AB}\xi^\mu_A\xi^\nu_B\,. \tag{4.10}$$

It is then easily checked that the condition $\Sigma^{\mu\nu}\Sigma_{\mu\nu} = -2$ normalizing the bivector implies that $\beta = -{}^2g$, where ${}^2g$ is the determinant of the two-metric g_{AB} induced on the timelike surface by the four-metric $g_{\mu\nu}$:

$$g_{AB} = g_{\mu\nu}\xi^\mu_A\xi^\nu_B\,. \tag{4.11}$$

[The identity

$$({}^2g)g^{AB} = \epsilon^{AC}\epsilon^{BD}g_{CD} \tag{4.12}$$

will be found useful in proving this and other later results.] $\Sigma^{\mu\nu}$ as defined by (4.10) obviously obeys the conditions for a simple surface-forming bivector, Eqs. (2.2) and (2.4). Thus, it only remains to show that the equation of motion (4.7b) is equivalent to the usual parametrized form of the string equation of motion. To do this we substitute (4.10) into Eq. (4.7b), getting

$$\epsilon^{AB}\xi_{\kappa A}\xi^\nu{}_B\nabla_\nu(\beta^{-1/2}\epsilon^{CD}\xi^\mu_C\xi^\kappa_D) = 0\,, \tag{4.13}$$

where we have raised the μ index, and eliminated an overall factor of $\beta^{-1/2}$. Now, putting $\epsilon^{AB}\xi_{\kappa A}$ inside the covariant derivative, and subtracting the extra term, we get, using (4.12) and the fact that $\beta = -{}^2g$,

$$\xi^\nu{}_B\nabla_\nu(\tilde{g}^{BC}\xi^\mu_C) - \beta^{-1/2}\epsilon^{AB}\epsilon^{CD}\xi^\mu_C\xi^\kappa_D\xi^\nu_B\nabla_\nu\xi_{\kappa A} = 0\,. \tag{4.14}$$

Here, $\tilde{g}^{AB} = (-{}^2g)^{1/2}g^{AB}$. Now, the second term in (4.14) vanishes. To see this, note that

$$\xi_{\kappa A} = g_{\kappa\lambda}\xi^\lambda_A\,, \tag{4.15}$$

and that $g_{\kappa\lambda}$ may be taken out of the covariant derivative. Then, expanding the covariant derivative and remembering (4.9), we get for the relevant factors in the second term

$$\epsilon^{AB}\xi^\nu_B\nabla_\nu\xi^\lambda_A = \epsilon^{AB}\left(\frac{\partial^2x^\lambda}{\partial\tau^A\partial\tau^B} + \left\{{\lambda \atop \nu\kappa}\right\}\xi^\kappa_A\xi^\nu_B\right); \tag{4.16}$$

hence it vanishes, due to the symmetry in AB of the terms in the parentheses and the antisymmetry of ϵ^{AB}. Expansion of the first term in (4.14) gives

$$\frac{\partial}{\partial\tau^B}\left(\tilde{g}^{BC}\frac{\partial}{\partial\tau^C}x^\mu\right) + \left\{{\mu \atop \nu\kappa}\right\}\xi^\nu_B\xi^\kappa_C\tilde{g}^{BC} = 0\,. \tag{4.17}$$

This is just the covariant form of the string equation of motion in an arbitrary background metric.[13]

Thus, each world sheet into which the "thickened" string naturally divides is an extremal sheet, obeying the string equation of motion; just as each world line into which our "thickened" particle naturally divided was a geodesic, obeying the particle equation of motion.

We have carried out our discussion using an arbitrary parametrization τ^A of the world sheet. However, it is easy to see that by a reparametrization of the world sheet, one can multiply the simple bivector $S^{\mu\nu}$ by any arbitrary factor; so that we could just as well have made β a constant, for example. In particular, we can adopt a normalization analogous to that in the particle case, where $V^\mu V_\mu = 1$ implies that $V^\mu = dx^\mu/d\lambda$, where λ is the proper time along the world line; i.e., the arc length along the world line is given directly by $\int d\lambda$. Correspondingly, $\Sigma^{\mu\nu}\Sigma_{\mu\nu} = -2$ implies that there is a parametrization for which

$$\Sigma^{\mu\nu} = \epsilon^{AB}\frac{\partial x^\mu}{\partial\tau^A}\frac{\partial x^\nu}{\partial\tau^B}\,,$$

with $2g = -1$, so that "area" on the world sheet is given directly by

$$A = \int d^2\tau\,.$$

We now turn back to the conservation law (4.6b). This is the differential form of a conservation law, which we can transform by the Gauss-Stokes theorem into an integral form. But this time it relates a three-volume and a bounding two-surface, and tells us that the integral over any closed two-surface of $\tilde{\rho}\Sigma^{\mu\nu}$ must vanish:

$$\oint \tilde{\rho}\Sigma^{\mu\nu} d\sigma_{\mu\nu} = 0\,, \qquad (4.18)$$

where $d\sigma_{\mu\nu}$ is the element of two-surface area. Of course, just as in the application of the ordinary Stokes's theorem in three-dimensional space a closed curve bounds an infinity of two-surfaces, in our case the closed two-surface bounds an infinity of three-spaces. But this does not matter: We need only fix our attention on the closed two-surface.

One similar integral conservation law is very familiar: the conservation of charge, resulting from the fact that the divergence of the Maxwell tensor is equal to the charge-current vector. But in that case, we usually take our closed two-surface entirely in a spacelike three-hypersurface ("at a given time") to calculate the total enclosed charge. In our case, since the two-surfaces representing the movement of the strings are timelike, the most important application of the integral conservation law (4.18) will be to closed two-surfaces with timelike portions; in particular when a timelike portion is formed by the motion of a string, there will be no flux through such a portion. We picture

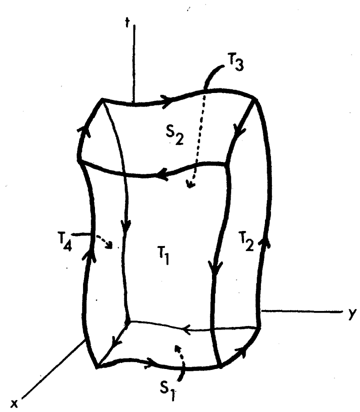

FIG. 1. Closed two-dimensional hypersurface in three-dimensional space-time. With an additional dimension of space, the enclosed three-space would not be unique. This can represent a one-parameter family of open strings, with T_1 and T_3 representing bounding strings, and T_2 and T_4 representing one-parameter families of null curves formed by the string end points (see Appendix A). S_1 and S_2 are two spacelike two-surfaces intersecting the family of strings.

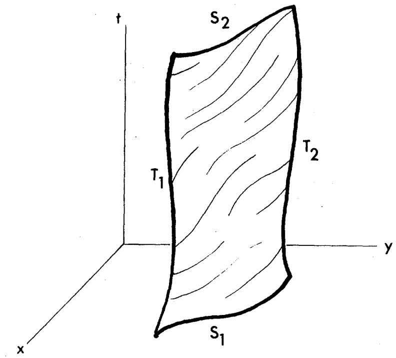

FIG. 2. Closed one-dimensional curve in three-dimensional space-time, with one possible two-space it bounds. Here, it is easy to visualize how the same boundary can enclose different spaces.

such a closed two-surface in Fig. 1 (with one spatial dimension suppressed, unfortunately). The possible shifting of the "enclosed" three-space cannot be easily visualized; to help visualize this, we look at a corresponding one-dimensional closed curve (Fig. 2), where it is easy to see how the two-surface bounded by the curve may be shifted without altering the boundary. The reader hopefully will be helped by the similarities and contrasts between these two figures to "visualize" (?) the case of a two-dimensional boundary in four-dimensional space-time—admittedly no easy task.

So, for example, we could imagine that T_1 and T_3 are two-surfaces formed by motions of the string (i.e., by the $\Sigma^{\mu\nu}$ field). Then (4.18) would tell us that the flux through $T_2 + T_4 + S_1 + S_2$ must vanish, with orientations on each as indicated. Of-course, if we reverse the orientations on S_2 and T_4, say, so that they have the same orientation as S_1 and T_2, respectively, we see that the flux through $S_1 + T_2$ must equal the flux through $S_2 + T_4$. Notice that this integral conservation law connecting an integral over a three-volume with the integral over its bounding two-surface does not force us, as did the thickened particle integral conservation law, to thicken our string into a world tube. We could just thicken it by one dimension, to get a three-dimensional thickened string. Of course, this would mean that the density factor $\tilde{\rho}$ for the string would still have a one-dimensional δ function in it, which would create problems in the extension to general relativity to be discussed in the last section. On the other hand, nothing prevents us from thickening the string into a world tube, eliminating all δ functions from the string stress-energy tensor. In any case, for our discussion of the conservation law, we need only consider a onefold infinity of strings chosen in such a way as to fill up some

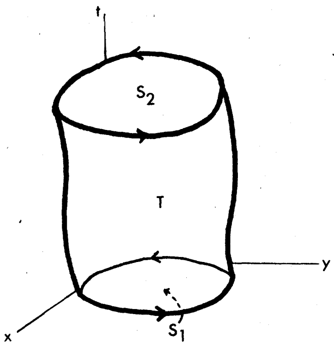

FIG. 3. One-parameter family of closed strings. The boundary T represents one such string. S_1 and S_2 are two spacelike two-surfaces intersecting the family of strings.

three-volume bounded by the two-surface in question.

Now let us apply these ideas to the usual models of classical closed and open strings. For the closed string the whole timelike part of the boundary (corresponding to $T_1+T_2+T_3+T_4$ in Fig. 1) is formed by the $\Sigma^{\mu\nu}$ field (see Fig. 3). Thus, the integral form of the conservation law is

$$\int_{S_1} \tilde{\rho}\Sigma^{\mu\nu} d\sigma_{\mu\nu} = \int_{S_2} \tilde{\rho}\Sigma^{\mu\nu} d\sigma_{\mu\nu}, \qquad (4.19)$$

with similar orientation of S_1 and S_2 as pictured in Fig. 3.

For the open string, we must add the usual boundary conditions, which assure that there is no flux of momentum out of the end points of the string. An analysis of this condition shows that it requires that an end point of the open string move at the speed of light, i.e., that it sweep out a null world line (not a null geodesic, of course). Such an analysis, starting from the stress-energy tensor for the string dust, is given in Appendix A; it shows that the bivector field $S^{\mu\nu}$ must become null at the string end points. The usual derivations of this condition[14] based on a variational principle leave something to be desired in rigor, since the parametrization on which they are based breaks down just at the end points.

We may again take a onefold infinity of such open strings, chosen in such a way as to fill up some three-volume bounded by a closed two-surface like that in Fig. 1. Now T_2 and T_4 are the timelike two-surfaces formed by the one-parameter family of null curves swept out by the end points. (It is also possible they could be null surfaces, i.e., be built up of two-surface elements tangent to the null cone at each point of the surface.) The integrals over T_2 and T_4 still vanish, because the end point null vector is common to the surface element and to the bivector; so once again we get Eq. (4.19), but applied to Fig. 1.

These are the conservation laws for the closed and open thickened strings.

We can again formally pass to the limit of the "thin" string by writing $\tilde{\rho}$ as a δ function over the string[15]:

$$\tilde{\rho}(x) = \iint \mu(\tau^A)\delta^4[x - z(\tau^A)]\, d^2\tau .$$

We now integrate this over a two surface bounding a three-volume which just encloses the portion of our string between two spacelike curves on it, s_1 and s_2. Inserting the above $\tilde{\rho}$ into the conservation law (4.19), we see that it gives us

$$\int_{s_1} \mu[\tau^A(s)]\, ds = \int_{s_2} \mu[\tau^A(s)]\, ds ,$$

where the integral is to be taken over a closed curve for the closed string, and between the limits of s that correspond to the end points of the open string. Hence, the constancy of the integral for any such curve on the string world sheet is the content of the conservation law in this case. It is also possible to convert the bivector conservation law we have been discussing into two vector conservation laws, as shown in Appendix B.

V. CONCLUSIONS

Now that we have shown how to form a perfect dust of strings, quite analogous to the perfect dust of particles long familiar in special and general relativity, the question naturally arises whether strings can be used as building elements for more complicated phenomenological models of matter. The answer is yes. In another paper, it will be shown that a relativistic theory of a perfectly elastic medium composed of strings can be set up by analogy with the usual elastic theory built up out of particles. In particular, the theory of a string perfect fluid and its derivation from a Clebsch-type variational principle will be developed.

The question of possible quantization of such models of matter is also of interest. Such a quantization presents some interesting problems compared to the usual fluid case. In the fluid model composed of particles, each particle has only a finite number of degrees of freedom, of course, and an infinite number of degrees of freedom arises only from the passage to the continuum model. In the case of the string fluid, each string possesses an infinite number of degrees of freedom to begin with, which already presents difficulties in their quantization.[16]

It is also clear that there is nothing in our theory of the string dust which cannot be generalized to an m-dimensional "string" in an n-dimensional space-time.[17] This generalization, and its connection with the theory of "congruences" of m-dimensional submanifolds in an n-dimensional Riemann space of arbitrary signature will be discussed in a joint paper with J. Plebanski now in preparation.

Another case of interest is the classical null string[18]; if the string two-surface element is tangent to the null cone at each point, our previous considerations must be modified, just as in the case of point particles following null geodesics. This problem will be treated in the next paper of this series.

All of this work is based on the mathematical representation of the string subspaces by a bivector (or multivector, in the case of higher dimensional subspaces). But consideration of the particle case suggests another possibility. The timelike unit tangent vector to a family of particle world lines can be analyzed in terms of its rotation, shear, expansion, and acceleration.[19] The free-particle world lines (timelike geodesics) are then completely characterized by vanishing acceleration. Is any generalization of this approach possible for higher-dimensional subspaces? It turns out that there is. Three-index tensorial rotation, shear, and acceleration tensors may be defined for congruences of timelike subspaces of any number of dimensions.[20] The string equations of motion then prove to be equivalent to the requirement of vanishing of the (unique) trace of the three-index acceleration of the (holonomic) congruence of subspaces. This will be proved elsewhere.[21]

Up to now we have only discussed strings moving in an arbitrary background space-time metric. Now we come to the question of a possible fully general-relativistic theory of the string, in which the string stress-energy tensor acts, via the Einstein equations, as a source of the gravitational field. One might think that all that had to be done was to put the "thin" string stress-energy tensor, formed with a four-dimensional δ function over the two-dimensional world sheet in question, into the field equations. However, there are a number of problems with such a procedure. In the first place, there are mathematical problems involved with the use of the δ function in nonlinear equations; they do not show up when one considers only the conservation laws for the stress-energy tensor (the integrability conditions for the Einstein field equations) which are linear equations, but would have to be faced in treating the full Einstein equations. Secondly, if one looks at the corresponding structureless-point-particle problem, one sees that the exact solution for the "particle" at rest (the Schwarzschild solution) has quite a different structure, involving topological complexities and singularities of a nonpointlike nature (i.e., not along a timelike world line). This has been emphasized by Dirac, who was led to postulate quite a different model for a particle in the attempt to derive gravitational equations of motion for such a particle.[22]

Of course, in a linearized approximation scheme to the field equations, it is perfectly acceptable to introduce a δ function source, since the field equations solved in each approximation are linear. If one does this in the spherically symmetric case, one gets in the first approximation a Coulomb-like solution with only a singularity at the point mass. Indeed, as Trautman has shown,[23] one can iterate this solution to all orders and sum the resulting series to get the Schwarzschild solution. But, of course, this just emphasizes the point that the exact solution has quite a different structure in the neighborhood of the "particle" from the approximate ones.

If we try to use the linearized field equations with the string δ-function stress-energy tensor, we meet with another problem, at least in one simple case we were able to investigate in detail.[24] The simplest solution for the open string is the rigidly rotating string, in which none of the internal degrees of freedom of the string are excited. The linearized field of this string looks roughly like that of a rigidly rotating rod in the linearized approximation, except near and at the end points, which move with the speed of light as mentioned earlier (see Appendix A). The linearized field becomes singular, not only at these end points, but along a line emanating from each end point and extending out to infinity, so that the total radiated power diverges. Clearly, the linear approximation has broken down. By taking the "realistic" (or massive) string model of Takabayashi,[25] and going to the limit of the "geometrical" (or massless) string model, one can verify that the infinity only arises in the massless limit for which the end points move at the speed of light. There is reason to believe that this behavior will persist for more complicated motions of the open string in the linearized approximation. Thus, it becomes a very interesting question whether the exact theory of the thickened open string will exhibit a similar pathology; or whether the thickening—plus nonlinearity—will allow regular solutions.

Returning for the moment to the particle case, the problems of the point particle source are alleviated by use of the thickened particle, or perfect dust model. In the case of spherical symmetry, the external field of the dust will continue to be the Schwarzschild field (by Birkhoff's theorem), while the collapse (or expansion) of the dust may be studied. Collapse beyond the Schwarzschild horizon has been extensively studied, of course, in connection with the black-hole problem. Various nonspherical dust motions have also been investigated.

We hope that the use of thickened string sources will enable us to investigate similar collapse problems, the possible formation of horizons, the nature of singularities, etc., for a new class of stress-energy tensors, which represent in one sense the simplest possible generalization of the particle perfect-dust model.

What hope is there to actually construct such solutions? All one can say at this point is that there is some hope. As we have showed elsewhere,[26] the perfect-dust stress-energy tensor conservation equations (the integrability conditions for the field equations) can be solved without any essential restrictions on the metric, before the field equations are tackled. It seems that a similar technique may work for the thickened-string stress-energy tensor conservation equations. This would at least constitute a big first step toward the solution of the field equations.

Beyond that, one may hope that general considerations will enable one to make some progress in discussing such questions as the existence of solutions, whether there exists a correctly posed Cauchy problem, whether regular solutions exist for open strings, and the nature of the horizons and singularities that may arise in the collapse of a string dust.

APPENDIX A: BOUNDARY CONDITIONS FOR THE OPEN STRING

As mentioned in the text, most proofs of the boundary conditions for the end points of an open string are based upon use of a parametrization of the string which breaks down just at the end points. Our parameter-independent thickened-string technique enables us to use the standard jump conditions for a stress-energy tensor to establish the boundary conditions.

We shall show that the only boundaries on which the thickened-string stress-energy tensor can end are either hypersurfaces formed of string two-sheets; or, in the case of open strings, hypersurfaces formed by the end points of the string which sweep out null curves in space-time.

For this purpose we assume that we have a two-parameter family of string world sheets filling some four-dimensional world tube with timelike boundary, and investigate the junction conditions that must be satisfied at the boundary, outside of which the string dust vanishes. Since the equations of motion of the string dust follow from the conservation law for its stress-energy tensor, it is sufficient to examine the junction conditions for the stress-energy tensor at the boundary. As is well known,[27] if $\phi=\text{const}$ is the equation of a bounding hypersurface such that the stress-energy tensor vanishes on one side of the hypersurface, then

$$T^{\nu}_{\mu}\phi_{,\nu}=0 \tag{A1}$$

must hold on the other side. Physically, this is the requirement that there be no flow of energy-momentum across the boundary, i.e., that the hypersurface not act as a source or sink of energy-momentum. Thus, the boundary condition basically is dictated by global conservation of energy-momentum.

We take the string stress-energy tensor in the form [Eq. (4.1)]

$$T^{\nu}_{\mu}=S_{\mu}^{\cdot\kappa}S_{\kappa}^{\cdot\nu}; \tag{A2}$$

we do not adopt the normalization condition Eq. (4.2) because this would presuppose that the string remained timelike on all portions of the bounding hypersurface, and thus preclude an important possibility, as we shall see. Combining (A1) and (A2), we see that the boundary condition can be satisfied if (and only if) one of the following conditions holds:

$$\text{I}:\ S^{\nu}_{\kappa}\phi_{,\nu}=0;\quad \text{II}:\ S^{\nu}_{\kappa}\phi_{,\nu}=\lambda_{\kappa},\ \text{and}\ S^{\kappa}_{\mu}\lambda_{\kappa}=0.$$

We shall examine these two possibilities in turn.

Case I. This condition implies that the two-space spanned by the timelike bivector $\underline{S}$ lies in the hypersurface $\phi=\text{const}$. Thus, this hypersurface is formed by a one-parameter family of string world sheets. So, one portion of the boundary of the string world tube can be formed of such a one parameter family. For closed strings, this is the only possibility.

Case II. The first part of condition II implies that λ_{κ} lies in the two-space spanned by the bivector $\underline{S}$. The second part of condition II implies that it nevertheless has a vanishing scalar product with all vectors in $\underline{S}$. Clearly, $\underline{S}$ cannot be a timelike bivector. Indeed, since λ_{κ} lies in $\underline{S}$, we can write $\underline{S}$ as

$$S^{\mu\nu}=\lambda^{\mu}W^{\nu}-\lambda^{\nu}W^{\mu}, \tag{A3}$$

where $\vec{W}$ is some other vector in $\underline{S}$. Then, the second part of condition II shows that

$$\lambda^{\kappa}\lambda_{\kappa}=0,\quad \lambda^{\kappa}W_{\kappa}=0. \tag{A4}$$

Thus, $\vec{\lambda}$ is a null vector, and $\vec{W}$ must be an orthogonal spacelike vector. $\underline{S}$ is therefore a *null* bivector. At a portion of the boundary characterized by condition II, $\underline{S}$ must therefore degenerate from timelike to null. To see what this implies for an individual string, note that it implies that $\underline{S}$ does *not* lie in the hypersurface $\phi=$ const, and that the intersection of $\underline{S}$ with $\phi=$ const is formed by the null trajectories of the λ_κ field. To see this, insert (A3) into the first part of condition II, which yields

$$W^\nu \phi_{,\nu}=1\,,\quad \lambda^\nu \phi_{,\nu}=0\,. \tag{A5}$$

Thus, $\vec{W}$ does not lie in $\phi=$ const, while $\vec{\lambda}$ does, proving our assertion. This means that an open string can end at a null curve (which will lie in the timelike surface $\phi=$ const if we consider a two-parameter family of such open strings). In this case the stress-energy tensor at such a portion of the boundary takes the form [gotten by inserting (A3) into (A2) and using (A4)]

$$T_\mu^\nu=-(W_\kappa W^\kappa)\lambda_\mu\lambda^\nu\,. \tag{A6}$$

Thus, energy-momentum flows along the boundary of each string, which is why the boundary does not act as a source or sink of energy-momentum.

A simple example shows that case II may easily be realized. Consider the spinning string, which is kept extended by the tension produced by its own rotation. We place it in the x-y plane of some special-relativistic inertial system, with its center at the origin. It may then be parametrized by distance along the strong, r, and inertial time t; i.e., $\tau^A=r,t$, with $-\infty\le t\le\infty$, $-1/\omega\le r\le 1/\omega$, where ω is the angular velocity of the string ($c=1$ in our units). Then

$$\begin{aligned} x&=r\cos\omega t\,,\quad z=0\,,\\ y&=r\sin\omega t\,,\quad t=t\,,\end{aligned} \tag{A7}$$

express the space-time inertial coordinates in terms of the two parameters and ω.

The string bivector is spanned by

$$\xi^\mu=\frac{\partial x^\mu}{\partial r}=(0,\,\cos\omega t\,,\,\sin\omega t\,,\,0)\,,$$

$$\eta^\mu=\frac{\partial x^\mu}{\partial t}=(1,\,-\omega r\sin\omega t\,,\,\omega r\cos\omega t\,,0)\,.$$

Note that $\vec{\xi}$ and $\vec{\eta}$ are orthogonal, and that while $\vec{\xi}$ is always spacelike, $\vec{\eta}$ is timelike for $r\ne\pm 1/\omega$, but becomes null as $r\to\pm 1/\omega$. Thus, the string bivector $S^{\mu\nu}=\xi^\mu\eta^\nu-\xi^\nu\eta^\mu$ is timelike except at the end points, where it becomes null. Thus, except at the end points, we can normalize $S^{\mu\nu}$, to get

$$\Sigma^{\mu\nu}=(1-\omega^2r^2)^{-1/2}S^{\mu\nu}\,.$$

It is easily checked that (A7) obey the parametrized form of the string equations of motion (4.17). It is also easily verified that the string stress-energy tensor at the string end points assumes the form (A6), where λ^κ is the null limiting form of η^κ at the two end points.

By taking a two-parameter family of such spinning strings we can fill a timelike world tube in various ways. For example, by filling a sphere uniformly with such strings, we can get a spherically symmetric string dust, etc.

APPENDIX B: REDUCTION OF BIVECTOR CONSERVATION LAW TO TWO VECTORIAL CONSERVATION LAWS

If we choose a particular pair of bivectors v_A^μ to span the bivector $\Sigma^{\mu\nu}$,

$$\Sigma^{\mu\nu}=\epsilon^{AB}v_A^\mu v_B^\nu\,, \tag{B1}$$

we can rewrite the conservation law (4.6b) as

$$\nabla_\mu(\rho\Sigma^{\mu\nu})=\nabla_\mu(\rho v_1^\mu)v_2^\nu-\nabla_\mu(\rho v_2^\mu)v_1^\nu+\rho[v_1,v_2]^\nu=0\,, \tag{B2}$$

where $[v_1,v_2]$ is the Lie bracket of the two vector fields. Since they are surface forming, their bracket is always a linear combination of the two fields [indeed, this condition on the two vector fields is just equivalent to condition (2.4) on the bivector field Σ]. Now it is always possible to choose the two vector fields so that this bracket vanishes, while preserving the normalization condition (4.2) on Σ. (This is just equivalent to saying that a parametrization exists such that $v_A^\mu=\partial x^\mu/\partial\tau_A$ and

$$g_{AB}=g_{\mu\nu}\frac{\partial x^\mu}{\partial\tau^A}\frac{\partial x^\nu}{\partial\tau^B}$$

has determinant -1.) In that case the last term in (B2) vanishes; and since v_1^μ and v_2^μ are linearly independent, the two conservation laws

$$\nabla_\mu(\rho v_1^\mu)=0\,,\quad \nabla_\mu(\rho v_2^\mu)=0 \tag{B3}$$

must hold.

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*A preliminary announcement of the results of this paper appeared in *GR8: Abstracts of the Contributed Papers, 8th International Conference on General Relativity and Gravitation* (Univ. of Waterloo, Ontario, Canada, 1977), p. 324.

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[1]For recent reviews, see M. S. Marinov, Usp. Fiz. Nauk 121, 377 (1977) [Sov. Phys. Usp. 20, 179 (1977)] and J. Scherk, Rev. Mod. Phys. 47, 123 (1975). These include references to most of the relevant papers.

[2]In addition to the papers cited in the previous note, see

T. Takabayashi, Prog. Theor. Phys. 52, 1910 (1974) for a careful treatment of the classical string.

[3]While the generalization of the special relativistic string model to an arbitrary background space-time is easily worked out [see M. Gürses and F. Gürsey, Phys. Rev. D 11, 967 (1975)], the only work known to me on a general-relativistic string theory in which the string acts as a source of gravitational field is work with P. Letelier; this, however, treats only the linearized theory (see reference in Ref. 24).

[4]See J. A. Schouten, *The Ricci Calculus* (Springer, Berlin, 1954). We follow his notational conventions in the main. Our signature is -2.

[5]See, e.g., H. Cartan, *Differential Forms* (Houghton-Mifflin, Boston, 1970). The treatment of string congruences using differential forms will be discussed in a paper with J. Plebanski.

[6]Such a surface element if often called a blade. In four dimensions, it can be proved that every bivector is the sum of two such blades, or simple bivectors.

[7]Indeed, this result can be proved for any space with affine connection. See P. Havas, J. Math. Phys. 5, 373 (1964).

[8]Dirac has emphasized this point in the course of putting forth another model for sources of the gravitational field, in his discussion of the equation of motion problem: P. A. M. Dirac, Proc. R. Soc. London A270, 354 (1963).

[9]This form of the perfect-fluid equations of motion was given earlier in J. Stachel, Phys. Rev. 180, 1256 (1969).

[10]We assume that the space-time manifold is oriented. Then an inner orientation of any submanifold induces an outer orientation, needed for the Gauss-Stokes formula (see Schouten, Ref. 4).

[11]For a discussion of source multipole moments in curved space-times, see the work of W. G. Dixon, Proc. R. Soc. London A314, 499 (1970); A319, 509 (1970); Philos. Trans. R. Soc. A277, 59 (1974).

[12]As we shall see in Appendix A, it is possible for an open string bivector to degenerate into a null bivector (i.e., the alternating product of a null vector and an orthogonal spacelike vector) at the end points of the string. Equation (4.1) is still appropriate for the stress-energy tensor even at such points.

[13]This form is a slight generalization of that given by Gürses and Gürsey (Ref. 3). A more detailed discussion and a derivation of (4.17) from a variational principle will be published elsewhere.

[14]Such derivations, with references to earlier papers, can be found in the references in Ref. 1.

[15]Aragone and Deser have derived the usual parametrized form of the string equations of motion from the application of the conservation laws to such a δ-function stress-energy tensor. See C. Aragone and S. Deser, Nucl. Phys. B92, 327 (1975).

[16]For discussions of the quantization of the relativistic string, see the reviews referenced in Ref. 1, and papers referred to in them. See also D. C. Salisbury and P. G. Bergmann, Syracuse University report (unpublished) and references therein.

[17]Aragone and Deser (Ref. 15) have already noted the possibility of generalizing the single string equations of motion this way.

[18]See A. Schild, Phys. Rev. D 16, 1722 (1977).

[19]See, e.g., C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 566.

[20]For preliminary discussions of this decomposition, see J. Stachel in Proceedings of the Third Latin American Symposium on Relativity and Gravitation (unpublished); J. Math. Phys. (to be published).

[21]See J. Stachel, Ref. 20, first item, for a preliminary discussion of this result.

[22]See reference in Ref. 8.

[23]A. Trautman, Bull. Acad. Polon. Sci. Cl. III 4, 443 (1956).

[24]P. Letelier and J. Stachel (unpublished).

[25]T. Takabayashi, in *Quantum Mechanics, Determinism, Causality and Particles*, edited by M. Flato *et al.* (Reidel, Dordrecht, 1976), pp. 179–216.

[26]J. Stachel, Phys. Rev. 180, 1256 (1969).

[27]See, for example, J. L. Synge, *Relativity, The Special Theory* (North-Holland, Amsterdam, 1964), 2nd edition, pp. 384–385.