PHYSICAL REVIEW D VOLUME 21, NUMBER 8 15 APRIL 1980

Aspects of seven-dimensional relativity

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(Received 6 August 1979)

Two versions of a Kaluza-Klein model of seven-dimensional relativity are discussed. These models are characterized by having a fixed geometry for the internal parts of their manifolds. In version (1) the internal part is flat (T_3) and in version (2) it is curved (S_3). The physical interpretation of the internal coordinates is given in both versions. The main differences are that in version (2) the model features a cosmological constant given by the curvature constant of the sphere and the gauge fields depend in a definite way on the internal coordinates [they do not in version (1)]. These gauge fields appear as part of the metric tensor in seven dimensions. The result by Kerner and Cho which states that seven-dimensional relativity contains as a special case four-dimensional gravity coupled to Yang-Mills fields is rederived. The Dirac Lagrangian is given for both versions. It is defined to be the free-field Lagrangian in seven dimensions, i.e., it contains spinors coupled to seven-dimensional "gravity" only. Again as a special case it contains a gauge-invariant Lagrangian featuring a minimal coupling and a Fierz-Pauli term. The latter can be eliminated by choosing a particular way for the dimensional reduction procedure. Spinors carry an internal degree of freedom originating in the use of higher dimensions. For both versions this internal degree of freedom may be identified with the gauge degree of freedom. For version (1), scalar fields are also discussed and some restrictions concerning the inclusion of higher groups are given.

I. INTRODUCTION

During the past few years there has been a renewed interest in high- (>4) dimensional field-theoretical models.[1-9] As far as the present renaissance is concerned, the first motivation came from the natural appearance of higher dimensions in dual models.[1,2,5] Very soon, however, it was remembered[3] that high-dimensional models may sometimes be looked upon as unified models; the classical example being the Kaluza[10]-Klein[11] (KK) model which gives a unified Lagrangian for gravity coupled to electromagnetism. The basic assumption of this model is to identify part of the metric tensor of the high-dimensional space with the four-potential of the electromagnetic field.

More generally the present philosophy is the following: Given a Lagrangian in $4+N$ dimensions, $\mathcal{L}(x,\vec{x})$ (x are the ordinary space-time variables, $\vec{x}$ are the internal variables), one derives an effective (dimensionally reduced) Lagrangian $\mathcal{L}_I(x) = \int \mathcal{L}(x,\vec{x})d^N\vec{x}$. This Lagrangian may exhibit restrictions as compared to a Lagrangian for the same system which has been written down *ad hoc*. For example, if one tries in this way to unify Yang-Mills fields and Higgs fields into a single type of field (compare Refs. 7–9) one hopes to get relations between, say, the masses of the vector bosons and the Higgs bosons. [N.B.: Quite generally there is the hope that spontaneous symmetry breakdown may be achieved by using higher dimensions (compare Ref. 6).]

The intentions of this paper, however, are somewhat more technical, and I will return to more physical questions later. I will discuss a KK model in seven dimensions exhibiting gravity coupled to a non-Abelian Yang-Mills field and including scalar and spinor fields which are coupled to the Yang-Mills field in a gauge-invariant way. The Yang-Mills field, like the electromagnetic field before, appears as part of the metric tensor for the high-dimensional space. For the five-dimensional case (the gauge group being Abelian) this has been done by Thirring.[12] A Lagrangian which, after dimensional reduction, describes gravity coupled to a Yang-Mills field has been derived by Kerner[13] and Cho.[14] Furthermore, some sequences of an effective Dirac Lagrangian derived from a high-dimensional one have been discussed by Rayski[15]; in this case, however, the minimal coupling between spinors and gauge field is inserted by hand.

For the five-dimensional Abelian case it is known that a gauge-invariant effective Lagrangian for scalars and/or spinors coupled to the electromagnetic field can be derived from a free-field Lagrangian in five dimensions (compare, for example, Ref. 12). In this paper I will demonstrate that the same is possible for the seven-dimensional (non-Abelian) case.

Concerning the interpretation of the extra (internal) dimensions I will consider two choices. I will choose a fixed geometry for the internal part of the space (some further remarks on this point below in this section). The first choice is a flat internal space (a three-dimensional torus) and the gauge transformations *act* on the coordinates in a nonlinear way. The way this works will be made precise in Sec. VI. The second choice is to identify the internal part of the space with the gauge group as a manifold. I will explicitly consider the non-

Abelian Lie group SO(3) [SU(2)]. As a manifold this group has nonvanishing curvature and is identified with the sphere S_3.

Mathematically this discrimination seems to be that one between an associated fiber bundle (first choice) and a principal fiber bundle (second choice).[16] The first choice may be understood by analogy with the way Lorentz transformations and Minkowski space are related to each other. A difference to be kept in mind of course is that in the present case the action of the group is nonlinear.

Technically the most obvious difference between the two choices will be that in the first case the gauge fields do not depend on the internal coordinates, whereas in the second case they do, and the covariant derivatives of the gauge fields enter in a different way. The fact that in high-dimensional theories the gauge fields may depend on the internal coordinates has been observed before (compare Refs. 6 and 17); in the present case I give this dependence explicitly; that is, the gauge field is determined completely modulo its dependence on the Minkowskian variables (see Sec. VII). Similarly, I give the dependence of the extra fields (scalars, spinors) on the internal coordinates explicitly (see Secs. III, IV, and VIII).

In more detail, the organization of the paper is as follows: Sections II and VII (for the first and second choice for the internal part of the manifold, respectively) contain in the language of differential forms the rederivation of the result given by Kerner[13] and Cho,[14] namely, that an Einstein-Yang-Mills Lagrangian may be derived from a high-dimensional curvature scalar. An important distinction between the two cases is that for the second there is a (very large) cosmological term which does not appear in the first case.

It has been observed by Trautmann[18] that if the gauge fields appear as part of the metric tensor and the internal geometry is fixed in the way to be described in Secs. II and VII, the high-dimensional manifold is a fiber bundle with ordinary space-time as a base manifold and the gauge group as the structure group. It is worthwhile noting that the statement "a KK model exhibits the geometry of a fiber bundle" may as well be understood as "a KK model is a concrete visualization of a fiber bundle as a high-dimensional manifold." Technically this means that though one starts with a curvature scalar for, say, seven dimensions as a Lagrangian, nevertheless the group of allowed coordinate transformations is no longer the group of general coordinate transforms in seven dimensions, but rather a restricted group, which leaves the geometry of the internal (non-Minkowskian) part of the space fixed.[12]

In Secs. III and VIII (for the first and second choice of manifolds, respectively) an effective four-dimensional Dirac Lagrangian including gauge fields coupled in a gauge-invariant way is derived from a free-field Dirac Lagrangian in seven dimensions. The Dirac field is coupled to seven-dimensional "gravity," of course. The central technical problem in these sections is the following: The coupling between spinors and gauge fields in a KK model appears as

$$\int \bar{\Psi}(x,\tilde{x})\Gamma^\alpha \sum_{s=5}^{7} A_\alpha^s \partial_s \Psi(x,\tilde{x})\, dV_I\,, \tag{1.1}$$

where the integral is an invariant one over the internal part of the manifold. This has to be converted into

$$\bar{\psi}(x)\Gamma^\alpha \sum_{s=5}^{7} A_\alpha^s T_s \psi(x)\,, \tag{1.2}$$

where T_s give a matrix representation of the gauge group algebra on the spinor fields, (1.2) being the expression one is used to from the usual formulation of gauge models. The conversion from (1.1) to (1.2) is constrained by the fact that the dependence of the gauge fields on the internal coordinate is given. The conversion therefore has to be achieved by a suitable choice for the $\tilde{x}$ dependence of the spinor fields.

This problem is solved explicitly in both cases, and the effective four-dimensional Lagrangian indeed contains a minimal-coupling term. Since we are working in seven dimensions, in (1.1) and (1.2) the spinors are eight-dimensional objects; they may be looked upon as a doublet of ordinary Dirac spinors. For both choices under consideration, the internal label arising this way may be identified with the one that the T_s are acting upon. This means that the gauge degree of freedom may be identified with the spinors' internal degree of freedom stemming from higher dimensions. That this identification is possible has been assumed before.[3,15]

Furthermore, the Lagrangian contains a Fierz-Pauli term, as has been observed before for the five-dimensional case.[12,19] What is new in the present case is that this term can be eliminated by a suitable solution $\Psi(x,\tilde{x})$ (for both choices of internal coordinates). The appearance of the Fierz-Pauli term means that one is dealing with a nonrenormalizable Lagrangian, a not surprising result considering that one has started with a nonrenormalizable gravity-type theory. It would be an interesting question (though beyond the scope of this paper) whether the elimination of the Fierz-Pauli term is preserved in higher orders.

A Fierz-Pauli term furthermore violates CP invariance. (Phenomenologically, like in the five-dimensional case,[12] the Fierz-Pauli term occurs with too small a coefficient to account for the observed *CP* violation.) This does not mean, however, that a five-dimensional free-field Dirac Lagrangian would be less symmetric than a four-dimensional one. There exist three discrete symmetry transformations but they are not the same as in the usual (four-dimensional) theory. The observation that higher dimensions have nontrivial consequences for discrete symmetry transformations is due to Rayski[15] and Thirring.[12] The paper by Thirring, in particular, contains an exhaustive discussion of the five-dimensional case. I will discuss discrete symmetry transformations in a forthcoming paper.[20]

The last term in the Lagrangian originating from higher dimensions is a mass shift for the spinor field of the order of Planck's mass, i.e., 10^{19} GeV. The appearance of such high masses in KK models in well known (see, e.g., Ref. 12) and is one of the undesirable features of these models.

The remaining Secs. IV, V, and VI are all written for the case of the flat internal geometry. Section IV contains a discussion of the Lagrangian for scalar fields. In Sec. V an argument is given which, for gauge groups of rank 2 or higher, restricts the type of representations given by the spinor fields which can consistently be used. In case of SU(3) the representations excluded are those with noninteger triality.

In Sec. VI I discuss the relation between (global) gauge transformations and internal coordinate transformations. This need not be done for the alternative choice of manifold since, if the internal part of the space is identified with the group as a manifold, this relation is obvious.

Section IX contains some concluding remarks. Finally, the Appendices contain some calculational details which have been left out from the actual text. The last Appendix, E, is a glossary, where I compile the notations used in this paper.

II. THE KALUZA-KLEIN *ANSATZ* FOR THE METRIC: FLAT INTERNAL SPACE

I will consider a seven-dimensional space with signature $(- - - +; - - -)$. The restriction to seven dimensions is made with hindsight to the later sections of this paper; the results of this section can be formulated for higher dimensions as well. The internal dimensions are chosen to be spacelike in order to reproduce the right coupling of gravity to the Yang-Mills field (see below).

The starting point for a Kaluza-Klein theory is to assume the following decomposition of the seven-dimensional metric tensor G_{mn}:

$$G_{mn}=\begin{array}{c} a \\ \\ u \\ \\ \end{array}\left[\begin{array}{cc} \bar{g}_{ab}+g_{tu}A^t_aA^u_b & -g_{wt}A^t_a \\ & \\ -g_{ut}A^t_b & g_{uw} \\ b & w \end{array}\right]. \tag{2.1}$$

The conventions about indices are as follows: Latin indices are world indices, Greek indices are frame indices (for a definition of frames and other basic information on differential forms see, e.g., Ref. 21). Indices early in the alphabet run from 1 to 4: $\alpha,\beta,\gamma,\delta=1,\ldots,4$; $a,b,c,d=1,\ldots,4$; middle indices run from 1 to 7: μ, ν, λ, $\rho=1,\ldots,7$; $m,n,l,k=1,\ldots,7$; late indices from 5 to 7: ω, φ, $\sigma,\tau=5,6,7$; $u,v,w,s,t=5,6,7$.

For this section as well as for Secs. III–VI the internal part of the space will have the structure of a three-dimensional torus, $T_3=T_1\times T_1\times T_1$. Since T_3 is flat, it is no restriction to write

$$g_{uw}=-\delta_{uw}. \tag{2.2}$$

The motivation for this choice is that (2.2) constitutes the Killing-Cartan metric of the group SO(3), which appears in the setup of a Yang-Mills theory.

Our aim now is to work out the curvature scalar corresponding to the metric tensor (2.1). It will be demonstrated that the $N+4=7$ dimensional curvature scalar R_{N+4} decomposes as

$$R_{N+4}=\bar{R}_4-F^2/4, \tag{2.3}$$

where $\bar{R}_4$ is the four-dimensional curvature scalar constructed from $\bar{g}_{ab}$ and F^2 is the ordinary Yang-Mills Lagrangian density. This means that $\bar{g}_{ab}$ rather than G_{ab} is the physical four-dimensional metric tensor, i.e., the metric tensor which describes gravity (see below as well). The expression on the right-hand side of (2.3) is independent of the internal coordinates and thus gives the ordinary Einstein-Yang-Mills Lagrangian.

The derivation of (2.3) will involve the following steps. The curvature scalar in seven dimensions is defined as

$$R_{N+4}=R^{\mu}{}_{\lambda}{}^{\gamma}{}_{\mu}=R^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha}+2R^{\alpha}{}_{\tau}{}^{\tau}{}_{\alpha}+R^{\sigma}{}_{\tau}{}^{\tau}{}_{\sigma}, \tag{2.4}$$

where the curvature tensor $R_{\mu\nu\rho\lambda}$ is defined via the expansion of the curvature two-forms $\Omega_{\mu\nu}$ in terms of the basic one-forms θ^ρ,

$$\Omega_{\mu\nu}=\tfrac{1}{2}R_{\mu\nu\rho\lambda}\theta^\rho\wedge\theta^\lambda. \tag{2.5}$$

The curvature two-forms will be determined from the connection one-forms $\omega_{\mu\nu}$ via Cartan's second identity:

$$d\omega_{\mu\nu}+\omega_{\mu\lambda}\wedge\omega^{\lambda}{}_{\nu}=\Omega_{\mu\nu}. \tag{2.6}$$

The connection one-forms in turn are determined from the basic one-forms via Cartan's first identity

$$d\theta^{\mu}+\omega^{\mu}{}_{\nu}\wedge\theta^{\nu}=0\,. \tag{2.7}$$

The first thing to be done is therefore to make a convenient choice for the basic one-forms, i.e., to choose a frame.

The seven-dimensional line element corresponding to (2.1) is

$$\begin{aligned} dS^2&=G_{mn}dx^m dx^n=\eta_{\mu\nu}\theta^{\mu}\theta^{\nu}\\ &=\bar g_{ab}\,dx^a dx^b+g_{tu}(dx^t-A^t_a dx^a)(dx^u-A^u_b dx^b)\,, \end{aligned} \tag{2.8}$$

where

$$\eta_{\mu\nu}=\mathrm{diag}(-\,-\,-\,+\,;\,-\,-\,-)\,. \tag{2.9}$$

The first line of (2.8) defines the frame θ^{μ}. A suitable frame for us will be one with

$$dS^2=\eta_{\alpha\beta}\bar\theta^{\alpha}\bar\theta^{\beta}+\eta_{\sigma\tau}\theta^{\sigma}\theta^{\tau}\,, \tag{2.10}$$

where the basic one-forms are given by[22]

$$\begin{aligned}\bar\theta^{\alpha}&=\theta^{\alpha}\,,\\ \theta^{\tau}&=dx^{\tau}-A^{\tau}_{\alpha}dx^{a}\,.\end{aligned} \tag{2.11}$$

Barred quantities such as $\bar\theta^{\alpha}$ are to be understood in such a way that they refer to quantities defined with respect to the physical metric tensor $\bar g_{ab}$, rather than with respect to $G_{ab}=\bar g_{ab}+g_{vt}A^v_a A^t_b$. To simplify formulas in the text I extend the definition of A^u_a to A^n_m, where the extra components are defined to be

$$A^{\tau}_{u}=A^{\alpha}_{a}=0\,. \tag{2.12}$$

In order to apply Cartan's first identity to (2.11) it is necessary to define the one-form dA^t_b. I will set

$$dA^t_b=D_nA^t_b\,dx^n=D_aA^t_b\,dx^a+D_sA^t_b dx^s\,, \tag{2.13}$$

where (the Γ^t_{rs} are the structure constants of the gauge group)

$$D_nA^t_b=\nabla_nA^t_b+\Gamma^t_{rs}A^r_nA^s_b \tag{2.14}$$

contains the usual partial derivative and the Christoffel symbols constructed from $\bar g_{ab}$ and g_{uv}. One has, in particular [remember (2.12)],

$$D_sA^t_b=\nabla_sA^t_b=\partial_sA^t_b\,. \tag{2.15}$$

The statement (2.13) and (2.14) is the requirement that the A behave like gauge fields, and the two-form $d(A^t_m dx^m)$ is constructed by means of the gauge-covariant derivative D_n. Mathematically one may understand this so that $A^t_b dx^b$ is treated as a connection one-form and the covariant derivative is defined to give the gauge-covariant curvature two-form. If one proceeds this way, one may as in the five-dimensional case consistently assume that the components of the metric tensor G_{mn} do not depend on the internal coordinates. Therefore in particular from (2.13) and (2.14),

$$dA^t_b=D_aA^t_b dx^a\,. \tag{2.16}$$

It is worthwhile to remark that for a curved internal manifold the gauge fields have to depend on the internal coordinates, and the gauge-covariant derivative can be introduced in quite a different way (see Sec. VII). From (2.11), (2.16):

$$\begin{aligned} d\theta^{\sigma}&=-\tfrac12 D_aA^{\sigma}_b dx^b\wedge dx^a\\ &=-\tfrac12 F^{\sigma}_{ab}dx^a\wedge dx^b=-\tfrac12 F^{\sigma}{}_{\alpha\beta}\theta^{\alpha}\wedge\theta^{\beta}\,. \end{aligned} \tag{2.17}$$

Cartan's first identity reads [see (2.7)]

$$d\theta^{\sigma}=-\omega^{\sigma}{}_{\alpha}\wedge\theta^{\alpha}-\omega^{\sigma}{}_{\tau}\wedge\theta^{\tau}\,, \tag{2.18}$$

from which, comparing coefficiencts, one finds

$$\omega^{\sigma}{}_{\tau}=0\,, \tag{2.19}$$

$$\omega^{\sigma}{}_{\alpha}=-\tfrac12 F^{\sigma}{}_{\alpha\beta}\theta^{\beta}\,. \tag{2.20}$$

The barred connections $\bar\omega^{\alpha}{}_{\beta}$ are introduced by

$$d\bar\theta^{\alpha}=-\bar\omega^{\alpha}{}_{\beta}\wedge\bar\theta^{\beta}\,. \tag{2.21}$$

This equation separates the effect of physical gravity ($\bar g_{ab}$) from the remainder of the metric tensor: $\bar\omega^{\alpha}{}_{\beta}$ is that part of the connection which is due to the presence of physical gravity (which is described by $\bar g_{ab}$ rather than G_{ab}). On the other hand, from (2.11)

$$d\bar\theta^{\alpha}=d\theta^{\alpha}=-\omega^{\alpha}{}_{\beta}\wedge\theta^{\beta}-\omega^{\alpha}{}_{\tau}\wedge\theta^{\tau}\,, \tag{2.22}$$

and one finds

$$\omega_{\alpha\beta}=\bar\omega_{\alpha\beta}+\tfrac12\eta_{\sigma\tau}F^{\tau}{}_{\alpha\beta}\theta^{\sigma}\,. \tag{2.23}$$

All connections are now known in terms of physically identifiable quantities. One may therefore proceed to work out the curvature two-forms.[23]

It is convenient to work out $\Omega_{\alpha\beta}, \Omega_{\alpha\tau}, \Omega_{\varphi\tau}$ separately. From Cartan's second identity [see (2.6)],

$$\Omega_{\alpha\beta}=d\omega_{\alpha\beta}+\omega_{\alpha\gamma}\wedge\omega^{\gamma}{}_{\beta}+\omega_{\alpha\tau}\wedge\omega^{\tau}{}_{\beta}\,. \tag{2.24}$$

Using the information given above the terms on the right-hand side turn out to be

$$d\omega_{\alpha\beta}=d\bar\omega_{\alpha\beta}-\tfrac14\eta_{\sigma\tau}F^{\tau}{}_{\alpha\beta}F^{\sigma}{}_{\gamma\delta}\theta^{\gamma}\wedge\theta^{\delta}\,, \tag{2.25}$$

$$\begin{aligned}\omega_{\alpha\gamma}\wedge\omega^{\gamma}{}_{\beta}=(\bar\omega_{\alpha\gamma}\wedge\bar\omega^{\gamma}{}_{\beta})&+\tfrac14\eta_{\sigma\tau}\eta_{\varphi\omega}\eta^{\gamma\gamma'}F^{\tau}{}_{\alpha\gamma}F^{\varphi}{}_{\gamma'\beta}\theta^{\sigma}\wedge\theta^{\omega}\\ &-\tfrac12\eta_{\sigma\tau}F^{\tau}{}_{\alpha\gamma}\theta^{\sigma}\wedge\bar\omega^{\gamma}{}_{\beta}-\tfrac12\eta_{\varphi\omega}\eta^{\gamma\gamma'}F^{\varphi}{}_{\gamma'\beta}\bar\omega_{\alpha\gamma}\wedge\theta^{\omega}\,,\end{aligned} \tag{2.26}$$

$$\omega_{\alpha\tau}\wedge\omega^{\tau}{}_{\beta}=-\tfrac14\eta_{\tau\sigma}F^{\sigma}{}_{\alpha\gamma}F^{\tau}_{\beta\delta}\theta^{\gamma}\wedge\theta^{\delta}\,. \tag{2.27}$$

The other elements of the curvature two-forms are

$$\Omega_{\alpha\tau}=d\omega_{\alpha\tau}+\omega_{\alpha\beta}\wedge\omega^{\beta}{}_{\tau}+\omega_{\alpha\varphi}\wedge\omega^{\varphi}{}_{\tau}\,, \tag{2.28}$$

where

$$d\omega_{\alpha\tau}=-\tfrac14\eta_{\tau\sigma}\big(D_{\gamma}F^{\sigma}{}_{\alpha\beta}\big)\theta^{\gamma}\wedge\theta^{\beta}+\tfrac12\eta_{\tau\sigma}F^{\sigma}{}_{\alpha\beta}\bar\omega^{\beta}{}_{\gamma}\wedge\theta^{\gamma}\,, \tag{2.29}$$

$$\omega_{\alpha\beta}\wedge\omega^{\beta}{}_{\tau}=-\tfrac{1}{2}\eta_{\omega\tau}F^{\omega}{}_{\beta\gamma}\overline{\omega}_{\alpha\beta}\wedge\theta^{\gamma}+\tfrac{1}{4}\eta_{\sigma\varphi}\eta_{\omega\tau}F^{\varphi}{}_{\alpha\beta}F^{\omega}{}_{\beta\gamma}\theta^{\sigma}\wedge\theta^{\gamma}, \tag{2.30}$$

and

$$\omega_{\alpha\varphi}\wedge\omega^{\varphi}{}_{\tau}=0, \tag{2.31}$$

and finally

$$\Omega_{\varphi\tau}=d\omega_{\varphi\tau}+\omega_{\varphi\alpha}\wedge\omega^{\alpha}{}_{\tau}+\omega_{\varphi\sigma}\wedge\omega^{\sigma}{}_{\tau}, \tag{2.32}$$

$$d\omega_{\varphi\tau}=0=\omega_{\varphi\sigma}\wedge\omega^{\sigma}{}_{\tau}, \tag{2.33}$$

and

$$\omega_{\varphi\alpha}\wedge\omega^{\alpha}{}_{\tau}=-\tfrac{1}{4}\eta_{\varphi\tau'}\eta_{\tau\tau''}\eta^{\alpha\alpha'}F^{\tau'}{}_{\alpha\beta}F^{\tau''}{}_{\alpha'\gamma}\theta^{\beta}\wedge\theta^{\gamma}, \tag{2.34}$$

so that there are no contributions to the seven-dimensional curvature scalar from $\Omega_{\varphi\tau}$. The following terms contribute to the curvature scalar R_{4+N}:

(i) To $\overline{R}_4$: The first terms on the right-hand sides of (2.25) and (2.26), respectively.

(ii) Extra contributions from higher dimensions, giving F^2—the second terms on the right-hand sides of (2.25) and (2.30), respectively, and the right-hand side of (2.27).

Putting everything together one finds eventually[24]

$$R_{4+N}=\overline{R}_4-F^2/4, \tag{2.35}$$

where

$$F^2=-\sum_{\sigma=5}^{7}F^{\sigma}{}_{\alpha\beta}F^{\alpha\beta}_{\sigma}. \tag{2.36}$$

Some remarks are in order.

(1) Had we chosen a signature $(- - - +;+++)$ rather than $(- - - +;\ - - -)$, then the minus sign in (2.35) would be reversed. The relative sign as in (2.35) would be reversed. The relative sign, as in (2.35) is, however, the wanted one.

(2) The formalism allows the following freedom: to replace

$$G_{wa}=-g_{wt}A^t_a \tag{2.37}$$

by

$$G_{wa}=-fg_{wt}A^t_a, \tag{2.38}$$

where f is a constant. Furthermore (2.14) may be replaced by ($e=$ const)

$$D_nA^t_m=\nabla_nA^t_m+e\Gamma^t_{rs}A^r_nA^s_m. \tag{2.39}$$

Then instead of (2.35),

$$R_{4+N}=\overline{R}_4-(f^2/4)F^2, \tag{2.40}$$

where now

$$F^t_{ab}=\nabla_aA^t_b-\nabla_bA^t_a+e\Gamma^t_{rs}A^r_aA^s_b. \tag{2.41}$$

(3) From (2.40) one concludes that f must be related to the gravitational constant κ via

$$\kappa=f^2. \tag{2.42}$$

(4) It is straightforward to generalize the results of this section to more than seven dimensions.

III. THE DIRAC EQUATION

The covariant derivative $\varphi_{|\mu}$ of a spinor and/or tensor field is given by[25]

$$\varphi_{|\mu}=\varphi_{\mu}+\Sigma^{\nu\lambda}\omega_{\mu\nu\lambda}\varphi. \tag{3.1}$$

Here the $\Sigma^{\nu\lambda}$ give a finite-dimensional representation of the frame group (see Appendix A) with the φ as representation vectors. The $\omega_{\mu\nu\lambda}$ are the Ricci rotation coefficients. They are related to the connection one-forms via $\omega_{\mu\nu}=\omega_{\mu\nu\lambda}\theta^{\lambda}$. If no torsion is present and the natural basis $\theta_{\mu}=dx_{\mu}$ is being used, the $\omega_{\mu\nu\lambda}$ are the Christoffel symbols.

It is elucidating to work out (3.1) for a known case. In fact, if φ is a vector field, then the matrices $\Sigma_{\mu\nu}$ have elements

$$[\Sigma_{\mu\nu}]^{\lambda}{}_{\rho}=\eta_{\nu\rho}\delta^{\lambda}{}_{\mu}-\eta_{\mu\rho}\delta^{\lambda}{}_{\nu} \tag{3.2}$$

and for $\theta_{\mu}=dx_{\mu}$—the $\omega_{\mu\nu\lambda}$ being the Christoffel symbols—(3.1) reduces to the ordinary covariant derivative of a vector. An intuitive meaning of (3.1) emerges if one thinks of $\varphi_{|\mu}$ as a gauge-covariant derivative, the gauge group being the frame group.

In the case under consideration, the frame group is SO(6, 1) (see Appendix A) and φ is a spinor with eight components. The generators $\Sigma^{\mu\nu}$ can be written as

$$\Sigma^{\mu\nu}=\tfrac{1}{8}[\Gamma^{\mu},\Gamma^{\nu}], \tag{3.3}$$

where the Γ_{μ}, $\mu=1,\ldots,7$ are a set of generalized Dirac matrices. They are explicitly given in Appendix B. It is important to note that the generalized Clifford algebra has the form

$$\{\Gamma^{\mu},\Gamma^{\nu}\}=2\eta^{\mu\nu}, \tag{B1}$$

which is just the usual form. Using the basis dx^m rather than the frame θ^{μ} would mean that one would have to deal with a space-time-dependent Clifford algebra:

$$\{\Gamma^m,\Gamma^n\}=2G^{mn}. \tag{3.4}$$

For the following we rewrite (3.1) in terms of forms. Using (3.3) one has

$$D\varphi=d\varphi+\tfrac{1}{8}\,\omega_{\nu\lambda}[\Gamma^{\nu},\Gamma^{\lambda}]\varphi, \tag{3.5}$$

with

$$D\varphi=\varphi_{|\mu}\theta^{\mu},\quad d\varphi=\varphi_{,\mu}\theta^{\mu}. \tag{3.6}$$

Here the relation between the Ricci rotation coefficients and the connection one-forms has been used. It may be stated as

$$\langle \omega_{\nu\lambda} \cdot \theta_\mu \rangle = \omega_{\nu\lambda\mu}\,, \tag{3.7}$$

where the scalar product $\langle \cdot \rangle$ is defined by

$$\langle \theta^\lambda \cdot \theta_\mu \rangle = \delta^\lambda{}_\mu\,. \tag{3.8}$$

We are now in position to write down the generalized Dirac equation for the generalized Dirac field Ψ. It reads

$$im_0\Psi = \Gamma^\mu \Psi_{|\mu} = (\Gamma^\mu d_\mu + \tfrac{1}{8}\langle \omega_{\mu\nu} \cdot \theta_\sigma \rangle \Gamma^\sigma [\Gamma^\mu, \Gamma^\nu])\Psi\,. \tag{3.9}$$

Here d_μ are the frame components of the seven-dimensional gradient (the partial derivatives). Their explicit form is given below. The Ψ is an eight-dimensional spinor field depending on seven variables, $\Psi = \Psi(x, \vec{x})$. I will use the notation that $x = (x^a)$ means the Minkowskian part of the generalized space-time and $\vec{x} = (x^s)$ means the internal part. Defining the generalized Dirac-adjoint spinor by $\bar{\Psi} = \Psi^\dagger \Gamma^0$, the four-dimensional effective Dirac Lagrangian is given by[26]

$$\mathcal{L}(x) = \int d^3x\, (im_0 \bar{\Psi}\Psi - \bar{\Psi}\Gamma^\mu \Psi_{|\mu})\,. \tag{3.10}$$

Note that d^3x is indeed the invariant volume element for T_3. It has been shown[12] that in the case of a five-dimensional KK theory an *Ansatz* $\Psi(x, x^5)$ can be found such that (3.10) reduces to the ordinary Lagrangian density for a generalized eight-component Dirac field coupled to the electromagnetic field. This coupling contains a minimal-coupling term and a Fierz-Pauli term. The corresponding result will now be demonstrated for the present case.

As a first step it is convenient to rewrite the generalized Dirac equation (3.9) in terms of Ψ, A, and F. For convenience I will in the following assume that no gravity is present. Formally this means $\theta^\alpha = dx^\alpha, \bar{g}^{ab} = \eta^{ab}$.

The frame components of the partial derivatives are given by

$$d_\mu \Psi = e^m{}_\mu d_m \Psi\,, \tag{3.11}$$

where the components of the $(4+N)$-beins are introduced by

$$\theta^\mu = e^\mu{}_m dx^m\,, \quad dx^m = e^m{}_\mu \theta^\mu\,. \tag{A3}$$

They are explicitly written down in (A4) and (A5). In (3.11) d_m is the ordinary partial derivative, $d_1\Psi = (\partial/\partial x^1)\Psi = \Psi_{,1}$. Explicitly, one then finds for the frame components of the partial derivative for $m=1$ [remember (2.12)]:

$$\Psi_{,\underline{1}} = \Psi_{,1} + A_1^m \Psi_{1m} = \Psi_{,1} + A_1^s \Psi_{,s} \tag{3.12}$$

and similarly for $m = 2, 3, 4$. In (3.12) the numerical frame index has been written with a bar. For $m = 5, 6, 7$ one has

$$\Psi_{,\underline{5}} = \Psi_{,5}\,, \quad \Psi_{,\underline{6}} = \Psi_{,6}\,, \quad \Psi_{,\underline{7}} = \Psi_{,7}\,. \tag{3.13}$$

In order to work out the second half of the right-hand side of (3.9) one has to remember that the absence of gravity means for the connection one-forms

$$\begin{aligned} &\omega^\sigma{}_\tau = 0\,, \\ &\omega_{\alpha\beta} = \tfrac{1}{2} F^\tau{}_{\alpha\beta}\theta_\tau\,, \\ &\omega^\tau{}_\alpha = \tfrac{1}{2} F^\tau{}_{\alpha\beta}\theta^\beta\,. \end{aligned} \tag{3.14}$$

Using (3.8) one then finds

$$\langle \omega_{\mu\nu} \cdot \theta_\sigma \rangle \Gamma^\sigma [\Gamma^\mu, \Gamma^\nu] = \tfrac{1}{2} F^\varphi{}_{\beta\gamma} \Gamma_\varphi [\Gamma^\beta, \Gamma^\gamma]\,. \tag{3.15}$$

Using (3.12), (3.13), and (3.15) the Dirac equation (3.9) becomes

$$\begin{aligned} im_0\Psi = {} & \Gamma^\alpha (\delta^a_\alpha d_a + A^s_\alpha d_s)\Psi \\ & + \Gamma^{\underline{5}} \Psi_{,5} + \Gamma^{\underline{6}} \Psi_{,6} + \Gamma^{\underline{7}} \Psi_{,7} \\ & + \tfrac{1}{16} F^\varphi{}_{\beta\gamma} \Gamma_\varphi [\Gamma^\beta, \Gamma^\gamma]\,. \end{aligned} \tag{3.16}$$

The corresponding Lagrangian density is

$$\begin{aligned} \mathcal{L}(x) = \int d^3\vec{x}\, \{ & \eta^{\alpha\beta} \bar{\Psi} \Gamma_\alpha (\partial_\beta + A^\tau_\beta \partial_\tau)\Psi - im_0 \bar{\Psi}\Psi \\ & + \bar{\Psi}\Gamma^{\underline{5}}\Psi_{,5} + \bar{\Psi}\Gamma^{\underline{6}}\Psi_{,6} + \bar{\Psi}\Gamma^{\underline{7}}\Psi_{,7} \\ & + \tfrac{1}{16} \bar{\Psi} F^\varphi{}_{\beta\gamma} \Gamma_\varphi [\Gamma^\beta, \Gamma^\gamma]\Psi \}\,, \end{aligned} \tag{3.17}$$

where $\partial_\beta = d_b$ if $\beta = b$ and $\partial_\tau = d_t$ if $\tau = t$. In the approach taken here one now searches for an $\vec{x}$ dependence for $\Psi(x, \vec{x})$ such that

$$\int d^3\vec{x}\, \eta^{\alpha\beta} \bar{\Psi} \Gamma_\alpha (\partial_\beta + A^\tau_\beta \partial_\tau)\Psi$$

becomes the ordinary minimal-coupling term and

$$\bar{\Psi}\Gamma^{\underline{5}}\Psi_{,5} + \bar{\Psi}\Gamma^{\underline{6}}\Psi_{,6} + \bar{\Psi}\Gamma^{\underline{7}}\Psi_{,7}$$

can be absorbed into the mass term.

I do not know of any systematical way of solving this problem. In the Abelian case [the gauge group for the Yang-Mills field being $\mathrm{U}(1)^N$] a simple Fourier expansion will work.[12,15] The reason is that in this case the gauge group not only acts transitively on T_N but can be identified also with T_N as a manifold (modulo multiple coverings).

For the present case, however, I have found a special ansatz which gives the desired result. I put

$$\begin{aligned} \Psi(x, \vec{x}) = {} & [\exp(i\mu T^5 x_5) + \exp(i\mu T^6 x_6) \\ & + \exp(i\mu T^7 x_7)]\psi(x)\,. \end{aligned} \tag{3.18}$$

As will become more apparent later, μ must have the dimension of a mass. Accordingly x^s will have the dimension of a length. Formally this may be taken care of by writing

$$\mu = \mu_0 M_0\,, \quad x^s = x_0^s / M_0$$

so that

$$x^s \mu = x_0^s \mu_0$$

and M_0 sets the mass scale. Referring to μ as an integer, as will be done below, strictly refers to μ_0, provided x_0^s takes values out of the interval $[0, 2\pi]$. Consider now the relation

$$i\mu T_v \int_0^{2\pi} \exp(i\mu T_v x^v) = \exp(i\mu T_v 2\pi) - 1 \qquad (3.19)$$

(no sum over v).

For integer μ (even integer μ) the right-hand side of (3.19) vanishes provided ψ carries integer isospin (half-integer isospin), since then $\exp(i\mu T_v 2\pi)$ describes a rotation around the v axis by an angle $2\pi\mu$ which in turn is described by the unit matrix. Thus for integer (even integer) μ,

$$\int d^3x \overline{\Psi} \Gamma^\alpha A_\alpha^s \partial_s \Psi = (2\pi)^3 i\mu \overline{\psi} (\Gamma^\alpha A_\alpha^s T_s)\psi\,, \qquad (3.20)$$

which is just the form required for minimal coupling and

$$\int d^3x\, \overline{\Psi}[\Gamma^{\underline{5}}\partial_5 + \Gamma^{\underline{6}}\partial_6 + \Gamma^{\underline{7}}\partial_7]\Psi$$
$$= (2\pi)^3 i\mu \psi [\Gamma^{\underline{5}} T_5 + \Gamma^{\underline{6}} T_6 + \Gamma^{\underline{7}} T_7]\psi\,. \quad (3.21)$$

For nonsingular T_v one has

$$\int_0^{2\pi} dx^v \exp(i\mu T_v x^v) = 0 \quad \text{(no sum over } v)\,, \qquad (3.22)$$

and (3.17) reduces to

$$\mathcal{L}(x) = (2\pi)^3 \{\eta^{\alpha\beta} \overline{\psi} \Gamma_\alpha (3\partial_\beta + i\mu A_\beta^s T_s)\psi - 3im_0 \overline{\psi}\psi$$
$$+ i\mu\overline{\psi}[\Gamma^{\underline{5}} T_5 + \Gamma^{\underline{6}} T_6 + \Gamma^{\underline{7}} T_7]\psi$$
$$+ \tfrac{1}{16} r \overline{\psi} F^\varphi{}_{\beta\gamma} \Gamma_\varphi [\Gamma^\beta \Gamma^\gamma]\psi\}\,. \qquad (3.23)$$

Here $r = 3$ if all T and Γ commute. If $T_v = t_v$, where the t_v are defined in equation (B3), then $r = 1$. Note that the condition that T_v are nonsingular is sufficient to ensure (3.22) but not necessary. In fact, if T_v in the adjoint representation of SU(2), $(T_v)^{st} \sim \epsilon_{vst}$, then $\det(T_v) = 0$ but still (3.22) holds true as may be checked by explicit calculation. For the fundamental two-dimensional representation of SU(2) the T_v are the Pauli matrices and nonsingular. As a side remark I note that (3.22) holds true as well for the fundamental three-dimensional representation of the SU(3) algebra given by the Gell-Mann matrices.

For the remaining discussion of (3.23) I will restrict myself to the case $T_v = t_v$. The motivation for this is the following: If the eight-dimensional spinors are looked upon as a doublet of ordinary Dirac spinors, then this means that this internal degree of freedom of the eight-dimensional spinors is being identified with the gauge degree of freedom. This seems to be the most economical way of making use of this internal degree of freedom.

In this case, i.e., if $T_v = t_v$,

$$\Gamma^5 t_5 = \Gamma^6 t_6 = \Gamma^7 t_7 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}. \qquad (3.24)$$

Furthermore, using the explicit form for the $(4+N)$-beins, numerically

$$\Gamma^{5,6,7} = \Gamma^{\underline{5},\underline{6},\underline{7}}\,. \qquad (3.25)$$

Thus, eventually from (3.23),

$$\mathcal{L}(x) = (2\pi)^3 \{\eta^{\alpha\beta} \overline{\psi} (\Gamma_\alpha \partial_\beta + \tfrac{1}{3} i\mu f A_\beta^s t_s)\psi$$
$$- im_0 \overline{\psi}\psi + i\mu\overline{\psi}\Gamma^5 t_5 \psi$$
$$+ \tfrac{1}{48} f \overline{\psi} F^\varphi{}_{\beta\gamma} \Gamma_\varphi [\Gamma^\beta, \Gamma^\gamma]\psi\}\,, \qquad (3.26)$$

where the constants $f = \sqrt{\kappa}$ and e (the latter within the definition of the field-strength tensor) have been introduced again. In order to have gauge invariance the identification

$$2\mu f/3 = e\,, \quad \mu = 3e/(2\sqrt{\kappa}) \qquad (3.27)$$

has to be made. Here the extra factor 2 in $(2\mu f/3) = e$ stems from the normalization of t_φ,

$$[t_\varphi, t_\omega] = 2\Gamma^\tau{}_{\varphi\omega} t_\tau\,, \quad \Gamma^\tau{}_{\varphi\omega} = -i\epsilon_{\tau\varphi\omega}\,,$$

which I have chosen in consistency with later parts of this paper. Since

$$\sqrt{\kappa} \sim 10^{-19}\ \text{GeV}^{-1}\,, \qquad (3.28)$$

μ is a mass parameter of the order 10^{18} to 10^{20} GeV (Planck's mass). It is by this way of identifying coupling constants that Planck's mass enters into a KK theory, thus setting a scale for the tightness of the internal coordinates.

In order to get the final form for $\mathcal{L}(x)$ the term $i\mu\overline{\psi}\Gamma^5 t_5 \psi$ in (3.26) has to be absorbed into the mass term. This can be done by means of the substitution

$$\psi(x) \to e^{\phi\Gamma^5 t_5}\psi(x)\,. \qquad (3.29)$$

The phase ϕ is determined by the requirement that the mass term be a multiple of the unit matrix. The Dirac equation following from (3.26) changes under the substitution (3.29) into

$$-im_0\psi + e^{-2\phi\Gamma^5 t_5}\Gamma^\alpha(\partial_\alpha + \tfrac{1}{3} i\mu f A_\alpha^\varphi t_\varphi)\psi$$
$$+ i\mu\Gamma^5 t_5 \psi + \tfrac{1}{48} f F^\varphi{}_{\beta\gamma}\Gamma_\varphi[\Gamma^\beta, \Gamma^\gamma]\psi = 0\,. \quad (3.30)$$

Multiplying the whole equation by $\exp(2\phi\Gamma^5 t_5)$ the mass matrix M of the spinor field ψ is given by

$$M = [\exp(2\phi\Gamma^5 t_5)](m_0 - \mu\Gamma^5 t_5)\,. \qquad (3.31)$$

Inserting this into (3.29) and multiplying again by $\bar{\psi}$, the final form for the Lagrangian is $[\tilde{\partial}=\frac{1}{2}(\vec{\partial}-\overleftarrow{\partial})]$

$$\mathcal{L}(x)=\left\{-i\bar{\psi}M\psi+\bar{\psi}\Gamma^{\alpha}[\tilde{\partial}_{\alpha}+(ie/2)A_{\alpha}^{\varphi}t_{\varphi}]\psi + \frac{f}{48}\bar{\psi}\frac{1}{M}(m_0+\mu\Gamma^5 t_5)F^{\varphi}{}_{\beta\gamma}\Gamma_{\varphi}[\Gamma^{\beta}\Gamma^{\gamma}]\psi\right\}, \tag{3.32}$$

with

$$M=\sqrt{m_0{}^2+\mu^2}=\sqrt{MM^{\dagger}} \tag{3.33}$$

for a suitable phase ϕ and $(2\mu f/3)=e$ [cf. (3.27)]. In fact, using the Hermitian matrix $\tilde{\Gamma}^5=\Gamma^5 t_5/i$,

$$\begin{aligned}M&=e^{2i\phi\tilde{\Gamma}^5}(m_0-i\mu\tilde{\Gamma}^5)\\&=(m_0\cos2\phi-\mu\sin2\phi)+i\tilde{\Gamma}^5(m_0\sin2\phi+\mu\cos2\phi).\end{aligned} \tag{3.34}$$

In order for M to be diagonal,

$$\mu/m_0=-\tan2\phi, \tag{3.35}$$

thus

$$M=m_0(1+\tan^2 2\phi)^{1/2}=(m_0{}^2+\mu^2)^{1/2}. \tag{3.36}$$

Accordingly the mass of the spinor field in (3.32) is of the order of Planck's mass. One might ask whether the result that the mass of the spinor field is shifted by Planck's mass is dependent on the Ansatz being used for $\psi(x,\vec{x})$. I therefore discuss a few modifications of this *Ansatz* in Appendix C, but none of them changes the result in question.

IV. SCALARS AND A MODIFICATION OF THE DIRAC EQUATION

Principally in the same manner as for spinors it is possible to include scalars into the theory. Unlike for spinors, the Lagrangian for the scalars will be just a minimal-coupling Lagrangian.

The free Lagrangian for a scalar field (coupled to seven-dimensional "gravity") is given by

$$\mathcal{L}(x)=\int d^3x\,(G^{mn}\Phi^{\dagger}{}_{,m}\Phi_{,n}-m_0{}^2\Phi^{\dagger}\Phi), \tag{4.1}$$

where

$$\Phi=\Phi(x,\vec{x}) \tag{4.2}$$

and G^{mn} is the inverse of the seven-dimensional metric tensor G_{mn}. Its explicit form is given in Appendix A. Correspondingly [see (A2)]

$$\begin{aligned}G^{mn}\Phi^{\dagger}{}_{,m}\Phi_{,n}&=\Phi^{\dagger}{}_{,a}\Phi_{,b}\bar{g}^{ab}+\Phi^{\dagger}{}_{,u}\Phi_{,v}g^{uv}+\Phi^{\dagger}{}_{,u}\Phi_{,a}\bar{g}^{ab}A^{u}{}_{b}\\&\quad+\Phi^{\dagger}{}_{,a}\Phi_{,u}\bar{g}^{ab}A^{u}{}_{b}+\Phi^{\dagger}{}_{,u}\Phi_{,v}A^{u}{}_{b}A^{v}{}_{a}\bar{g}^{ab}.\end{aligned} \tag{4.3}$$

The analog of the simple *Ansatz* given for the spinor field does not work. For the fundamental representation of SU(2) I have found, however, that the following *Ansatz* allows to reproduce the minimal-coupling Lagrangian for the scalar field:

$$\begin{aligned}\Phi(x,\vec{x})=\{&\alpha[c_5^{(1)}+c_6^{(1)}+c_7^{(1)}]\\&+i\beta[t_5 s_5^{(2)}+t_6 s_6^{(2)}+t_7 s_7^{(2)}]\}\,\phi(x).\end{aligned} \tag{4.4}$$

In this section $t_{5,6,7}$ are the Pauli matrices

$$[t_s,t_v]=\sum_w 2i\epsilon_{svw}t_w,\quad \{t_s,t_v\}=2\delta_{sv}.$$

The notations in (4.4) are the following:

$$\left.\begin{aligned}c_v^{(i)}&=\cos\mu_i x^v t_v,\quad i=1,2\\ s_v^{(i)}&=\sin\mu_i x^v t_v,\quad i=1,2\end{aligned}\right\}\text{(no sum over } v) \tag{4.5}$$

and α and β are real constants. $\int d^3x$ is now to be understood as

$$\int_{-a\pi/2}^{a\pi/2}\int_{-a\pi/2}^{a\pi/2}\int_{-a\pi/2}^{a\pi/2}dx^5dx^6dx^7, \tag{4.6}$$

and $a\mu_i=m_i$ are odd integers. Therefore

$$\int_{-a\pi/2}^{a\pi/2}(\sin\mu_i x)dx=0. \tag{4.7}$$

The terms in (4.3) containing derivatives with respect to the internal coordinates will combine into the minimal-coupling term

$$|(\partial_a+i\mu f A_a^v t_v)\phi|^2 \tag{4.8}$$

if certain algebraic constraints are fulfilled. In fact, the first, third, and fifth term on the right-hand side of (4.3) have the respective coefficients

$$\begin{aligned}P_1&=(a\pi)^3[\tfrac{3}{2}\alpha^2+\tfrac{3}{2}\beta^2+(24\alpha^2)/(m_1{}^2\pi^2)],\\P_2&=8\alpha\beta a\pi/\mu_1,\\P_3&=\frac{a}{2}\pi^3(\alpha^2m_1{}^2+\beta^2m_2{}^2),\end{aligned} \tag{4.9}$$

which have to be the coefficients A^2, AB, and B^2 of a complete square in order to combine to (4.8). Therefore one has to fulfill the constraint

$$(P_2)^2/P_1=P_3, \tag{4.10}$$

which reads explicitly

$$am_1\pi^4\left(3\frac{\alpha}{\beta}+3\frac{\beta}{\alpha}+\frac{12}{m_1{}^2\pi^2}\frac{\alpha}{\beta}\right)=64/(\alpha^2m_1{}^2+\beta^2m_2{}^2). \tag{4.11}$$

Furthermore, in order to compensate for the smallness of f, P_2/P_1 has to be of the order of Planck's mass, i.e.,

$$m_1\pi a\left(\frac{3}{2}\frac{\alpha}{\beta}+\frac{3}{2}\frac{\beta}{\alpha}+\frac{24}{m_1{}^2\pi^2}\frac{\alpha}{\beta}\right) \tag{4.12}$$

has to be of the order 10^{-18} GeV^{-1}. P_3/P_1 has to be of the order of Planck's mass squared. Both are achieved by choosing $a \sim 10^{-18}$ and α and β roughly of the same order of magnitude and m_1 not too large. The constraint (4.11) then puts α and β at the value of the square root of Planck's mass. Again, from the second term on the right-hand side of (4.3) one gets a contribution to the mass term of the order of Planck's mass.[27]

For the given representation the *Ansatz* (4.4) is quite closely related to the *Ansatz* (3.18) for the spinors. In fact for $\alpha=\beta=1$, and $\mu_1=\mu_2=\mu$ the *Ansätze* agree apart from the change in the integration range.

The *Ansatz* (4.3) for the scalar field may be used for the spinor field as well. One gets the same Lagrangian as in Sec. III, with

$$\mu \to \tilde{\mu} = P_2/P_1 \tag{4.13}$$

and for the Fierz-Pauli term,

$$f \to \tilde{f} = (a\pi)^3 f\left(\frac{3}{2}\,\alpha^2 + \frac{24\alpha^2}{m_1{}^2\pi^2} - \frac{\beta^2}{2}\right)\Big/P_1\,. \tag{4.14}$$

Note that $|\tilde{f}| \leq |f|$. It is interesting to note that as a function of α and β, f has a zero, i.e., the Fierz-Pauli term can be eliminated.

V. A REMARK ON HIGHER GROUPS

An ansatz for the spinor field like that of Sec. III is easily generalized to the case of more than seven dimensions. An interesting restriction occurs, however, if groups of rank 2 or higher are being used.

For definiteness consider the case of SU(3). Being of rank 2 the group has a two-dimensional center with generators Y and T_3. Consider the case that for $\mathcal{L}=\mathcal{L}(\bar{\psi},\psi)$, ψ is a vector $\psi=\psi_z$ such that

$$T_3\psi_z = 0 = Y\psi_z\,. \tag{5.1}$$

Then the relative factors of the different terms in the Lagrangian density (3.26) will change as compared to those for ψ's which have $Y\psi \neq 0$ and/or $T_3\psi \neq 0$. This means that universality is lost and for such representations the concept of universality becomes obsolete.

One would therefore exclude representations from consideration which contain a vector ψ_2 satisfying (5.1). In the case of SU(3) this excludes the representations

$$\underline{8}=D(1,1)\,,\ \underline{10}=D(3,0)\,,\ \underline{27}=D(2,2)\,,\ \underline{28}=D(6,0)\,,\ldots \tag{5.2}$$

(and complex conjugated ones). It leaves, however,

$$\underline{3}=D(1,0)\,;\ \underline{6}=D(2,0)\,;\ \underline{15}=D(4,0)\,;\ \underline{21}=D(5,0)\,;\ldots \tag{5.3}$$

(and complex conjugated).

The sets (5.2) and (5.3) are discriminated by the triality of their elements. For a representation $D(p,q)$ the triality n is defined as $p-q=3n$. The elements of (5.2) therefore have integer triality whereas the elements of (5.3) do not. The quark representation $\underline{3}=D(1,0)$ is among the allowed representations.

VI. GAUGE TRANSFORMATIONS VERSUS COORDINATE TRANSFORMS

It has been remarked by Rayski[15] for the five-dimensional case that global gauge transformations and coordinate transformations of the internal coordinates may be related in the following way: The scalar field in five dimensions $\Phi(x,x^5)$ is given by $\Phi(x,x^5)=\exp(i\mu x^5)\phi(x)$. Under a gauge transformation, $\phi(x)\to\exp(ia)\phi(x)$. This gauge transformation may be interpreted as a coordinate transformation $x_5 \to x_5 + a/\mu = x_5'$. Thus under a gauge transformation

$$\Phi(x,x_5) \to \Phi(x,x_5') = \Phi(x, x_5 + a/\mu)\,. \tag{6.1}$$

The purpose of this section is to demonstrate a similar phenomenon for the present seven-dimensional case. In fact, consider the ansatz (3.18) with $T_v \to t_v$, the latter being defined in Appendix B:

$$\begin{aligned}\Psi(x,\vec{x}) = [&\exp(i\mu x^5 t_5) + \exp(i\mu x^6 t_6)\\ &+ \exp(i\mu x^7 t_7)]\psi(x)\,.\end{aligned} \tag{6.2}$$

I will consider a specific gauge transformation

$$\psi(x) \to \exp(i\lambda^5 t_5)\psi(x) \equiv \exp(i\lambda t_5)\psi(x)\,. \tag{6.3}$$

A more general gauge transformation may be obtained by successively repeating (6.3) for other directions. I now proceed to demonstrate that the equality

$$\begin{aligned}[\exp(i\mu x^5 t_5) + \cdots]\exp(i\lambda t_5)\psi(x) &= \exp(i\lambda' t_5)[\exp(i\mu x'^5 t_5) + \cdots]\psi(x)\\ &= \exp(i\lambda' t_5)\Psi(x,\vec{x}')\end{aligned} \tag{6.4}$$

holds. Namely, one concludes from (6.4) by comparing coefficients of $t_v\psi(x)$ and $\psi(x)$

$$\begin{aligned}&(\cos\lambda')\Sigma' - (\sin\lambda')s_5' = z_1\,,\\ &(\sin\lambda')\Sigma' + (\cos\lambda')s_5' = z_2\,,\end{aligned} \tag{6.5}$$

$$\begin{aligned}&(\cos\lambda')s_6' + (\sin\lambda')s_7' = z_3\,,\\ &(-\sin\lambda')s_6' + (\cos\lambda')s_7' = z_4\,,\end{aligned} \tag{6.6}$$

where

$$\Sigma = \cos\mu x^5 + \cos\mu x^6 + \cos\mu x^7 ,\qquad \Sigma' = \cos\mu x'^5 + \cos\mu x'^6 + \cos\mu x'^7 , \tag{6.7}$$

$$s_v = \sin\mu x^v, \quad s'_v = \sin\mu x'^v , \tag{6.8}$$

$$z_1 = (\cos\lambda)\Sigma - (\sin\lambda)s_5 ,\qquad z_2 = (\sin\lambda)\Sigma + (\cos\lambda)s_5 , \tag{6.9}$$

$$z_3 = (\cos\lambda)s_6 - (\sin\lambda)s_7 ,\qquad z_4 = (\cos\lambda)s_7 + (\sin\lambda)s_6 . \tag{6.10}$$

The subsystems (6.5) and (6.6), respectively, are solvable for arbitrary λ'. For given real λ one finds λ' to be real. For consistency of (6.5) and (6.6) λ' has to be adjusted for given λ.

Motivated by numerical analysis I furthermore give the following conjecture: The subsystems (6.5) and (6.6) are compatible for $\lambda' = 0$.

An exact proof of this statement would be difficult since in either case ($\lambda' = 0$ or not), (x_v, λ) and (x'_v, λ') are related in a nonlinear way. A consistent solution of (6.5) and (6.6) gives a representation of a gauge transformation as a coordinate transformation. The case $\lambda' = 0$ would give the exact analog of (6.1). An important difference, however, is that the gauge transformations are now related to nonlinear coordinate transformations (which are not translations) of the internal coordinates.

Another interpretation of the results obtained in this section is that the gauge transformations can always be compensated by a coordinate transformation so that as a net effect $\Psi(x, x_5)$ $[\Psi(x, \vec{x})]$ does not change at all ($\lambda' = 0$ assumed).

VII. THE KALUZA-KLEIN *ANSATZ* FOR THE METRIC: CURVED INTERNAL SPACE

Again our starting point is the Kaluza-Klein *Ansatz* for the metric (2.1):

$$G_{mn} = \begin{pmatrix} \bar{g}_{ab} + g_{tv} A^t_a A^v_b & -g_{wt} A^t_a \\ -g_{vt} A^t_b & g_{uw} \end{pmatrix} .$$

What will be different as compared with the situation discussed so far (Secs. II–VI) is that the internal part of the space will from now on be identified with the gauge group as a manifold; g_{uw} is therefore now the metric of the sphere S_3 [the gauge group being SO(3)]. In order to derive the usual formulation of a Yang-Mills theory one has to require

$$g_{uv} \to -\delta_{\varphi\omega}$$

This was true from the very beginning in the previous approach. For a curved space such as S_3 this can be done only if one chooses a noncoordinate basis. This is a basis whose elements do not necessarily commute. Using as a parametrization of S_3 the Eulerian angles x^u such a basis is constructed as follows:

$$e^\varphi = h^\varphi_u dx^u, \quad e_\varphi = h^u_\varphi \frac{\partial}{\partial x^u} . \tag{7.1}$$

Explicitly (barred indices are frame indices),

$$e_{\bar 5} = -i\left(-\cos x^5 \cot x^6 \partial_5 - \sin x^5 \partial_6 + \frac{\cos x^5}{\sin x^6}\partial_7\right),$$
$$e_{\bar 6} = -i\left(-\sin x^5 \cot x^6 \partial_5 + \cos x^5 \partial_6 + \frac{\sin x^5}{\sin x^6}\partial_7\right), \tag{7.2}$$
$$e_{\bar 7} = -i\partial_5 ,$$

and

$$\begin{array}{cc} & \begin{array}{ccc} \varphi = \quad \bar 5 & \bar 6 & \bar 7 \end{array} \\ h^\varphi_u = i & \begin{bmatrix} 0 & 0 & 1 \\ -\sin x^5 & \cos x^5 & 0 \\ \cos x^5 \sin x^6 & \sin x^5 \sin x^6 & \cos x^6 \end{bmatrix} \end{array} . \tag{7.3}$$

I have chosen the following conventions: $e_{\bar 5}, e_{\bar 6}, e_{\bar 7}$ coincide with L_x, L_y, L_z of Edmonds,[28] respectively, thus

$$[e_\varphi, e_\omega] = i\sum_{\tau=5}^{7} \epsilon_{\varphi\omega\tau} e_\tau, \quad \epsilon_{\bar 5\bar 6\bar 7} = 1 .$$

The l_φ are related to e_1, e_2, e_3 of Misner, Thorne, and Wheeler[29] by

$$e_1 = -ie_{\bar 6}, \quad e_2 = ie_{\bar 5}, \quad e_3 = ie_{\bar 7} ,$$

and the Eulerian angles are $\alpha = x^5$, $\beta = x^6$, $\gamma = x^7$. The h, respectively, have the following properties:

$$h^\tau{}_t h^t{}_\sigma = \delta^\tau{}_\sigma, \quad h^t{}_\tau h^\tau{}_s = \delta^t{}_s . \tag{7.4}$$

Furthermore,

$$g_{\varphi\omega} = e_\varphi \cdot e_\omega = -\delta_{\varphi\omega} ,\qquad g^{\varphi\omega} = e^\varphi \cdot e^\omega = -\delta^{\varphi\omega} . \tag{7.5}$$

The frame of the seven-dimensional space is now given by

$$\bar\theta^\alpha = \theta^\alpha , \tag{7.6}$$

$$\theta^\tau = h^\tau{}_t dx^t - A^\tau_a dx^a . \tag{7.7}$$

From (7.7) and (7.4) in particular,

$$dx^t = h^t{}_\tau(\theta^\tau + A^\tau_a dx^a) . \tag{7.8}$$

Following the same line of argument as in Sec. II one first determines the connection one-forms from $d\theta^\tau$:

$$d\theta^\tau = h^\tau_{v,w} dx^w \wedge dx^v - A^\tau_{a,b} dx^b \wedge dx^a - A^\tau_{a,w} dx^w \wedge dx^a$$

$$= \tfrac{1}{2} f^\tau_{vw} h^w_\varphi h^v_\sigma \{\theta^\varphi \wedge \theta^\sigma + A^\sigma_\alpha \theta^\varphi \wedge \theta^\alpha - A^\varphi_\alpha \theta^\sigma \wedge \theta^\alpha + A^\varphi_\alpha A^\sigma_\beta \theta^\alpha \wedge \theta^\beta\} - A^\tau_{\alpha,\beta} \theta^\beta \wedge \theta^\alpha - A^\tau_{\alpha,w} h^w_\varphi (\theta^\varphi + A^\varphi_\beta \theta^\beta) \wedge \theta^\alpha , \tag{7.9}$$

where

$$f^{\tau}_{vw}=(h^{\tau}_{v,w}-h^{\tau}_{w,v}). \qquad (7.10)$$

The Christoffel symbols of the sphere S_3 are defined by

$$\Gamma^{\tau}{}_{\varphi\sigma}=\tfrac{1}{2}f^{\tau}_{vw}(h^{w}_{\sigma}h^{v}_{\varphi}-h^{w}_{\varphi}h^{v}_{\sigma}). \qquad (7.11)$$

If one imposes the requirement

$$\partial^{*}_{\varphi}A^{\tau}_{\alpha}\equiv h^{w}_{\varphi}A^{\tau}_{\alpha,w}=\Gamma^{\tau}{}_{\varphi\sigma}A^{\sigma}_{\alpha}, \qquad (7.12)$$

the second and third terms in curly brackets and the first term in parentheses in (7.9) vanish and one obtains from (7.9) using (7.11)

$$d\theta^{\tau}=\tfrac{1}{2}\Gamma^{\tau}{}_{\varphi\sigma}\theta^{\varphi}\wedge\theta^{\sigma}-\tfrac{1}{2}F^{\tau}_{\alpha\beta}\theta^{\alpha}\wedge\theta^{\beta}, \qquad (7.13)$$

where

$$F^{\tau}_{\alpha\beta}=\{-A^{\tau}_{\alpha,\beta}+A^{\tau}_{\beta,\alpha}+\Gamma^{\tau}{}_{\varphi\sigma}A^{\varphi}_{\alpha}A^{\sigma}_{\beta}\}. \qquad (7.14)$$

As one can see the commutator term in $F^{\tau}_{\alpha\beta}$ appears now because A^{τ}_{α} depends on the internal coordinates. Since the world components A^{t}_{α} and the frame components A^{τ}_{α} are related via $A^{\tau}_{\alpha}=h^{\tau}_{t}A^{t}_{\alpha}$ for a curved internal space the gauge fields have to depend on the internal coordinates. The explicit form of this dependence is determined below.

The quantities $\Gamma^{\tau}{}_{\varphi\sigma}$ which are the Christoffel symbols of S_3 are at the same time the structure constants of SO(3). This statement depends of course on the choice of the basis e_{φ}. Explicitly

$$\Gamma^{\tau}{}_{\varphi\sigma}=-\Gamma_{\tau\varphi\sigma}=-i\epsilon_{\tau\varphi\sigma}. \qquad (7.15)$$

Therefore (7.14) is the usual field-strength tensor for the Yang-Mills fields. Comparing (7.13) with Cartan's first identity [cf. (2.7)]

$$d\theta^{\tau}=-\omega^{\tau}{}_{\nu}\wedge\theta^{\nu}$$

and using the expansion formula

$$\omega^{\tau}{}_{\nu}=\omega^{\tau}{}_{\nu\lambda}\theta^{\lambda}, \qquad (7.16)$$

one finds

$$\omega^{\tau}{}_{\alpha\varphi}=0=\omega^{\tau}{}_{\varphi\alpha}, \qquad (7.17)$$

$$\omega^{\tau}{}_{\alpha\beta}=-\tfrac{1}{2}F^{\tau}{}_{\alpha\beta},\quad \omega^{\tau}{}_{\varphi\sigma}=\tfrac{1}{2}\Gamma^{\tau}{}_{\varphi\sigma}. \qquad (7.18)$$

I now proceed to determine the dependence of the gauge fields on the internal coordinates. This dependence is given as a solution of (7.12) which is looked upon as a system of partial differential equations. It is solvable as one may check by rewriting (7.12) as

$$A^{\tau}_{\alpha,w}=h^{\varphi}_{w}\Gamma^{\tau}{}_{\varphi\sigma}A^{\sigma}_{\alpha}, \qquad (7.19)$$

and verifying the integrability condition

$$A^{\tau}_{\alpha,vw}-A^{\tau}_{\alpha,wv}=0. \qquad (7.20)$$

This is not too surprising since the ∂^{*}_{φ} have just the same commutation relations as the Γ as matrices. Indeed the ∂^{*}_{φ} have the commutation relations of the angular momentum operators L_{φ}:

$$\partial^{*}_{5}=\tfrac{1}{2}(L_{+}+L_{-}),\quad \partial^{*}_{6}=\frac{-i}{2}(L_{+}-L_{-}),\ \partial^{*}_{7}=L_{z}, \qquad (7.21)$$

where $L_{\pm}=L_{x}\pm iL_{y}$, L_{z} are the angular momentum operators in the notation of Edmonds[28] (see above).

Iterating (7.12) one realizes that the A^{σ}_{α} are the components of a spherical tensor of rank 1 and thus one finds the following explicit solution for the spherical components ${}_{s}A^{\sigma}_{\alpha}$ of the gauge fields in terms of Wigner's rotation functions:

$${}_{s}A^{\pm}_{\alpha}={}_{s}\hat{B}^{\sigma'}_{\alpha}D^{1}_{\pm\sigma'},\quad {}_{s}A^{7}_{\alpha}={}_{s}\hat{B}^{\sigma'}_{\alpha}D^{1}_{0\sigma'}. \qquad (7.22)$$

Here the spherical components ${}_{s}A^{\sigma}_{\alpha}$ of a vector A^{σ}_{α} are defined as

$${}_{s}A^{\pm}_{\alpha}=\mp\frac{1}{\sqrt{2}}(A^{5}_{\alpha}\pm iA^{6}_{\alpha}),\quad {}_{s}A^{7}_{\alpha}=A^{7}_{\alpha}. \qquad (7.23)$$

The difference in the respective definitions of $L_{\pm}$ and ${}_{s}A^{\pm}_{\alpha}$ will be of some importance below. In (7.23) I have dropped the bar for framed indices again: All tensor components will be understood as frame components.

In (7.22), ${}_{s}\hat{B}^{\sigma}_{\alpha}$ does not depend on the internal coordinates but is otherwise arbitrary. The ˆ on ${}_{s}\hat{B}_{\alpha}$ indicates that the differential equation (7.12), being homogeneous, determines its solution only up to an overall factor. From (7.22) one finds for the Cartesian components A^{σ}_{α}:

$$A^{5}_{\alpha}=\frac{1}{\sqrt{2}}\,{}_{s}\hat{B}^{\sigma'}_{\alpha}(D^{1}_{-\sigma'}-D^{1}_{+\sigma'}),$$

$$A^{6}_{\alpha}=\frac{i}{\sqrt{2}}\,{}_{s}\hat{B}^{\sigma'}_{\alpha}(D^{1}_{-\sigma'}+D^{1}_{+\sigma'}), \qquad (7.22')$$

$$A^{7}_{\alpha}={}_{s}\hat{B}^{\sigma'}_{\alpha}D^{1}_{0\sigma'}.$$

From (7.12) and (7.14) one finds that $F^{\tau}_{\alpha\beta}$ satisfies the same differential equation as A^{τ}_{α}:

$$\partial^{*}_{\varphi}F^{\tau}{}_{\alpha\beta}=\Gamma^{\tau}{}_{\varphi\sigma}F^{\sigma}{}_{\alpha\beta}. \qquad (7.24)$$

Furthermore,

$$\partial^{*}_{\varphi}F^{\tau}{}_{\alpha\beta}F_{\tau}{}^{\alpha\beta}=0, \qquad (7.25)$$

a fact which is important for the calculation of the Lagrangian density: It expresses the invariance under rotations for the square of the Yang-Mills curvature tensor.

The remaining arguments are as given before. The constants e, f are introduced in essentially the same manner. That is, one first replaces (7.8) by

$$dx^{t}=h^{t}{}_{\tau}(\theta^{\tau}+fA^{\tau}_{\alpha}dx^{a}). \qquad (7.26)$$

Further one substitutes

$$\vec{x}\to\mu\vec{x}, \qquad (7.27)$$

where in order to retain the periodicity of the rotation functions, μ has to be an integer (integer L). As before one identifies

$$\mu f \sim e, \quad f=\sqrt{\kappa}\,. \tag{7.28}$$

The Lagrangian density is given by

$$\begin{aligned}\mathcal{L}(x) &= \int \sqrt{|\bar{g}|}\, R_{4+N} \\ &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} dx^5 \sin x^6 dx^6 dx^7 \sqrt{|\bar{g}|}\, R_{4+N}\,,\end{aligned}$$

whence, after substituting (7.27),

$$\mathcal{L}(x)=\tfrac{1}{2}\pi\sqrt{|\bar{g}|}\,(\bar{R}_4-\tfrac{1}{4}f^2F^2+12\mu^2)\,, \tag{7.29}$$

the integral over the internal volume being the invariant integral over S_3. (7.29) is the Lagrangian density for gravity coupled to Yang-Mills fields with a cosmological term present. This term originates in the extra term in (7.13) (or, equivalently, the fact that the $\omega^{\tau}{}_{\sigma\varphi}$ do not vanish). The cosmological term is very large,

$$12\mu^2 \sim 10^{40}\ \text{GeV}\,. \tag{7.30}$$

The appearance of such a large cosmological term has been noted before.[14]

VIII. DIRAC EQUATION–CURVED INTERNAL SPACE

The general arguments in deriving the Dirac Lagrangian are the same as in Sec. III. That is, instead of (3.17) one gets

$$\begin{aligned}\mathcal{L}(x)=\int_{S_3}\{&\eta^{\alpha\beta}\bar{\Psi}\Gamma_\alpha(\partial_\beta+i\mu f A^{\varphi}_{\beta}\partial^*_{\varphi})\Psi - im_0\bar{\Psi}\Psi \\ &+i\mu\bar{\Psi}\Gamma^{\varphi}\partial^*_{\varphi}\Psi+\tfrac{1}{16}i\mu\bar{\Psi}\Gamma_{\varphi\omega\tau}\Gamma^{\varphi}[\Gamma^{\omega},\Gamma^{\tau}]\Psi \\ &+\tfrac{1}{16}f\bar{\Psi}F^{\varphi}_{\beta\gamma}\Gamma_{\varphi}[\Gamma^{\beta},\Gamma^{\gamma}]\Psi\}\,.\end{aligned} \tag{8.1}$$

The factor "i" in the terms containing ∂^*_φ in (8.1) appears in the following way: For convenience, ∂^*_φ in the previous section had been defined to be Hermitian, contrary to the definition of ∂_α. The factor i is now written down to get matching Hermiticity properties. Formally this can be done by a change in the definition of h^{φ}_{ν}, but keeping the definition of ∂^*_φ.

Again we are facing the problem to find an *Ansatz* for the $\vec{x}$ dependence of $\Psi(x,\vec{x})$ such that

$$\int_{S_3}[\eta^{\alpha\beta}\Psi\Gamma_\alpha(\partial_\beta+i\mu f A^{\varphi}{}_{\beta}\partial^*{}_{\varphi})\Psi] \tag{8.2}$$

gives the minimal-coupling term and the next three terms in (8.1) combine into a mass term. This will be achieved by means of the following *Ansatz*:

$$\Psi(x,\vec{x})=\sum_{M,N=-L}^{L} D^{L}_{MN}(\vec{x})\psi_{MN}(x)\,. \tag{8.3}$$

(8.3) may be looked upon as a Fourier expansion over S_3. The normalized (invariant) integral is

$$\int_{S_3}=\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} dx^5\, dx^6\, dx^7 \sin x^6\,. \tag{8.4}$$

Now

$$A^{\varphi}{}_{\beta}\partial^*_{\varphi}=(-1)^{\varphi'}({}_sA^{\varphi'}_{\beta})({}_s\partial^*_{-\varphi'})\,, \tag{8.5}$$

where the subscript s has been introduced again to indicate the spherical components of a tensor. Then, by using (8.3) and (7.22)

$$\begin{aligned}\int_{S_3}\Psi(x,\vec{x})({}_sA^{\varphi'}{}_{\beta}\,{}_s\partial^*_{\varphi'})(-1)^{\varphi'}\Psi(x,\vec{x}) &= \sum_{M',N',M,N,\varphi,\varphi'}\int_{S_3}\bar{\psi}_{MN}(x)D^{*L}_{M'N'}({}_s\hat{B}^{\varphi}_{\beta})D^{1}_{\varphi\varphi'}({}_s\partial^*_{-\varphi'})(-1)^{\varphi'}D^{L}_{MN}\psi_{MN}(x) \\ &= (-1)^{M'-N'}(-1)^{\varphi'}\bar{\psi}_{M'N'}\psi_{MN}({}_s\hat{B}^{\varphi}_{\beta})\begin{pmatrix}L & L & 1\\ M & -M' & \varphi'\end{pmatrix}\begin{pmatrix}L & L & 1\\ N & -N' & \varphi\end{pmatrix}({}_sT^{(L)}_{-\varphi'})^{MM}\,,\end{aligned} \tag{8.6}$$

where from now on proper summations will be understood. The matrix ${}_sT^{(L)}_{\varphi}$ is defined in Appendix D and the preceeding factors in (8.6) are Wigner's $3j$-symbols in the convention of Edmonds.[28] Comparing the definition of ${}_sT^{(L)}_{\varphi}$ and the explicit expressions for the $3j$-symbols[28] one finds

$$\begin{pmatrix}L & L & 1\\ M & -M' & \varphi\end{pmatrix}=({}_sT^{(L)}_{\varphi})^{MM'}\frac{(-1)^{L-M'}}{[(2L+1)(L+1)L]^{1/2}}\,. \tag{8.7}$$

Using (8.7) one finds the expression (8.6) equal to

$$\begin{aligned}\frac{(-1)^{2L}}{(2L+1)(L+1)L}\bar{\psi}_{M'N'}\psi_{MN}\,{}_s\hat{B}^{\varphi}_{\beta}({}_sT^{(L)}_{\varphi'}\,{}_sT^{(L)}_{-\varphi'})^{MM'}(-1)^{\varphi'}({}_sT^{(L)}_{\varphi})^{NN'} &= \frac{(-1)^{2L}}{2L+1}\bar{\psi}_{M'N'}[\delta^{M'M}({}_s\hat{B}^{\varphi}_{\beta})({}_sT^{(L)}_{\varphi})^{NN'}\psi_{MN} \\ &= \frac{(-1)^{2L}}{2L+1}\bar{\psi}_{MN'}({}_s\hat{B}^{\varphi}_{\beta})({}_sT^{(L)}_{-\varphi})^{N'N}(-1)^{\varphi}\psi_{MN}\,,\end{aligned} \tag{8.8}$$

where in the last line the Hermiticity properties of ${}_ST_\varphi^L$ have been used. Choosing now the special tensor field

$$\psi_{MN}(x)=\psi^0_M\psi_N(x)\,, \tag{8.9}$$

where only the factor $\psi_N(x)$ carries spinor indices, with a normalization $\sum_M\psi^{0*}_M\psi^0_M=1$, the last line of (8.8) becomes

$$\frac{(-1)^{2L}}{2L+1}\,\bar\psi_{N'}(x)({}_S\hat B^\varphi_\beta)({}_ST^{(L)}_{-\varphi})^{N'N}(-1)^\varphi\psi_N(x)\,. \tag{8.10}$$

Setting

$${}_S\hat B^\varphi_\beta(-1)^{2L}={}_SB^\varphi_\beta \tag{8.11}$$

one can now identify B^φ_β with a standard Yang-Mills field. The internal degree of freedom for the eight-dimensional spinors may be identified with the gauge degree of freedom in the following way: I choose $L=1$ and write

$$\psi_\sigma(x)=t_\sigma\psi(x)\,, \tag{8.12}$$

where I use the nonspherical notation again,

$$\{t_\sigma\}=\{t_5,t_6,t_7\}$$

are the matrices defined in Appendix B, and $\psi(x)$ is an eight-dimensional spinor field. Since

$$(T_\varphi{}^1)^{\sigma'\sigma}=\Gamma_{\sigma'\varphi\sigma}\,, \tag{8.13}$$

(8.10) becomes (remember the commutation relation for t_N)

$$\tfrac{2}{3}\bar\psi(x)B^\varphi_\beta t_\varphi\psi(x)\,, \tag{8.14}$$

which is the desired result.

The remaining terms are now comparatively easy to evaluate, since no further constraints will be imposed. In particular, the Fierz-Pauli term

$$\int_{S_3}\Psi F^\varphi{}_{\beta\gamma}\Gamma_\varphi[\Gamma^\beta,\Gamma^\gamma]\Psi=\int_{S_3}\bar\psi_{M'N'}\overset{*}{D}{}^L_{M'N'}(\vec x)(-1)^\varphi[{}_SF^{\varphi'}_{\alpha\beta}(B)]D^1_{\varphi\varphi'}({}_S\Gamma_{-\varphi})[\Gamma^\beta,\Gamma^\gamma]D^L_{MN}(\vec x)\psi_{MN}$$
$$=\frac{(-1)^{2L+\varphi+\varphi'}}{(2L+1)(L+1)L}\bar\psi_{M'N'}({}_ST^{(L)}_\varphi)^{MM'}({}_ST^{(L)}_{-\varphi'})^{N'N}[{}_SF^{\varphi'}_{\alpha\beta}(B)]({}_S\Gamma_{-\varphi'})[\Gamma^\beta\Gamma^\gamma]\psi_{MN}\,. \tag{8.15}$$

From the symmetry properties of the $3j$-symbols,

$$({}_ST^{(L)}_\varphi)^{MM'}=(-1)^{2L+1}({}_ST^{(L)}_\varphi)^{M'M} \tag{8.16}$$

[remember (8.13)], and (8.15) becomes

$$\frac{-1}{(2L+1)(L+1)L}\psi^{0*}_{M'}(T^{(L)}_\varphi)^{M'M}\psi^0_M\bar\psi_{N'}(T^{(L)}_{\varphi'})^{NN'}F^{\varphi'}_{\alpha\beta}\Gamma^\varphi\times[\Gamma^\beta\Gamma^\gamma]\psi_N\,. \tag{8.17}$$

Defining

$$a^{(L)}_\varphi=\psi^{0*}_{M'}(T^{(L)}_\varphi)^{M'M}\psi^0_M\,, \tag{8.18}$$

this being a vector under internal rotations, one has for the Fierz-Pauli term

$$\int_{S_3}\bar\Psi F^\varphi_{\beta\gamma}\Gamma_\varphi[\Gamma^\beta\Gamma^\gamma]\Psi=\frac{-a^{(L)}_\varphi}{(2L+1)(L+1)L}\bar\psi T^{(L)}_{\varphi'}F^{\varphi'}_{\alpha\beta}\Gamma^\varphi[\Gamma^\beta\Gamma^\gamma]\psi\,. \tag{8.19}$$

For the case where the gauge degree of freedom is being identified with the internal degree of freedom [cf. Eq. (8.12)] one gets instead of (8.19) ($L=1$)

$$\frac{-a^{(1)}_\varphi}{(2L+1)(L+1)L}\bar\psi t_{N'}\Gamma_{N'\varphi'N}F^{\varphi'}_{\alpha\beta}\Gamma^\varphi[\Gamma^\beta\Gamma^\gamma]t_N\psi\,. \tag{8.20}$$

Note that by some suitable particular choice of ψ^0_N one can achieve

$$a^{(L)}_\varphi=0\,, \tag{8.21}$$

so that again the Fierz-Pauli term can be eliminated.

Using the relations

$$\Gamma^5t_5=\Gamma^6t_6=\Gamma^7t_7=\begin{pmatrix}\gamma^5&0\\0&\gamma^5\end{pmatrix} \tag{8.22}$$

and, in particular,

$$\Gamma_{\varphi\omega\tau}\Gamma^\varphi[\Gamma^\omega,\Gamma^\tau]=6\Gamma^5t_5\,, \tag{8.23}$$

the Lagrangian for the case where the gauge degree of freedom and internal degree of freedom stemming from higher dimensions for the spinors are identified is given by

$$\begin{aligned}\mathcal{L}(x)=\frac{1}{2L+1}&\times\Big\{\eta^{\alpha\beta}\bar\psi\Gamma_\alpha\Big[\Big(\sum t_N{}^2\Big)\partial_\beta+2i\mu fB^\varphi_\beta t_\varphi\Big]\psi\\&-im_0\bar\psi\Big(\sum t_N{}^2\Big)\psi+i\mu\bar\psi\Gamma^5t_5\Big[3+\frac{6(\sum t_N{}^2)}{16}\Big]\psi\\&-\frac{2a^{(1)}_\varphi f}{(2L+1)(L+1)L}\bar\psi t_{N'}\Gamma_{N'\varphi'N}F^{\varphi'}_{\alpha\beta}\Gamma^\varphi[\Gamma^\beta,\Gamma^\gamma]t_N\psi\Big\}\,,\end{aligned} \tag{8.24}$$

where one has to set $L=1$.

In a normalization $\hat t=t/2$ such that $[\hat t,\hat t]=i\epsilon\hat t$, $\sum(\hat t_N{}^2)=\frac{3}{4}$, and the first line on the right-hand side of (8.24) would read

$$\tfrac{3}{4}\eta^{\alpha\beta}\bar\psi\Gamma_\alpha(\partial_\beta+\tfrac{4}{3}\mu fB^\varphi_\beta t_\beta)\psi\,, \tag{8.25}$$

where μf is the self coupling of the gauge field.

The relative factor $\frac{3}{4}$ in (8.25) (which would destroy gauge invariance) can be gotten rid of in the following way: Replace (8.12) by

$$\psi_N(x)=\tilde{t}_N\psi(x), \quad \tilde{t}_N=t_N+\alpha_N 1 . \tag{8.26}$$

Then

$$\Gamma_{N'\varphi N}\tilde{t}_{N'}\tilde{t}_N=2t\varphi , \tag{8.27}$$

but

$$\bar{\psi}_N\psi_N=\bar{\psi}\left[\left(\sum t_N{}^2\right)+\left(\sum|\alpha_N|^2\right)+2\left(\sum t_N\,\mathrm{Re}\alpha_N\right)\right] \tag{8.28}$$

and similarly for the kinetic term. For $\mathrm{Re}\alpha_N=0$ (remember $t_N{}^2=1$) one can therefore get the required normalization factor. The mass terms change in a similar manner whereas in the Fierz-Pauli term one simply replaces $t_N\to\tilde{t}_N$.

The two mass terms in (8.24) (second line) can be diagonalized as before, i.e., by setting

$$\psi(x)\to[\exp(\phi\Gamma^5t_5)]\psi(x) . \tag{8.29}$$

The Dirac Lagrangian following from (8.24) is

$$\begin{aligned}\mathfrak{L}(x)=\bar{\psi}\Big\{&\eta^{\alpha\beta}\Gamma_\alpha(\partial_\beta+\tfrac{1}{3}i2\mu fB^\varphi_\beta t_\varphi)-iM\\&-\frac{fa_\varphi^{(1)}}{9M}\Gamma_{N'\varphi'N}[m_0+\mu\Gamma^5t_5(1+\tfrac{6}{16})]t_{N'}\\&\times F^{\varphi'}_{\alpha\beta}\Gamma^\varphi[\Gamma^\beta\Gamma^\gamma]t_N\Big\}\psi ,\end{aligned} \tag{8.30}$$

with

$$\begin{aligned}&M=e^{2\phi\Gamma^5t_5}[m_0-\mu\Gamma^5t_5(1+\tfrac{6}{16})] ,\\&M^2=M^\dagger M=[m_0{}^2+\mu^2(1+\tfrac{6}{16})^2] .\end{aligned} \tag{8.31}$$

The detailed form of the Fierz-Pauli term is apparently different from the previously (Sec. III) given one. Principally, this can serve as a discriminating feature of the two models.

IX. CONCLUDING REMARKS

The purpose of this paper may be stated as to give a supplement to the papers by Kerner[13] and Cho[14] who demonstrated that an Einstein-Yang-Mills Lagrangian can be derived from a high-dimensional "gravity" Lagrangian. [For reasons of historical justice one should include the remark that the generalized KK *Ansatz* for the metric (2.1) has already been written down by DeWitt.[25]] The unification thus achieved would remain incomplete could one not demonstrate—as has been done in this paper—that matter fields can be included by means of Lagrangians which contain the coupling of matter fields to high-dimensional "gravity" only. These Lagrangians contain indeed as special cases gauge-invariant Lagrangians as one is used to.

In comparing the two approaches given here the second seems to be superior in the way the gauge-covariant derivative of the gauge fields is introduced. The crucial condition (7.12) is less strong than it appears to be on first sight: It states in fact only that the A^a_α are to behave like components of rank-1 tensor operators under internal rotations. The papers by Kerner[13] and Cho[14] are both formulated for this particular type of geometry and consequently both contain the constraint (7.12).

In the first case I have taken full advantage of the assumption that the internal part of the manifold is not identified with the group as a manifold by choosing the internal part of the manifold to be flat. The price one has to pay for this advantage is that in this case the gauge-covariant derivative for the gauge fields appears more in an *ad hoc* manner. One may, however, always feel motivated by the pragmatical point of view: This is just the way things have to be defined to give the right answer. What is gained is mainly that no cosmological constant appears—due to the vanishing of the curvature scalar for the flat internal space. Otherwise the dimensionally reduced Lagrangians are similar enough.

One can always try to compensate the large cosmological constant of Sec. VII by starting with a Lagrangian in seven dimensions which already contains a cosmological term. A less crude method has recently been advocated, for instance, by Kopcyński,[31] namely the inclusion of a (quadratic) torsion term.

Apparently the appearance of these very large parameters is the biggest obstacle confronting KK theories. This applies not only to the cosmological constant and the masses of the matter fields but also to the coupling constant of the Fierz-Pauli term which due to the mass parameter being very large is too small to give a phenomenologically useful theory of *CP* violation[12]—the coupling constant for the Fierz-Pauli term has the upper bound $f=\sqrt{\kappa}$ (cf. also Appendix C).

The latter statement on the other hand has to be taken with some care insofar as nobody (as far as I know) has tried to connect the phenomenology of *CP* violation with a Fierz-Pauli-type interaction.

Another field where a KK-type approach has proved useful recently is supergravity.[6] The technical results of this paper may be useful here, too.

It may, however, as well turn out that a more useful line of investigation is to take higher dimensions more seriously than in an approach which would eventually use only dimensionally reduced Lagrangians. As an example one might consider the (more general) Lagrangian (8.1) rather than the dimensionally reduced one (8.30) as the "true" one. I will return to this question later.

Note added in proof. After the proofs for these papers were returned, my attention was drawn to the following papers: G. Domokos and S. Kövesi-Domokos, Phys. Rev. D 16, 3060 (1977); R. Casalbuoni, G. Domokos, and S. Kövesi-Domokos, *ibid.* 17, 2048 (1978). In particular, much of the contents of Sec. VIII of the present paper has been discussed by these authors. I wish to thank Dr. G. Domokos for drawing my attention to these papers. Furthermore, my attention was also drawn to the following paper: L. N. Chang, K. I. Macrae, and F. Mansouri, Phys. Rev. D 13, 235 (1976). Again this paper overlaps partially with the present one, in particular Sec. VII.

ACKNOWLEDGMENTS

Thanks are due to Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for the hospitality extended to me at the International Centre for Theoretical Physics, Trieste. T. N. Sherry helped me through the papers by Kerner, Thirring, and Cho. He furthermore read large portions of a preliminary version of the manuscript and made a number of useful comments. I am similarly indebted to S. Downes-Martin and Professor A. Trautmann for reading parts of the manuscript and some discussions on topics of this paper. A. Namazie helped me to use the computer of the Centro di Calcolo of the University of Trieste for the numerical calculation mentioned in Sec. VI. Furthermore, I discussed parts of this paper at an early stage with Professor H. Doebner. N. S. Baaklini helped me by reading the entire manuscript and by a number of encouraging discussions. Finally, I wish to thank F. Gretsch for making it possible for me to review certain pleasureful studies on his Roc Jet model, without which this paper hardly could have been written.

APPENDIX A

Remember Eq. (2.1). In short form I write the metric tensor as

$$G_{mn}=\begin{bmatrix} \bar g_{ab}+A_a\cdot A_b & -A_{va} \\ -A_{wb} & g_{wv}\end{bmatrix}. \tag{A1}$$

Its inverse G^{mn} is given by

$$\begin{aligned} G^{mn}&=\begin{bmatrix} G^{bc} & G^{bw} \\ G^{vc} & G^{vw}\end{bmatrix} \\ &=\begin{bmatrix} \bar g^{bc} & \bar g^{ba}A_a^w \\ \bar g^{ca}A^v{}_a & g^{vw}+A^v{}_bA^{wb}\end{bmatrix}. \end{aligned} \tag{A2}$$

The $(4+N)$-beins $l^\mu{}_m, l^m{}_\mu$ are defined by

$$\begin{aligned} \theta^\mu &= l^\mu{}_m dx^m, \\ dx^m &= l^m{}_\mu \theta^\mu. \end{aligned} \tag{A3}$$

For the case of vanishing gravity they are given by [remember (2.11) and (2.12)]

$$\begin{aligned} l^\mu{}_m &= \delta^\mu{}_m - \delta^\mu{}_\tau \delta^a{}_m A_a^\tau, \\ l^m{}_\mu &= \delta^m{}_\mu + \delta^m{}_\tau \delta^a{}_\mu A_a^\tau. \end{aligned} \tag{A4}$$

In matrix form one may write

$$[l^m{}_\mu]=\begin{bmatrix} \delta^a{}_\alpha & A^t_\alpha \\ 0 & \delta^t{}_\tau\end{bmatrix}. \tag{A5}$$

The components $l^m{}_\mu$ of the vector l^m transform like vector components under SO(6,1). We shall refer to the symmetry group of the frame metric as the frame group. In this case it is SO(6,1). It is an important point to note that the frame group is not the product group SO(3,1) × SO(3) [SO(3,1) being the homogeneous Lorentz group], this would be the case only if instead of (A5)

$$[l^m{}_\mu]=\begin{bmatrix} \delta^a{}_\alpha & 0 \\ 0 & \delta^t{}_\tau\end{bmatrix}. \tag{A6}$$

To see this one considers the action of the infinitesimal generators of SO(6,1), $L^\mu{}_\nu e^m{}_\nu$, and observes that due to the presence of A^τ_α in (A5), $L^{\tau\alpha}e^t{}_\alpha \neq 0$.

APPENDIX B

In this appendix the explicit representation of the generalized Dirac algebra in seven dimensions is shown. For a space of M dimensions, spinors of rank 1 have the dimension $2^{[M/2]}$, where $[M/2]$ is the biggest integer smaller than or equal to $M/2$. In our case there are seven generalized Dirac matrices with commutation relations

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \tag{B1}$$

which may be written down explicitly as

$$\begin{aligned} \Gamma^\alpha &= \begin{bmatrix} \gamma^\alpha & 0 \\ 0 & \gamma^\alpha \end{bmatrix}, \quad \Gamma^5=\begin{bmatrix} 0 & \gamma^5 \\ \gamma^5 & 0\end{bmatrix}, \\ \Gamma^6 &= \begin{bmatrix} 0 & -i\gamma^5 \\ i\gamma^5 & 0 \end{bmatrix}, \quad \Gamma^7=\begin{bmatrix} \gamma^5 & 0 \\ 0 & \gamma^5\end{bmatrix}. \end{aligned} \tag{B2}$$

The numerical indices are frame indices here. In (B2) the γ are any representation of the ordinary Dirac matrices $[(\gamma^5)^2 = -1]$. I define the commutators $\sigma^{\mu\nu} = [\Gamma^\mu, \Gamma^\nu]$. Of particular interest are

$$\tfrac{1}{2}\sigma^{56} = -it_7, \quad \tfrac{1}{2}\sigma^{67} = -it_5, \quad \tfrac{1}{2}\sigma^{57} = -it_6,$$

where explicitly

$$t_5=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad t_6=i\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad t_7=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{B3}$$

These matrices satisfy the commutation relations of SU(2):

$$[t_\sigma, t_\varphi]=2\sum_{\tau=5}^{7}\epsilon_{\sigma\varphi\tau}t_\tau\,,\quad \epsilon_{567}=1\,, \tag{B4}$$

and they anticommute. The following commutation relations are also being used:

$$[t_\varphi, \Gamma^\alpha]=0\,, \tag{B5}$$

$$[t_5, \Gamma^5]=[t_6, \Gamma^6]=[t_7, \Gamma^7]=0\,, \tag{B6}$$

$$\{t_5, \Gamma^{6,7}\}=\{t_6, \Gamma^{5,7}\}=\{t_7, \Gamma^{6,5}\}=0\,. \tag{B7}$$

I am using a representation of γ matrices such that the following properties of Γ^μ under Hermitian conjugation (+), transposition (tr), and complex conjugation (*) are being given:

$$\begin{aligned}&\Gamma^0=+\Gamma^{0+}=-\Gamma^{0\mathrm{tr}}=-\Gamma^{0*}\,,\\&\Gamma^i=-\Gamma^{i+}=+\Gamma^{i\mathrm{tr}}=-\Gamma^{i*}\,,\\&\Gamma^5=-\Gamma^{5+}=-\Gamma^{5\mathrm{tr}}=+\Gamma^{5*}\,,\\&\Gamma^6=-\Gamma^{6+}=+\Gamma^{6\mathrm{tr}}=-\Gamma^{6*}\,,\\&\Gamma^7=-\Gamma^{7+}=-\Gamma^{7\mathrm{tr}}=+\Gamma^{7*}\,.\end{aligned} \tag{B8}$$

Furthermore, for the t_v,

$$\begin{aligned}&t_{5,7}=+t_{5,7}^{+}=+t_{5,7}^{\mathrm{tr}}=+t_{5,7}^{*}\,,\\&t_6=+t_6^{+}=-t_6^{\mathrm{tr}}=-t_6^{*}\,.\end{aligned} \tag{B9}$$

APPENDIX C

In this appendix I discuss some modifications of the *Ansatz* for the spinor field as given in Sec. III, in particular, with hindsight to the questions: (1) Must the spinor field have a mass which is at least Planck's mass? (2) Is the upper limit for the *CP*-violation constant always $\sim f=\sqrt{\kappa}$? None of the modifications discussed changes the answers to the questions as given in Sec. III.

Instead of (3.18) one may try ($\mu_n=n$)

$$\Psi(x,\bar{x})=\sum_{n=1}^{N}\{\exp(i\mu_n x^5T_5)+\exp(i\mu_n x^6T_6)+\exp(i\mu_n x^7T_7)\}\psi_n(x)\,. \tag{C1}$$

In this case $\mathcal{L}(x)$ splits up into a direct sum of Lagrangian densities of the form (3.32). If all ψ_n are identical, however, a slight modification occurs. Nevertheless, N has to be of the order of Planck's mass. Furthermore, one may try ($\mu_n=n$)

$$\Psi(x,\bar{x})=\sum_{n=1}^{k}\beta_n\{\exp(i\mu_n x^5T_5)+\cdots\}\psi(x)\,. \tag{C2}$$

Inserting this into the Lagrangian and identifying constants as before, one finds that instead of μ now

$$\tilde{\mu}=\Big(\sum_{n=1}^{k}n|\beta_n|^2\Big)\Big/\Big(\sum_{n=1}^{k}|\beta_n|^2\Big) \tag{C3}$$

has to be of the order of Planck's mass. But from (C3),

$$\tilde{\mu}\leq\Big(k\sum|\beta_n|^2\Big)\Big/\Big(\sum|\beta_n|^2\Big)=k\,, \tag{C4}$$

which means that k has to be of the order of Planck's mass.

Finally the following *Ansatz* would give the right expression for the minimal coupling terms as well:

$$\Psi(x,\bar{x})=[\exp(i\mu x^5T_5)+\cdots+\alpha_k 1]\psi(x)\,. \tag{C5}$$

For the case $[\Gamma, T]=0$ ($r=3$) this simply results in the modification $\mu\to\tilde{\mu}$ with

$$\tilde{\mu}=(3\mu)/(3+|\alpha_N|^2)\,, \tag{C6}$$

$\tilde{\mu}$ of the order of Planck's mass so that μ has a *lower* limit, $\tilde{\mu}$.

For $T_v=t_v$ the situation is slightly different. In this case the Dirac equation reads

$$M\psi+\Gamma^\alpha(\partial_\alpha+[i\mu f/(3+|\alpha_N|^2)]A_\alpha^s t_s)\psi+\frac{1+|\alpha_N|^2}{16M(3+|\alpha_N|^2)}(m_0 f-e\Gamma^5 t_5)F^\varphi{}_{\beta\gamma}\Gamma_\varphi[\Gamma^\beta,\Gamma^\gamma]\psi=0\,, \tag{C7}$$

where

$$M=[m_0{}^2+\mu^2/(3+|\alpha_N|^2)^2]^{1/2}=[m_0{}^2+(e/f)^2]^{1/2}\,, \tag{C8}$$

and again μ has $3e/f$ as a lower limit. (Due to the normalization of t, e is half the self-coupling of the gauge field.)

APPENDIX D

Equation (7.22) follows from the observation

$$L_\pm D^L_{MN}=\sum_{\underline{M}=-L}^{L}(T_\pm^{(L)})^{M\underline{M}}D^L_{\underline{M}N}=[(L\mp M)(L\pm M+1)]^{1/2}D^L_{(M\pm1)N}\,, \tag{D1}$$

where explicitly

$$L_0 = -\frac{\partial}{\partial x^5}\ , \tag{D2}$$

$$L_\pm = -ie^{(\pm i x^5)}\left[-(\cot x^6)\frac{\partial}{\partial x^5} \pm i\,\frac{\partial}{\partial x^6} + \frac{1}{\sin x^6}\frac{\partial}{\partial x^7}\right],$$

and the D^L_{MN} are the Wigner rotation functions[28,32]

$$D^L_{MN} = D^L_{MN}(\tilde{x}) = e^{iMx^5} d^L_{MN}(x^6) e^{iNx^7}\ . \tag{D3}$$

The D^L_{MN} are eigenfunctions of L_0 with eigenvalue M and of $-i\partial/\partial x^7$ with eigenvalue N. Furthermore, D^L_{MN} is an eigenfunction of $L^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_0{}^2$ with eigenvalue $L(L+1)$. Note, however, that the step operators $L_\pm$ as defined in (D2) only effect the first index of D^L_{MN}.

It is elucidating to check these statements for a special case. This can be done for the particularly important case $L=1$ where the matrix d^1_{MN} is given by[32,33]

$$d^1_{MN} = \begin{array}{c|ccc} M \backslash N & +1 & 0 & -1 \\ \hline +1 & \frac{1}{2}(1+\cos x^6) & \frac{-1}{\sqrt{2}}\sin x^6 & \frac{1}{2}(1-\cos x^6) \\ 0 & \frac{\sin x^6}{\sqrt{2}} & \cos x^6 & \frac{-\sin x^6}{\sqrt{2}} \\ -1 & \frac{1}{2}(1-\cos x^6) & \frac{\sin x^6}{\sqrt{2}} & \frac{1}{2}(1+\cos x^6) \end{array}\ . \tag{D4}$$

APPENDIX E: NOTATIONS

World indices:

$a, b, c, d = 1, \ldots, 4$,

$l, m, n, k = 1, \ldots, 7$,

$s, t, u, w, v = 5, 6, 7$.

Frame indices:

$\alpha, \beta, \gamma, \delta = 1, \ldots, 4$,

$\mu, \nu, \lambda, \rho = 1, \ldots, 7$,

$\sigma, \tau, \varphi, \omega = 5, 6, 7$.

Numerical frame indices are sometimes written with a bar: $\bar{1}, \bar{2}, \ldots$.

Metric

G_{mn} : metric tensor in 7 dimensions. Definition: Eq. (2.1). The inverse G^{mn} is defined in Eq. (A2).

$\bar{g}_{ab}$: physical metric tensor describing gravity, cf. Eq. (2.1).

g_{uw} : metric tensor for internal part of manifold, cf. Eq. (2.1).

$\eta^{\mu\nu}$: diag $(---,+;---)$: frame components of metric tensor.

$e^m{}_\mu$: $(4+N)$-bein, Eqs. (A4), (A5).

θ^μ : basic one-forms, Eqs. (A3), (7.6), (7.7).

$\omega_{\mu\nu}$: connection one-forms.

$\bar{\omega}_{\alpha\beta}$: connection one-forms with respect to $\bar{g}_{ab}$.

Cartan's first identity:

$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0$.

Cartan's second identity:

$d\omega_{\mu\nu} + \omega_{\mu\lambda} \wedge \omega^\lambda{}_\nu = \Omega_{\mu\nu}$.

$\Omega_{\mu\nu}$: curvature two-forms.

Curvature tensor $R_{\mu\nu\rho\lambda}$ defined by

$\Omega_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\lambda}\theta^\rho \wedge \theta^\lambda$.

$\omega_{\mu\nu\lambda}$: Ricci rotation coefficients (3.1), (3.7).

$\varphi_{|\mu}$: covariant derivative of spinor or tensor fields (3.1), (3.5), (3.6).

$\Sigma^{\mu\nu}$: matrix representation of frame group (3.1), (3.2), (3.3).

A^m_n : gauge field (2.1), (2.12).

D_n : gauge-covariant derivative (2.14).

$F^\tau_{\alpha\beta}$: gauge field tensor (2.38), (2.39).

$\Psi(x, \tilde{x})$: generalized Dirac spinor depending on seven coordinates.

$\psi(x)$: generalized Dirac spinor depending on four coordinates.

Γ^μ : generalized Dirac matrices (Appendix A).

T_v : matrix representation of gauge group algebra on spinor (scalar) field.

t_v : special case of T_v, cf. Appendix B.

t_N : equivalent of t_v, spherical notation.

$\tilde{t}_N$: modification of t_N (Sec. VIII).

μ : mass parameter of order of Planck's mass. Secs. III, VII, VIII.

e : self-coupling of gauge field.

$\Phi(x, \tilde{x})$: scalar field depending on seven coordinates.

$\varphi(x)$: scalar field depending on four coordinates.

e_φ : noncoordinate base vectors on S_3, (7.1).

$\hbar^\varphi{}_t$: 3-bein on S_3, (7.3).

$\Gamma^\varphi{}_{\sigma\tau}$: Christoffel symbols on S_3 = structure constants of SO(3).

∂^*_φ : Eq. (7.12).

${}_sA^\varphi$: spherical components of a vector (7.22).

B^φ_α : gauge field independent of internal coordinates.

$D^L_{MN}, D^L_{\sigma\sigma'}$: Wigner rotation functions.

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[1]E. Cremmer and J. Scherk, Nucl. Phys. B103, 399 (1976); B108, 409 (1976); B118, 61 (1976).

[2]Z. Horvath, L. Palla, E. Cremmer, and J. Scherk, Nucl. Phys. B127, 57 (1977).

[3]J. Rayski, Lett. Nuovo Cimento 18, 422 (1976); 21, 425 (1978).

[4]W. Mecklenburg, Acta Phys. Pol. B9, 793 (1978).

[5]P. Forgács and Z. Horvath, Budapest Reports Nos. 384 and 385 (unpublished).

[6]J. Scherk and J. H. Schwarz, Phys. Lett. 82B, 60 (1979); Nucl. Phys. B153, 61 (1979), and references therein.

[7]W. Mecklenburg and D. P. O'Brien, Lett. Nuovo Cimento 23, 566 (1978).

[8]D. B. Fairlie, J. Phys. G 5, L55 (1979); Phys. Lett. 82B, 97 (1979).

[9]J. G. Taylor, Phys. Lett. 83B, 331 (1979).

[10]Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. 966 (1921).

[11]O. Klein, Zeitschr. Phys. 37, 895 (1926).

[12]W. Thirring, Acta Phys. Austriaca Suppl. 9, 256 (1972). This paper contains references to reviews on the older developments of the Kaluza-Klein theory.

[13]R. Kerner, Ann. Inst. Henri Poincaré 9, 143 (1968).

[14]Y. M. Cho, J. Math. Phys. 16, 2029 (1975).

[15]J. Rayski, Acta Phys. Pol. 27, 89 (1965); 27, 947 (1965); 28, 87 (1965).

[16]J. H. Rawnsley, Communications DIAS, Ser. A No. 25, 1978 (unpublished).

[17]P. Minkowski, Bern report 1977 (unpublished).

[18]A. Trautmann, Rep. Math. Phys. 1, 29 (1970).

[19]W. Pauli, Ann. Phys. 18, 337 (1933).

[20]W. Mecklenburg (unpublished).

[21]W. Israel, Communications DIAS, Ser. A, No. 19, 1970 (unpublished).

[22]Numerically, $A_\alpha^\tau = A_\alpha^t$ for $\tau = t$, since numerically $g_{tu} = \eta_{\sigma\tau}$ if $t = \sigma$ and $u = t$.

[23]The presentation given here is a direct generalization of the five-dimensional case as discussed in Ref. 12.

[24]The Einstein-Hilbert Lagrangian density is $\sqrt{|G|}\, R_{N+4}$. $|G|$ splits in a suitable local basis into a factor $|\bar{g}|$ and a factor $|g|$ (cf. Ref. 14).

[25]B. S. DeWitt, in *Relativity, Groups and Topology*, Les Houches Lectures 1963, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1964).

[26]The Lagrangian density as given here is Hermitian up to an overall factor i.

[27]For higher representations of SU (2) one would expect this mass term to be proportional to the eigenvalue of the Casimir operator in the given representation.

[28]A. R. Edmonds, *Drehimpulse in der Quantenmechanik* (Bibliographisches Institut, Mannheim, 1964).

[29]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Univ. of Maryland Press, College Park, Md., 1972).

[30]The functions $D^1_{\sigma\sigma'}$ are given explicitly in Appendix D.

[31]W. Kopcyński, Acta Phys. Pol. B10, 365 (1979).

[32]M. E. Rose, *Elementary Theory of Angular Momentum* (Wiley, New York, 1963).

[33]It seems that, at this point, the book by Edmonds (Ref. 28) contains an inconsistency in phase conventions.