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Where has the fifth dimension gone?

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We show that a simple solution to the vacuum field equations of general relativity in 4 + 1 space-time dimensions leads to a cosmology which at the present epoch has 3 + 1 observable dimensions in which the Einstein-Maxwell equations are obeyed. The large ratio of the electromagnetic to gravitational forces is a consequence of the age of the Universe, in agreement with Dirac's large-number hypothesis.

Over the years there has been a number of attempts[1-7] to construct unified field theories based on five-dimensional space-time, going back to that of Kaluza in 1921. The general scheme is that four dimensions, say $x^0, \ldots, x^3$, are identified with the observed space-time, and the associated ten components of the metric are used to describe gravity. The metric components connecting x^5 to $\{x^0, \ldots, x^3\}$ give four extra degrees of freedom which may be interpreted as the electromagnetic potential. In addition, there is a 15th degree of freedom (essentially g^{55}) which may either be set to one,[1] or else may be allowed to vary,[4,5] thereby introducing a scalar field into the problem.

It is remarkable that, under the assumption that a suitable Killing vector exists, Einstein's field equations in five dimensions look exactly like the Einstein-Maxwell system in four dimensions, with or without an extra scalar field. The great difficulty with this approach is to understand why such a Killing vector should exist, or, more loosely, why the observed Universe is four and not five dimensional. Indeed, the assumption of a Killing vector serves at least partially to dismantle the five-dimensional framework and the hoped-for unification of gravity and electromagnetism is severely compromised, if not entirely lost.

In this paper we discuss a model of a five-dimensional universe which naturally evolves into an effective four-dimensional one, even though all spatial dimensions are treated symmetrically in the field equations and the boundary conditions. As the reader will see, we achieve this at the cost of making some special choices in the solution to the field equations. In the absence of any justification for these choices,[8] we cannot claim to have proven, given our field equations, that the Universe must have evolved along the lines we suggest. Rather, it is our purpose to point out the possibility of such evolution, and to deduce therefrom certain interesting observational consequences.

We let time be a continuous parameter ranging over the real line, and we take the spatial x^i to be periodic; i.e., we assume that in a suitable coordinate system, they can be chosen to have the range

$$0 \leq x^j < L \ ,$$

where L is some parameter with the dimension of length. In our model, in the absence of matter (which we are temporarily disregarding), the basic equations are then

$$R_{\mu\nu} = 0, \tag{1}$$

where $R_{\mu\nu}$ is the five-dimensional Ricci tensor.

We assume that the Universe has evolved according to the Kasner solution[9] to Eq. (1), which is a particularly simple nontrivial homogeneous vacuum solution. Furthermore, there is reason to believe that many other five-dimensional homogeneous cosmologies will mimic the Kasner solution in much the same way that in four dimensions the mixmaster solutions exhibit Kasner-type behavior.[10] In d spatial dimensions, the Kasner solution has the form

$$ds^2 = -dt^2 + \sum_{i=1}^{d} (t/t_0)^{2p_i} (dx^i)^2, \tag{2}$$

which will solve Eq. (1) provided

$$\sum_{i=1}^{d} p_i = \sum_{i=1}^{d} p_i^2 = 1 \ . \tag{3}$$

Except in degenerate cases, these relations require at least one of the p_i to be negative. Thus in four dimensions the vacuum Kasner solution is a poor description of the Universe, since it predicts contraction in at least one dimension. Conventionally, this situation is improved[11] by adding matter on the right-hand side of Eq. (1); however, we propose to consider the Kasner solution in five dimensions, in which case the resolution of this difficulty is somewhat different.

In order to guarantee the appearance of isotropy, we take $p_1 = p_2 = p_3 = \frac{1}{2}$, $p_5 = -\frac{1}{2}$. It is possible that these choices may be justified by examining the stability of this solution compared to ones given by other values of the parameters. Then

$$ds^2 = -dt^2 + (t/t_0)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (t_0/t)(dx^5)^2 .$$

At the instant $t = t_0$, the Universe was spatially flat and appeared isotropic, with four space dimensions. The distance around any dimension was L. For $t \ll t_0$, the Universe had essentially only one spatial dimension, whose length approached infinity at the initial singularity $t = 0$. For $t \gg t_0$, the distance around the fifth dimension has shrunk to $(t_0/t)^{1/2}L$, while the other three spatial dimensions have grown to $(t/t_0)^{1/2}L$.

Assuming $(t_0/t)^{1/2}L$ is sufficiently small, i.e., that the Universe is sufficiently old, the fact that the fifth dimension is not observed is a consequence of the dynamics of the evolution of the cosmos, and not of the preordained existence of a Killing vector[1-5] or of spontaneous compactification[12] or some similar mechanism.

We can attempt to estimate the magnitude of $(t_0/t)^{1/2}L$ in a standard way. First we rescale our coordinates so that the Kasner metric takes the form

$$ds^2 = -dt^2 + (t/\tau)(d\vec{x})^2 + (\tau/t)(dx^5)^2 ,$$

where $t/\tau \approx 1$. In so doing, we must remember that our new x^5 satisfies

$$0 \leq x^5 < L(t_0/\tau)^{1/2} .$$

Now we add a small perturbation $h_{\mu 5}$ ($\mu = 0, 1, 2, 3$) to this metric, where $h_{\mu 5}$ is related to the electromagnetic potential A_μ by

$$\lambda_\mu \equiv h_{\mu 5} = (16\pi G)^{1/2} A_\mu .$$

[This can be derived by demanding that the space-time components of Eq. (1) take the form

$$\tilde{R}_{ab} - \tfrac{1}{2}\tilde{g}_{ab}\tilde{R} = 8\pi G(\tilde{T}^{(\text{em})}_{ab} + T'_{ab}),$$

where $\tilde{R}_{ab}$ is the usual four-dimensional Ricci tensor, $\tilde{T}_{ab}$ is the electromagnetic stress-energy tensor, and T'_{ab} is an extra contribution due to the scalar field.]

Following Souriau,[6] we then consider a quantum field ϕ coupled to this metric via the equation

$$\Box_5 \phi + a\phi = 0 , \tag{4}$$

where $\Box_5$ is the covariant five-dimensional d'Alembert operator. We demand that ϕ be periodic with period $L' = L(t_0/\tau)^{1/2}$ in the x^5 coordinate, and we ignore variations in the quantity t/τ, setting it equal to one. Then, through second order in λ we obtain

$$\Box\phi - (q^2 - a)\phi - 2iq(\lambda \cdot \partial\phi) - iq(\partial \cdot \lambda)\phi - q^2(\lambda \cdot \lambda)\phi - (\lambda_{,5}) \cdot \partial\phi + 2iq(\lambda \cdot \lambda_{,5})\phi = 0 , \tag{5}$$

where

$$\phi = \hat{\phi}(x^\mu)e^{iqx^5}, \quad q = 2\pi n/L' .$$

This is the same as the Klein-Gordon equation in the presence of an electromagnetic field (plus some other interactions due to the possible variation of λ in the fifth direction), provided we identify the charge e_n as

$$e_n = \frac{\hbar q_n}{c}(16\pi G)^{1/2} .$$

Thus the basic unit of charge obeys the relationship

$$\frac{e^2}{4\pi\hbar c}\,\frac{c^3}{16\pi^2 G\hbar} = \frac{1}{L'^2} = \frac{\tau}{t_0}\,\frac{1}{L^2} , \tag{6}$$

where τ is a time characteristic of the present age of the Universe and t_0 is the time at which the four dimensions of space were equally large. Putting in the known value of the left-hand side, we conclude that

$$(t_0/\tau)^{1/2}L = 2.38 \times 10^{-31} \text{ cm}. \tag{7}$$

Thus the distance around the fifth dimension is currently very small.

It is interesting that Eq. (6) is in conformity with Dirac's large-number hypothesis[13]: The large value of the ratio of the electromagnetic to gravitational coupling constants is a consequence of the age of the Universe. When t was $\approx t_0$, the two interactions were of approximately the same strength.

It should be noted, however, that Eq. (4) is not satisfactory as it stands as an equation to describe any of the known elementary particles (even in an approximation where spin is neglected). The reason is the $q^2\phi$ term in Eq. (5). With the value of q^2 that follows from Eq. (7), we find that this corresponds to a mass for the particle of 5.22 $\times 10^{20}$ MeV (9.3×10^{-7} g). One way out is to adjust the constant a so that $q^2 - a$ is in the range 0.5 to 5×10^3 MeV. This means fine tuning a to 20 decimal places. It also means that the particle under consideration was a tachyon in the past and will become very massive in the future. Altogether, this seems an unsatisfactory resolution to the problem. Other schemes for coupling matter to our five-dimensional metric are currently under investigation.

While the fifth dimension has been shrinking, the other three spatial dimensions have been expanding, with the radius of the Universe given by

$$R(t) = (t/t_0)^{1/2} L .$$

Thus $\dot{R}/R = H = 1/2t$.

The quantity H is the observed value of Hubble's constant. The reader may be worried that we have ignored an additional effect, namely, that according to Eq. (6), if we assume

$$\alpha = e^2/4\pi\hbar c$$

is constant over cosmological times (and choosing units so that $c = 1$) then the product $G\hbar$ must vary as $1/t$. If this leads to variations in atomic frequencies, then the observed value of H could be different from that predicted above.

Let m be a typical mass in atomic physics (e.g., the mass of electron). Then the ratio of electric to gravitational forces among atomic particles is $\sim e^2/Gm^2$. In atomic units, we have $\hbar = \text{const}$, and therefore $G \propto 1/t$. We also take m to be constant in atomic units. [Strictly speaking, this is an extra assumption that does not follow from Eq. (6), but it is a most natural one.] This gives $e^2/Gm^2 \propto t$, as expected from Dirac's hypothesis. In gravitational units, we have $G = \text{const}$, $\hbar \propto 1/t$, and since $\alpha = \text{const}$ we conclude that $e^2 \propto 1/t$. Then, in order to have $e^2/Gm^2 \propto t$, we conclude that $m \propto 1/t$ in gravitational units.

Now let us examine atomic frequencies. A typical atomic energy level will be given by

$$E = mf(\alpha)$$

and a typical frequency by

$$\nu = (1/h) m f(\alpha) .$$

We see that in both atomic and gravitational units the quantity $m/\hbar$ is constant, so that no correction is needed to our value of H, which is determined by measuring the red-shift of atomic spectral lines.[14] However, one must make the added assumption that the usual methods of determining galactic distances continue to hold in our cosmology.

Besides the value of Hubble's constant, we also predict the deceleration parameter:

$$q_0 \equiv -R\ddot{R}/\dot{R}^2 = 1 .$$

We have chosen to treat the case of five dimensions as the simplest generalization of the usual four. In order to include other interactions besides the gravitational and the electromagnetic in this scheme, it is necessary to generalize our picture to more dimensions. A vacuum Kasner solution of the form Eq. (2) exists in any number of spatial dimensions, provided that Eq. (3) is satisfied. If we choose the three positive powers to have the same value p_1 and let the remaining n powers have the same negative value p_2 we find

$$p_1 = \frac{3 + (3n^2 + 6n)^{1/2}}{3(n+3)} ,$$

$$p_2 = \frac{n - (3n^2 + 6n)^{1/2}}{n(n+3)} ,$$

so that for n large,

$$p_1 \approx 1/\sqrt{3} ,$$

and therefore

$$H \approx 1/\sqrt{3}\, t$$

and

$$q_0 \approx 0.73$$

in that case. Thus, modulo the assumptions mentioned above, we have

$$1/2t \leq H \leq 1/\sqrt{3}\, t$$

and

$$0.73 \leq q_0 \leq 1 .$$

In this paper we have attempted to convince the reader of the possibility that extra dimensions of space, which have appeared for technical reasons in the literature from time to time, may possess a hitherto unsuspected historical reality.

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