Thermal effects of acceleration through random classical radiation

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The thermal effects of acceleration found by Davies and Unruh within quantum field theory are shown to exist within random classical radiation. The two-field correlation functions for random classical radiation are used as the basis for investigating the spectrum of radiation observed at an accelerating point detector. An observer with proper acceleration a relative to the Lorentz-invariant spectrum of random classical *scalar* zero-point radiation finds a spectrum identical with that given by Planck's law for scalar thermal radiation where the temperature is related to the acceleration by $T = \hbar a/2\pi ck$. An observer with proper acceleration a relative to the Lorentz-invariant spectrum of random classical *electromagnetic* radiation finds a stationary radiation spectrum which is not Planck's spectrum. Rather, the observed spectrum in the electromagnetic case contains a term agreeing with Planck's electromagnetic spectrum plus an additional term. This spectrum for the electromagnetic case appears in the work of Candelas and Deutsch for an accelerating mirror and corresponds to thermal radiation in the non-Minkowskian space-time of the accelerating observer. The calculations reported here involve an entirely classical point of view, but are shown to have immediate connections with quantum field theory.

I. INTRODUCTION

Recent work[1-9] in astrophysics has suggested thermal effects due to acceleration. Specifically, an observer in uniform acceleration relative to the quantum vacuum is reported to observe radiation effects as though he were located in a thermal bath at absolute temperature $T = \hbar a/2\pi ck$, where a is the proper acceleration of the observer while $\hbar$ and k refer to Planck's and Boltzmann's constants, respectively. Sciama[10] fascinated a number of physicists by emphasizing this idea quite recently in an Einstein Centennial lecture for the New York Academy of Sciences.

At present it seems difficult for a physicist of ordinary training to get beyond the initial fascination with this idea to an understanding of the physics involved; the published literature on the thermal effects of acceleration tends to be mathematically sophisticated without providing any easy physical insight as to what is involved. The present article addresses this situation. It presents a physical picture as to why an accelerated observer should find an alteration in the vacuum of the universe. However, it does not suggest a physical reason as to why this alteration should take the form of a thermal bath. Indeed, the question of just what spectrum is thermal in an accelerating frame of reference requires sophistication beyond the level of this article, including the consideration of non-Minkowskian space-time.

The basis offered here for a physical insight into the effects of acceleration is that provided by a vacuum consisting of random classical radiation with a Lorentz-invariant spectrum. Such a model for the vacuum, although purely classical, can be shown[11-15] to describe quantitiatively a number of phenomena which are usually thought to require quantum descriptions. For the present problem, the model provides random fluctuations in the vacuum which are the same for every inertial observer,[12,13] yet which take on an altered appearance for an accelerating observer. Moreover, the classical model provides not only a physical picture but also detailed quantitative calculations applicable to the full quantum description. The classical correlation functions calculated within this classical model have values identical with the expectation values of corresponding products of quantum fields.[14] Thus the results can be taken over directly onto the quantum theory.

The organization of the article is as follows. First there is a qualitative discussion of random classical radiation with a Lorentz-invariant spectrum as a model for the vacuum. Next the change in this spectrum for an accelerating observer is considered. Then coordinates for an observer in hyperbolic motion are introduced. The remainder of the article is split into two separate cases involving first a massless scalar field and second a massless vector (electromagnetic) field. For the scalar field we obtain the Lorentz-invariant valcuum spectrum, calculate the correlation function for the field at the position of an accelerating detector, and then compare this correlation function with that for a detector at rest in Planck's spectrum of scalar thermal radiation. The correlation functions agree, provided the temperature is given by $T = \hbar a/2\pi ck$. This indeed bears out the notion of thermal effects of acceleration. Finally, for the scalar case we show the agreement between the classical correlation functions and the expectation values of corresponding products of quantum field operators. The same considerations are then carried through for the electromagnetic field. However, in this case the correlation functions for the fields at the position of a detector accelerating through the zero-point radiation turn out to be different from the correlation functions of the fields at a point at rest in the Planck spectrum of radiation. The spectrum seen by the accelerating observer is obtained from the correlation functions, and it is found that an additional contribution is present beyond the Planck thermal spectrum. This altered spectrum in the electromagnetic case is seen to be the same as that found earlier in work on an accelerating mirror, with the alteration related to unfamiliar space-time properties of an accelerating reference frame.

II. A CLASSICAL MODEL FOR THE VACUUM

A. Random classical radiation

Why should an accelerating observer regard himself as in a thermal bath? Surely this idea is foreign to Newtonian classical mechanics and remains puzzling from the perspective of naive quantum theory. Of course the analyses of this question in the literature start with the vacuum of quantum field theory, and the quantum vacuum seems hard to visualize. It involves fluctuations and virtual quanta but no real quanta. The physics literature tells us that acceleration somehow turns virtual quanta into real quanta.

Now a number of phenomena associated with the vacuum of quantum electromagnetic field theory can be understood in purely classical terms within classical electron theory, provided we change the homogeneous boundary conditions on Maxwell's equations to include random classical radiation with a Lorentz invariant spectrum.[11-15] In this new classical theory there is always present this random classical electromagnetic radiation irrespective of what physical situation is considered. This random radiation forms the "vacuum" state.

It should be noted that this classical model can be applied[14] to simulate some aspects of any quantum field theory, for example, a scalar field theory, not merely quantum electrodynamics. The random radiation introduced is not connected with temperature radiation but exists in the vacuum at the absolute zero of temperature; hence it is termed classical zero-point radiation. The zero-point radiation is treated just as any random classical radiation would be, and the fluctuations of the classical zero-point radiation are just as real as the fluctuations of classical thermal radiation. The only special aspect of zero-point radiation is its spectrum. It is Lorentz invariant,[12,13] and this aspect is crucial. Because the spectrum is Lorentz invariant, every inertial observer, no matter what his velocity, finds the same spectrum of random classical radiation. Every inertial frame is equivalent; there is no preferred frame. It turns out that the Lorentz-invariant spectrum of random classical radiation is unique up to a multiplicative constant.[12,13] Every other spectrum of random radiation has some preferred inertial frame. Now thermal radiation involves radiation above the zero-point spectrum; it involves a finite amount of energy and singles out a preferred frame of reference.

Planck's constant appears in the new classical theory as the scale factor in the Lorentz-invariant spectrum of random classical radiation. This is the one place where $\hbar$ is put into the classical theory. Every further appearance of Planck's constant is derived from its role as the scale factor of the zero-point radiation.

B. Acceleration relative to classical zero-point radiation

The idea of classical zero-point radiation is well suited for an understanding of a number of phenomena which are usually regarded as quantum mechanical, including diamagnetism[11,16] and Van der Waals forces.[17-19] In the present case it is ideally suited to giving one a sense of the change in fluctuation pattern when a detector undergoes accelerated motion—in other words, to giving one a sense of the thermal effects of acceleration.

Let us reconsider the basis for the Lorentz-invariant behavior of the vacuum fluctuations. In some given frame, which we will term the laboratory frame, a spectrum of completely random classical radiation can be written[13] as a sum over plane waves of various frequencies ω and wave vectors $\vec{k}$ with random phases. Another observer, moving with constant velocity with respect to the laboratory, sees the radiation pattern as composed of plane waves but now with frequencies ω' and wave vectors $\vec{k}'$ connected by a Lorentz transformation to the ω and $\vec{k}$ of the laboratory description. In general, the spectrum seen by the moving observer will be quite different from that seen by the laboratory observer. However, for a Lorentz-invariant spectrum the plane waves are shifted in precisely such a way as to reproduce the spectrum found by the laboratory observer. Every plane wave is Doppler shifted indeed, but for every wave Doppler shifted to a new frequency, some other wave is shifted down to take the place of the first wave. This balance which provides the Lorentz invariance is delicate and holds for a spectrum unique up to a multiplicative constant.

Now suppose we consider an accelerating observer. In this case a single plane wave in the laboratory frame seems, when observed by an accelerating detector, to be shifting its frequency continuously in time. A plane wave of frequency ω in the laboratory frame will thus involve a wide spectrum for an accelerating detector. The delicate balance for the invariant spectrum which was found by all inertial observers need not hold for an accelerating observer. It is precisely this change in spectrum away from the Lorentz-invariant radiation spectrum associated with the vacuum which is involved in the thermal effect of acceleration.

C. Hyperbolic motion

The simplest form of accelerated motion for a point is hyperbolic motion, also termed uniformly accelerated motion, in which the point has a constant acceleration relative to the inertial frame in which the point is instantaneously at rest. In this case there is no preferred inertial frame associated with the accelerating point.

For simplicity of notation we will choose an inertial frame referred to as the laboratory frame such that the accelerating point is at rest in this frame at proper time $\tau=0$ when the frame time t is also $t=0$. The accelerating point moves along the x axis of the laboratory frame with proper acceleration a, and reaches the point $x(0)=c^2/a$ at time $t=0$. Now the acceleration a of the point relative to its instantaneous inertial rest frame can be related[20] using standard Lorentz transformations to the acceleration $dv/dt=a(1-v^2/c^2)^{3/2}$ of the particle seen in the laboratory frame. Since the acceleration a is a constant, this expression can be integrated to give the velocity $v(t)=act(c^2+a^2t^2)^{-1/2}$ and position $x(t)=(c/a)(c^2+a^2t^2)^{1/2}$ seen in the laboratory frame as a function of the laboratory time t. Since the interval dt in the laboratory frame is related to the proper time interval $d\tau$ by $dt=d\tau(1-v^2/c^2)^{-1/2}$, we can use $v(t)$ above to solve for the laboratory time t as a function of the proper time τ, finding

$$t(\tau)=(c/a)\sinh(a\tau/c)\,. \tag{1}$$

This can then be used to obtain the position x and velocity v in the laboratory frame as functions of the proper time τ:

$$x=(c^2/a)\cosh(a\tau/c)\,, \tag{2}$$

$$v=c\tanh(a\tau/c)\,, \tag{3}$$

with

$$\gamma=(1-v^2/c^2)^{-1/2}=\cosh(a\tau/c)\,. \tag{4}$$

It should be emphasized that an observer undergoing uniform acceleration through the classical zero-point radiation finds a stationary spectrum of fluctuations, if not the vacuum spectrum. This occurs because of the assumed Lorentz invariance of the vacuum spectrum. In any inertial frame instanteously at rest with respect to the accelerating point, the spectrum of random classical radiation is always the same. There is no preferred inertial frame and no preferred time for the accelerating observer.

III. MASSLESS SCALAR FIELD

A. Lorentz-invariant vacuum radiation

In the case of a massless scalar field, $\phi(\vec{r},t)$, associated with the Lagrangian density[21] $\mathcal{L}=\frac{1}{2}[c^{-2}(\partial\phi/\partial t)^2-\nabla\phi\cdot\nabla\phi]$, the classical vacuum consists of random classical scalar radiation. The spatially homogeneous and isotropic distribution in empty space can be written as an expansion in plane waves with random phases:

$$\phi(\vec{r},t)=\int d^3k f(\omega)\cos[\vec{k}\cdot\vec{r}-\omega t-\theta(\vec{k})]\,, \tag{5}$$

where the $\theta(\vec{k})$ are random phases distributed uniformly on the interval $(0,2\pi)$ and independently for each wave vector $\vec{k}$.

The Lorentz-invariant spectrum can be found by requiring that the average value of the square of the field $\langle\phi(\vec{r},t)\phi(\vec{r},t)\rangle$ take the same spectral form in every inertial frame. Now the average value can be written from (5) as

$$\begin{aligned}\langle\phi(\vec{r},t)\phi(\vec{r},t)\rangle &= \int d^3k_1\int d^3k_2 f(\omega_1)f(\omega_2)\\ &\qquad\times\langle\cos[\vec{k}_1\cdot\vec{r}-\omega_1 t-\theta(\vec{k}_1)]\\ &\qquad\times\cos[\vec{k}_2\cdot\vec{r}-\omega_2 t-\theta(\vec{k}_2)]\rangle\\ &=\tfrac{1}{2}\int d^3k f^2(\omega)\,,\end{aligned} \tag{6}$$

where the average over the random phases can be written as

$$\begin{aligned}\langle\cos\theta(\vec{k})\cos\theta(\vec{k}')\rangle &=\langle\sin\theta(\vec{k})\sin\theta(\vec{k}')\rangle\\ &=\tfrac{1}{2}\delta^3(\vec{k}-\vec{k}')\,,\end{aligned} \tag{7}$$

$$\langle\cos\theta(\vec{k})\sin\theta(\vec{k}')\rangle=0\,, \tag{8}$$

and one integrates over the δ function. The scalar field observed from another inertial frame is unchanged in value so that

$$\begin{aligned}\phi'(\vec{r}',t')&=\phi(\vec{r},t)\\ &=\int d^3k f(\omega)\cos[\vec{k}'\cdot\vec{r}'-\omega' t'-\theta(\vec{k})]\,,\end{aligned} \tag{9}$$

where the last expression follows from the

Lorentz invariance of the phase of a plane wave. The wave vectors $(\omega, \vec{k})$, $(\omega', \vec{k}')$ seen in the two different inertial frames with relative velocity v along the x axis are connected by the usual Lorentz transformation[22]

$$\omega = \gamma(\omega' + v k_x'), \tag{10}$$

$$k_x = \gamma(k_x' + v\omega'/c^2), \tag{11}$$

$$k_y = k_y', \quad k_z = k_z', \tag{12}$$

with the Jacobian of the transformation giving

$$d^3k = d^3k'\gamma(1 + v k_x'/\omega'). \tag{13}$$

Thus from (9) we have

$$\begin{aligned}\langle\phi'(\vec{r}', t')\phi'(\vec{r}', t')\rangle &= \int d^3k f^2(\omega) \\ &= \int d^3k'\gamma\,(1 + v k_x'/\omega') \\ &\quad\times f^2(\gamma\omega' + \gamma v k_x').\end{aligned} \tag{14}$$

The last expression is exactly $\int d^3k' f^2(\omega')$, provided the function f^2 is linear in the inverse of its argument

$$f^2(\omega) = \text{const}/\omega. \tag{15}$$

Thus we have obtained the unique Lorentz-invariant spectrum for the scalar field. In order to make the connection with quantum field theory we will choose the constant in (15) so that the Lorentz-invariant spectral function $f_0(\omega)$ is

$$\pi^2 f_0^{\,2}(\omega) = \tfrac{1}{2}\hbar c^2/\omega. \tag{16}$$

B. Correlation function at an accelerating detector

The characteristics of a random classical field at a point $\vec{r}$ in space can be described by evaluating the correlations in the fluctuations. Thus for the scalar field $\phi(\vec{r}, t)$ of random classical radiation as seen in the laboratory frame, we would like to evaluate the average value $\langle\phi(\vec{r}, s - t/2)\phi(\vec{r}, s + t/2)\rangle$ involving the product of fields at the point $\vec{r}$ in space at times $s - t/2$ and $s + t/2$.

On the other hand, the fluctuating field φ at a point detector in hyperbolic motion should be evaluated as $\langle\varphi(0, \sigma - \tau/2)\varphi(0, \sigma + \tau/2)\rangle$, where we have chosen the detector at the origin of the accelerating frame and where $\varphi(0, \sigma \pm \tau/2)$ is the field at the accelerating detector point in the inertial frame instantaneously at rest with respect to the detector at proper time $\sigma \pm \tau/2$. From the knowledge of the Lorentz-transformation properties (9) of scalar fields, we find

$$\varphi(0, \sigma \pm \tau/2) = \phi\left[\frac{c^2}{a}\cosh\left(\frac{a(\sigma \pm \tau/2)}{c}\right), 0, 0, \frac{c}{a}\sinh\left(\frac{a(\sigma \pm \tau/2)}{c}\right)\right], \tag{17}$$

where ϕ is the field in the laboratory frame evaluated at the laboratory position (2) and the laboratory time (1) for the accelerating detector point.

The expressions (17) and (5) are now introduced into the correlation function, the average over the random phases performed, and the Lorentz-invariant spectral function $f_0(\omega)$ in (16) inserted so that

$$\begin{aligned}\langle\varphi_0(0, \sigma - \tau/2)\varphi_0(0, \sigma + \tau/2)\rangle = \int d^3k \frac{1}{\pi^2}\frac{\hbar c^2}{2\omega}\,\tfrac{1}{2}\cos\Big\{k_x\frac{c^2}{a}\left[\cosh\left(\frac{a(\sigma - \tau/2)}{c}\right) - \cosh\left(\frac{a(\sigma + \tau/2)}{c}\right)\right] \\ - \omega\frac{c}{a}\left[\sinh\left(\frac{a(\sigma - \tau/2)}{c}\right) - \sinh\left(\frac{a(\sigma + \tau/2)}{c}\right)\right]\Big\}.\end{aligned} \tag{18}$$

C. Stationary character of the correlation function

The stationary character of the correlation function is not exhibited here since the right-hand side of Eq. (18) seems to depend upon the choice of the proper time σ. However, the physical argument that there is no preferred time for hyperbolic motion indicates that actually the expression (18) must be independent of σ. The independence can be exhibited explicitly by changing the variables of integration $\vec{k}$ over to $\vec{k}'$ where

$$\omega' = \omega\cosh(a\sigma/c) - ck_x\sinh(a\sigma/c), \tag{19}$$

$$k_x' = k_x\cosh(a\sigma/c) - (\omega/c)\sinh(a\sigma/c), \tag{20}$$

$$k_y' = k_y, \quad k_z' = k_z, \tag{21}$$

which corresponds exactly to a Lorentz transformation from the unprimed laboratory frame over to the primed inertial frame in which the accelerating particle is instantaneously at rest at proper time σ. Under this transformation we find $d^3k/\omega = d^3k'/\omega'$ as from (10) and (13); and, after expansion of $\cosh[a(\sigma \pm \tau/2)/c]$ and $\sinh[a(\sigma \pm \tau/2)/c]$, we see that the argument of the cosine in Eq. (18) is just $-2(\omega'c/a)\sinh(a\tau/2c)$. Thus we have

$$\langle\varphi_0(0, \sigma - \tau/2)\varphi_0(0, \sigma + \tau/2)\rangle = \int \frac{d^3k'}{4\pi^2}\frac{\hbar c^2}{\omega'}\cosh\left[2\omega'\frac{c}{a}\sinh\left(\frac{a\tau}{2c}\right)\right], \tag{22}$$

which clearly depends only on the proper time interval τ and is independent of the central proper time σ.

D. Evaluation of the correlation function

The correlation function can be evaluated by first integrating over angles to give a factor of 4π and then carrying out the infinite integration in frequency by the use of a temporary cutoff. The integral is of the form

$$\int_0^\infty d\omega\,\omega\cos b\omega = \mathrm{Re}\lim_{\lambda\to 0}\int_0^\infty d\omega\,\omega\exp[(ib-\lambda)\omega] = -b^{-2}. \qquad (23)$$

In this fashion we find the correlation function measured at a point in uniform acceleration through random classical scalar radiation as

$$\langle\varphi_0(0,\sigma-\tau/2)\varphi_0(0,\sigma+\tau/2)\rangle = -\frac{\hbar}{\pi c}\left(\frac{a}{2c}\right)^2 \times\mathrm{csch}^2\left(\frac{a\tau}{2c}\right). \qquad (24)$$

E. Correlation function at a detector at rest in Planck's spectrum

The result (24) will now be compared with the correlation function $\langle\phi_T(0,s-t/2)\phi_T(0,s+t/2)\rangle$ for a particle at the origin of an inertial frame where there is present Planck's spectrum of random thermal radiation along with the zero-point radiation:

$$\pi^2 f_T{}^2(\omega) = \frac{\hbar}{\omega}\left(\frac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\right) = \frac{\hbar c^2}{2\omega}\coth\left(\frac{\hbar\omega}{2kT}\right). \qquad (25)$$

Employing the expression (5) for the scalar fields together with the thermal spectrum (25), and then averaging over the random phases as in (7) and (8) and integrating over the δ function and over angles, we obtain

$$\langle\phi_T(0,s-t/2)\phi_T(0,s+t/2)\rangle = \frac{\hbar}{\pi c}\int_0^\infty d\omega\,\omega\coth\left(\frac{\hbar\omega}{2kT}\right)\cos\omega t. \qquad (26)$$

The integral can be broken up into the form

$$\int_0^\infty d\omega\,\omega\coth\left(\frac{\hbar\omega}{2kT}\right)\cos\omega t = \int_0^\infty d\omega\,\omega\cos\omega t + \int_0^\infty d\omega\frac{2\omega\cos\omega t}{\exp(\hbar\omega/kT)-1} = -\frac{1}{t^2}+\left[\frac{1}{t^2}-\left(\frac{\pi kT}{\hbar}\right)^2\mathrm{csch}^2\left(\frac{\pi kTt}{\hbar}\right)\right], \qquad (27)$$

where we have used (23) for the singular part of the integral and a standard integral table[23] for the remaining term. Thus

$$\langle\phi_T(0,s-t/2)\phi_T(0,s+t/2)\rangle = \frac{-\hbar}{\pi c}\left(\frac{\pi kT}{\hbar}\right)^2 \times\mathrm{csch}^2\left(\frac{\pi kTt}{\hbar}\right). \qquad (28)$$

F. Agreement of the correlation functions

If we compare the correlation function $\langle\varphi_0(0,\sigma-\tau/2)\varphi_0(0,\sigma+\tau/2)\rangle$ in (24), found by the observer accelerating through the random classical scalar zero-point radiation, with the correlation function $\langle\phi_T(0,s-t/2)\phi_T(0,s+t/2)\rangle$ in (28), for an inertial observer in random classical scalar zero-point radiation with that for Planck's spectrum including zero-point radiation, we find they are identical in functional form provided the acceleration and the temperature are related by

$$T=\hbar a/(2\pi ck). \qquad (29)$$

It is precisely in this sense that one speaks of an observer accelerating through the inertial vacuum as finding himself in a thermal radiation bath. We notice that the relationship (29) between the temperature and the acceleration involves Planck's constant $\hbar$ which provides the scale of the zero-point radiation. The relationship in (29) seems most curious, and has been connected by various researchers to astrophysical phenomena involving black holes.

G. Connections with earlier quantum results

The calculations which we have presented hold for classical radiation. Earlier calculations are entirely in terms of quantum systems, often phrased in terms of quantum propagators. For free fields and harmonic-oscillator systems, there is an immediate connection[24] between the quantum field theory and random classical field theory. Specifically we can write the random classical field $\phi(\vec{r},t)$ in (5) in the form

$$\phi(\vec{r},t)=\int d^3k\,\tfrac{1}{2}f(\omega)\,[a(\vec{k})\exp(-iK\cdot x) + a^*(\vec{k})\exp(iK\cdot x)], \qquad (30)$$

where the four-vector product $K\cdot x$ is just $K\cdot x=\omega t-\vec{k}\cdot\vec{r}$ and the symbols $a(\vec{k})$ and its complex conjugate $a^*(\vec{k})$ involve the random phases $\theta(\vec{k})$;

$$a(\vec{k})=\exp[-i\theta(\vec{k})], \quad a^*(\vec{k})=\exp[i\theta(\vec{k})]. \qquad (31)$$

The spectral function $f^2(\omega)$ takes on the values in (16) and (25) depending upon whether or not the temperature is above zero. The averages over the random phases $\theta(\vec{k})$ give

$$\langle\langle a(\vec{k})a(\vec{k}')\rangle\rangle=\langle\langle a^*(\vec{k})a^*(\vec{k}')\rangle\rangle=0\,, \quad (32)$$

$$\langle\langle a(\vec{k})a^*(\vec{k}')\rangle\rangle=\langle\langle a^*(\vec{k}')a(\vec{k})\rangle\rangle=\delta^3(\vec{k}-\vec{k}')\,. \quad (33)$$

When written in this form the classical scalar field appears analogous[25] to the quantum free scalar field.

$$\underline{\phi}(\vec{r},t)=\int\frac{d^3k}{2\pi}\left(\frac{\hbar c^2}{\omega}\right)^{1/2}[\underline{a}(\vec{k})\exp(-iK\cdot x)+\underline{a}^\dagger(\vec{k})\exp(iK\cdot x)]\,, \quad (34)$$

where $\underline{a}(\vec{k})$ and $\underline{a}^\dagger(\vec{k})$ are the familiar annihilation and creation operators with commutators

$$[\underline{a}(\vec{k}),\underline{a}(\vec{k}')]=[\underline{a}^\dagger(\vec{k}),\underline{a}^\dagger(\vec{k}')]=0\,, \quad (35)$$

$$[\underline{a}(\vec{k}),\underline{a}^\dagger(\vec{k}')]=-[\underline{a}^\dagger(\vec{k}'),\underline{a}(\vec{k})]=\delta^3(\vec{k}-\vec{k}')\,, \quad (36)$$

and

$$\underline{a}(\vec{k})|0\rangle=0\,. \quad (37)$$

Using on the one hand the average over the random phases in (32) and (33), and on the other hand the properties of the annihilation and creation operators (35)–(37), it is easy to show that the classical vacuum correlation function is identical with the vacuum expectation value of the symmetrized product of quantum fields:

$$\langle\langle\phi_0(\vec{r},t)\phi_0(\vec{r}',t')\rangle\rangle=\tfrac{1}{2}\langle 0|\{\underline{\phi}(\vec{r},t)\underline{\phi}(\vec{r}',t')\}|0\rangle=\int\frac{d^3k}{4\pi^2}\frac{\hbar c^2}{\omega}\cos[\vec{k}\cdot(\vec{r}-\vec{r}')-\omega(t-t')]\,, \quad (38)$$

where

$$\{\underline{\phi}(\vec{r},t)\underline{\phi}(\vec{r}',t')\}=\underline{\phi}(\vec{r},t)\underline{\phi}(\vec{r}',t')+\underline{\phi}(\vec{r}',t')\underline{\phi}(\vec{r},t)\,. \quad (39)$$

Hence the values of these functions remain identical when we substitute the values (1), (2) for $\vec{r}, t$ and $\vec{r}', t'$ appropriate for an observer in uniform acceleration.

At finite temperatures the agreement between the classical correlation function and the vacuum expectation value still holds,[24] although the basis for the agreement is not immediate as it is for the vacuum case. In the classical case we need only change the spectral function from f_0 over to the Planck spectrum with zero-point radiation f_T in (25). The averages over random phases are just as in the previous case. Hence we obtain the correlation at finite temperature as

$$\langle\langle\phi_T(\vec{r},t)\phi_T(\vec{r}',t')\rangle\rangle=\int\frac{d^3k}{4\pi^2}\frac{\hbar c^2}{\omega}\coth\left(\frac{\hbar\omega}{2kT}\right)\times\cos[\vec{k}\cdot(\vec{r}-\vec{r}')-\omega(t-t')]\,. \quad (40)$$

We can rewrite this in a form which will be useful later:

$$\langle\langle\phi_T(\vec{r},t)\phi_T(\vec{r}',t')\rangle\rangle=\int\frac{d^3k}{4\pi^2}\frac{1}{Z}\sum_{n=0}^{\infty}\frac{\hbar c^2}{\omega}\exp\left[-\frac{\hbar\omega(n+\frac{1}{2})}{kT}\right]\{(n+\tfrac{1}{2})\exp[-iK\cdot(x-x')]+(n+\tfrac{1}{2})\exp[iK\cdot(x-x')]\}\,, \quad (41)$$

where we have written

$$Z=\sum_{n=0}^{\infty}\exp[-\hbar\omega(n+\tfrac{1}{2})/kT]\,, \quad (42)$$

and have used the expansion

$$\tfrac{1}{2}\coth\left(\frac{\hbar\omega}{2kT}\right)=\frac{\sum_{n=0}^{\infty}(n+\frac{1}{2})\exp[-\hbar\omega(n+\frac{1}{2})/kT]}{\sum_{n=0}^{\infty}\exp[-\hbar\omega(n+\frac{1}{2})/kT]}\,. \quad (43)$$

The quantum case stands in complete contrast with the classical one. The quantum field theory at finite temperature involves not a change in the field operator $\underline{\phi}(\vec{r},t)$ but rather a change in the state vector of the system. The expectation value at temperature T involves an incoherent sum over n-quantum states weighted by the Boltzmann factor

$$\langle|\underline{\phi}(\vec{r},t)\underline{\phi}(\vec{r}',t')|\rangle_T=\sum_{n_{\vec{k}_1}=0}^{\infty}\sum_{n_{\vec{k}_2}=0}^{\infty}\cdots\frac{1}{Z_1}\exp[-\epsilon_n(\vec{k}_1)]\frac{1}{Z_2}\exp[-\epsilon_n(\vec{k}_2)]\cdots\langle n_{\vec{k}_1}n_{\vec{k}_2}\cdots|\underline{\phi}(\vec{r},t)\underline{\phi}(\vec{r}',t')|n_{\vec{k}_1}n_{\vec{k}_2}\cdots\rangle\,, \quad (44)$$

where the state $|n_{\vec{k}_1}n_{\vec{k}_2}\cdots\rangle$ contains $n_{\vec{k}_1}$ quanta at wave vector $\vec{k}_1$, $n_{\vec{k}_2}$ quanta at wave vector $\vec{k}_2$, etc. Now from (35)–(37) the expectation value in the last line takes the form

$$\langle\{n\}|\underline{\phi}(\vec{r},t)\underline{\phi}(\vec{r}',t')|\{n\}\rangle=\int\frac{d^3k}{2\pi}\int\frac{d^3k'}{2\pi}\left(\frac{\hbar c^2}{\omega}\right)^{1/2}\left(\frac{\hbar c^2}{\omega'}\right)^{1/2}\langle\{n\}|[\underline{a}\exp(-iK\cdot x)+\underline{a}^\dagger\exp(iK\cdot x)]$$
$$\times[\underline{a}'\exp(-iK'\cdot x')+\underline{a}'^\dagger\exp(iK'\cdot x')]|\{n\}\rangle$$
$$=\int\frac{d^3k}{4\pi^2}\frac{\hbar}{\omega}\{(n_{\vec{k}}+1)\exp[-iK\cdot(x-x')]+n_{\vec{k}}\exp[iK\cdot(x-x')]\}. \tag{45}$$

The expression (45) is now substituted into (44), the order of integration and quanta summation is interchanged, and the sums carried out for all quanta which are not of the type $n_{\vec{k}}$. This cancels all but one of the factors of Z^{-1} leaving

$$\langle|\underline{\phi}(\vec{r},t)\underline{\phi}(\vec{r}',t')|\rangle_T=\int\frac{d^3k}{4\pi^2}\frac{1}{Z}\sum_{n=0}^{\infty}\frac{\hbar c^2}{\omega}\exp\left[-\frac{\hbar\omega(n+\frac{1}{2})}{kT}\right]\{(n+1)\exp[-iK\cdot(x-x')]+n\exp[iK\cdot(x-x')]\}. \tag{46}$$

If we now take the symmetrized product of the quantum operators, the factors of $(n+1)$ and n are both averaged to $(n+\frac{1}{2})$ so that the correlation function at finite temperature in (40)–(42) is in exact agreement with the quantum expectation value at the same temperature

$$\langle\phi_T(\vec{r},t)\phi_T(\vec{r}',t')\rangle=\tfrac{1}{2}\langle|\{\underline{\phi}(\vec{r},t)\underline{\phi}(\vec{r}',t')\}|\rangle_T. \tag{47}$$

From this identity of values, we see that our classical results for the thermal effects of acceleration have a direct connection with those of quantum field theory.

IV. ELECTROMAGNETIC FIELD

A. Fields of the classical electromagnetic vacuum

Having seen that for scalar fields, acceleration through the classical zero-point radiation gives rise to thermal effects, we would like to extend the investigation to the vector field provided by electromagnetism. The zero-point fields corresponding to a Lorentz-invariant spectrum of classical electromagnetic radiation have been investigated in a series of papers.[11-15] The fluctuating electric and magnetic fields can be written as

$$\vec{E}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)\mathfrak{h}(\omega)\times\cos[\vec{k}\cdot\vec{r}-\omega t-\theta(\vec{k},\lambda)], \tag{48}$$

$$\vec{B}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\hat{k}\times\hat{\epsilon}\,\mathfrak{h}(\omega)\times\cos[\vec{k}\cdot\vec{r}-\omega t-\theta(\vec{k},\lambda)], \tag{49}$$

where the $\theta(\vec{k},\lambda)$ are random phases distributed uniformly on the interval $(0,2\pi)$ and independently for each wave vector $\vec{k}$ and polarization λ. The function $\mathfrak{h}(\omega)$ gives the spectrum of radiation in terms of the frequency $\omega=ck$ and $\pi^2\mathfrak{h}^2(\omega)$ corresponds to the energy per normal mode at frequency ω. It has been shown[12,13] that random classical electromagnetic radiation has the same spectrum in every inertial frame if and only if

$$\pi^2\mathfrak{h}_0{}^2(\omega)=\text{const}\times\omega. \tag{50}$$

The choice of the constant as $\frac{1}{2}\hbar$ gives agreement with quantum theory:

$$\pi^2\mathfrak{h}_0{}^2(\omega)=\tfrac{1}{2}\hbar\omega. \tag{51}$$

B. Correlation functions for the fields at the accelerating detector

The fields $\vec{\mathcal{E}}(0,\sigma\pm\tau/2)$, $\vec{\mathcal{B}}(0,\sigma\pm\tau/2)$ observed at proper time $\sigma\pm\tau/2$ by the detector accelerating along the x axis are related by a Lorentz transformation[26] to those observed in the laboratory frame:

$$\vec{\mathcal{E}}(0,\sigma\pm\tau/2)=\sum_{\lambda=1}^{2}\int d^3k\{\hat{i}\epsilon_x+\hat{j}\gamma[\epsilon_y-v(\hat{k}\times\hat{\epsilon})_z/c]+\hat{k}\gamma[\epsilon_z+v(\hat{k}\times\hat{\epsilon})_y/c]\}\mathfrak{h}(\omega)\cos[k_xx-\omega t-\theta(\vec{k},\lambda)], \tag{52}$$

$$\vec{\mathcal{B}}(0,\sigma\pm\tau/2)=\sum_{\lambda=1}^{2}\int d^3k\{\hat{i}(\hat{k}\times\hat{\epsilon})_x+\hat{j}\gamma[(\hat{k}\times\hat{\epsilon})_y+v\epsilon_z/c]+\hat{k}\gamma[(\hat{k}\times\hat{\epsilon})_z-v\epsilon_y/c]\}\mathfrak{h}(\omega)\cos[k_xx-\omega t-\theta(\vec{k},\lambda)], \tag{53}$$

where γ, v, x, and t are related to the proper time $\sigma\pm\tau/2$ as in (1)–(4).

For the vector-field analysis where several field components are present, we must deal with several correlation functions rather than with a single one as in the scalar-field case. We will treat first $\langle\mathcal{E}_x(0,\sigma-\tau/2)\mathcal{E}_x(0,\sigma+\tau/2)\rangle$. We use the expressions from (52) and average over the random phases as

$$\langle\cos\theta(\vec{k},\lambda)\cos\theta(\vec{k}',\lambda')\rangle=\langle\sin\theta(\vec{k},\lambda)\sin\theta(\vec{k}',\lambda')\rangle=\tfrac{1}{2}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}'), \tag{54}$$

$$\langle \cos\theta(\vec{k},\lambda)\sin\theta(\vec{k}',\lambda')\rangle = 0\,. \tag{55}$$

After summing over the δ function in λ, integrating over the δ function in $\vec{k}'$, and inserting the Lorentz-invariant spectral function $\mathfrak{h}_0$ in (51) for classical electromagnetic zero-point radiation, we find

$$\langle \mathcal{E}_{0x}(0,\sigma-\tau/2)\mathcal{E}_{0x}(0,\sigma+\tau/2)\rangle = \sum_{\lambda=1}^{2}\int d^3k\,\epsilon_x{}^2\frac{1}{\pi^2}\frac{1}{2}\hbar\omega\,\tfrac{1}{2}\cos\left\{k_x\frac{c^2}{a}\left[\cosh\left(\frac{a(\sigma-\tau/2)}{c}\right)-\cosh\left(\frac{a(\sigma+\tau/2)}{c}\right)\right]\right.$$
$$\left.-\omega\frac{c}{a}\left[\sinh\left(\frac{a(\sigma-\tau/2)}{c}\right)-\sinh\left(\frac{a(\sigma+\tau/2)}{c}\right)\right]\right\}. \tag{56}$$

The sum over the polarizations may now be performed using

$$\sum_{\lambda=1}^{2}\epsilon_i(\vec{k},\lambda)\epsilon_j(\vec{k},\lambda) = \delta_{ij} - k_ik_j/k^2\,. \tag{57}$$

The stationary nature of the correlation function (56) is proven in a manner analogous to that used for the scalar case. Again, physically the expression must be independent of σ because there is no preferred proper time for a system in uniform acceleration through the Lorentz-invariant spectrum of classical electromagnetic zero-point radiation. We introduce exactly the change of variables given by the Lorentz transformation (19)–(21) and so find

$$d^3k(1-k_x{}^2/k^2)\omega = d^3k'(1-k_x'^2/k'^2)\omega' \tag{58}$$

where the exponential involves exactly the change found in the scalar case. Thus the correlation function (56) becomes just the expression we would find upon inserting $\sigma=0$,

$$\langle \mathcal{E}_{0x}(0,\sigma-\tau/2)\mathcal{E}_{0x}(0,\sigma+\tau/2)\rangle = \int\frac{d^3k'}{4\pi^2}\frac{(\omega'^2-ck_x'^2)}{\omega'}\hbar\cos\left[2\omega\frac{c}{a}\sinh\left(\frac{a\tau}{2c}\right)\right], \tag{59}$$

which is clearly independent of σ.

The angular integrations can be performed easily when the x axis is chosen as the polar axis,

$$\langle \mathcal{E}_{0x}(0,\sigma-\tau/2)\mathcal{E}_{0x}(0,\sigma+\tau/2)\rangle = \frac{2}{3}\frac{\hbar}{\pi c^3}\int_0^\infty d\omega\,\omega^3\cos\left[\omega\frac{2c}{a}\sinh\left(\frac{a\tau}{2c}\right)\right]. \tag{60}$$

The ω integration involves an oscillating divergent expression which can be treated in the same manner as that in Eq. (23); here we require

$$\int_0^\infty d\omega\,\omega^3\cos b\omega = 6b^{-4}\,. \tag{61}$$

Thus we obtain

$$\langle \mathcal{E}_{0x}(0,\sigma-\tau/2)\mathcal{E}_{0x}(0,\sigma+\tau/2)\rangle = \frac{4\hbar}{\pi c^3}\left(\frac{a}{2c}\right)^4\operatorname{csch}^4\left(\frac{a\tau}{2c}\right). \tag{62}$$

The correlation function $\langle \mathcal{E}_{0y}(0,\sigma-\tau/2)\mathcal{E}_{0y}(0,\sigma+\tau/2)\rangle$ for the y components of the electric field seen at the accelerating detector is evaluated in similar fashion. We read off from Eq. (52) the values of $\mathcal{E}_y(0,\sigma\pm\tau/2)$. Then we average over the random phases, sum and integrate over the δ functions, and insert the zero-point spectral function (51) to find

$$\langle \mathcal{E}_{0y}(0,\sigma-\tau/2)\mathcal{E}_{0y}(0,\sigma+\tau/2)\rangle = \sum_{\lambda=1}^{2}\int d^3k\frac{1}{\pi^2}\frac{1}{2}\hbar\omega\left[\epsilon_y\cosh\left(\frac{a(\sigma-\tau/2)}{c}\right)-(\hat{k}\times\hat{\epsilon})_z\sinh\left(\frac{a(\sigma-\tau/2)}{c}\right)\right]$$
$$\times\left[\epsilon_y\cosh\left(\frac{a(\sigma+\tau/2)}{c}\right)-(\hat{k}\times\hat{\epsilon})_z\sinh\left(\frac{a(\sigma+\tau/2)}{c}\right)\right]$$
$$\times\tfrac{1}{2}\cos\left\{k_x\frac{c^2}{a}\left[\cosh\left(\frac{a(\sigma-\tau/2)}{c}\right)-\cosh\left(\frac{a(\sigma+\tau/2)}{c}\right)\right]\right.$$
$$\left.-\omega\frac{c}{a}\left[\sinh\left(\frac{a(\sigma-\tau/2)}{c}\right)-\sinh\left(\frac{a(\sigma+\tau/2)}{c}\right)\right]\right\}. \tag{63}$$

Next we sum over polarizations using (57) and obtain here

$$\sum_{\lambda=1}^{2} \epsilon_y{}^2 = 1 - k_y{}^2/k^2, \quad \sum_{\lambda=1}^{2} (\hat{k}\times\hat{\epsilon})_z{}^2 = 1 - k_z{}^2/k^2, \quad \sum_{\lambda=1}^{2} \epsilon_y(\hat{k}\times\hat{\epsilon})_z = k_x/k . \tag{64}$$

The same change of variables (19)–(21) which was used earlier removes the σ dependence and reduces the correlation function to

$$\langle \mathcal{E}_{0y}(0,\sigma-\tau/2)\mathcal{E}_{0y}(0,\sigma+\tau/2)\rangle = \int \frac{d^3k'}{4\pi^2}\hbar\omega'\left[\left(1-\frac{k_y'^2}{k'^2}\right)\cosh^2\!\left(\frac{a\tau}{2c}\right)-\left(1-\frac{k_z'^2}{k'^2}\right)\sinh^2\!\left(\frac{a\tau}{2c}\right)\right]\cos\left[2\omega'\frac{c}{a}\sinh\left(\frac{a\tau}{2c}\right)\right]. \tag{65}$$

The angular integrations can be carried out easily and the use of the identity $\cosh^2 x - \sinh^2 x = 1$ leaves exactly the expression (60) found for $\langle \mathcal{E}_{0x}(0,\sigma-\tau/2)\mathcal{E}_{0x}(0,\sigma+\tau/2)\rangle$.

The correlation function $\langle \mathcal{E}_{0y}(0,\sigma-\tau/2)\mathcal{B}_{0z}(0,\sigma+\tau/2)\rangle$ can be obtained from (52) and (53). After averaging over random phases and inserting the Lorentz-invariant spectral function (51), this is

$$\begin{aligned}\langle \mathcal{E}_{0y}(0,\sigma-\tau/2)\mathcal{B}_{0z}(0,\sigma+\tau/2)\rangle = \sum_{\lambda=1}^{2}\int d^3k\frac{1}{\pi^2}\frac{1}{2}\hbar\omega&\left[\epsilon_y\cosh\left(\frac{a(\sigma-\tau/2)}{c}\right)-(\hat{k}\times\hat{\epsilon})_z\sinh\left(\frac{a(\sigma-\tau/2)}{c}\right)\right]\\ &\times\left[(\hat{k}\times\hat{\epsilon})_z\cosh\left(\frac{a(\sigma+\tau/2)}{c}\right)-\epsilon_y\sinh\left(\frac{a(\sigma+\tau/2)}{c}\right)\right]\\ &\times\tfrac{1}{2}\cos\left\{k_x\frac{c^2}{a}\left[\cosh\left(\frac{a(\sigma-\tau/2)}{c}\right)-\cosh\left(\frac{a(\sigma+\tau/2)}{c}\right)\right]\right.\\ &\left.\qquad-\omega\frac{c}{a}\left[\sinh\left(\frac{a(\sigma-\tau/2)}{c}\right)-\sinh\left(\frac{a(\sigma+\tau/2)}{c}\right)\right]\right\}.\end{aligned} \tag{66}$$

Now we sum over polarizations using (57) and obtain precisely the results in (64). Once again we can simplify the expression by the change of variables (19)–(21) to show the independence of the expression from the value of σ. We obtain, after use of the double angle identities for the hyperbolic functions,

$$\langle \mathcal{E}_{0y}(0,\sigma-\tau/2)\mathcal{B}_{0z}(0,\sigma+\tau/2)\rangle = \int\frac{d^3k'}{4\pi^2}\hbar\omega'\left\{\left[-\left(1-\frac{k_y'^2}{k'^2}\right)+\left(1-\frac{k_z'^2}{k'^2}\right)\right]\tfrac{1}{2}\sinh\left(\frac{a\tau}{c}\right)+\frac{k_x'}{k'}\right\}\cos\left[2\omega\frac{c}{a}\sinh\left(\frac{a\tau}{2c}\right)\right]. \tag{67}$$

When the angular integrations are carried out, the correlation function vanishes.

The three correlation functions (56), (63), and (66) are the only ones which require extended calculation. All the remaining functions can be obtained from these by symmetry considerations, or else vanish immediately from symmetries in the angular integrations. Thus $\langle \mathcal{E}_{0x}(0,\sigma-\tau/2) \times \mathcal{E}_{0y}(0,\sigma+\tau/2)\rangle$ involves k_xk_y and k_y after the sum over polarizations, and hence vanishes because it is odd in k_y. The function $\langle \mathcal{E}_{0y}(0,\sigma-\tau/2) \times \mathcal{E}_{0z}(0,\sigma+\tau/2)\rangle$ involves $\sum\epsilon_z(\hat{k}\times\hat{\epsilon})_z = \sum\epsilon_y(\hat{k}\times\hat{\epsilon})_y = 0$ and also k_yk_z; it vanishes because it is odd in both k_y and k_z. The function $\langle \mathcal{E}_{0x}(0,\sigma-\tau/2)\mathcal{B}_{0x}(0,\sigma+\tau/2)\rangle$ involves $\sum\epsilon_x(\hat{k}\times\hat{\epsilon})_x = 0$ and hence vanishes. Finally, $\langle \mathcal{E}_{0x}(0,\sigma-\tau/2)\mathcal{B}_{0y}(0,\sigma+\tau/2)\rangle$ involves k_z and k_xk_z and vanishes because it is odd in k_z. Rotational symmetry around the x axis and the symmetry between the electric and magnetic fields in free space provide all the remaining correlation functions in terms of those already considered.

Thus we have

$$\begin{aligned}\langle \mathcal{E}_{0i}(0,\sigma-\tau/2)\mathcal{E}_{0j}(0,\sigma+\tau/2)\rangle &= \langle \mathcal{B}_{0i}(0,\sigma-\tau/2)\mathcal{B}_{0j}(0,\sigma+\tau/2)\rangle\\ &= \frac{4\hbar}{\pi c^3}\left(\frac{a}{2c}\right)^4\operatorname{csch}^4\left(\frac{a\tau}{2c}\right)\end{aligned} \tag{68}$$

and

$$\langle \mathcal{E}_{0i}(0,\sigma-\tau/2)\mathcal{B}_{0j}(0,\sigma+\tau/2)\rangle = 0, \quad i,j=1,2,3. \tag{69}$$

C. Correlation functions for the fields at a detector at rest in Planck's spectrum

Pursuing our concern with the thermal effects of acceleration, we wish to compare the correlation functions (68) and (69) with those at a detector at rest in an inertial frame in Planck's spectrum of thermal radiation. The expressions for the classical electric and magnetic fields in thermal radiation are precisely (48) and (49), where now the spectral function is changed from the Lorentz-invariant zero-point spectrum $\mathfrak{h}_0$ in (51) over to the Planck spectrum including zero-point radiation given by

$$\pi^2\mathfrak{h}_T{}^2(\omega)=\hbar\omega\left(\tfrac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\right)=\tfrac{1}{2}\hbar\omega\coth\left(\frac{\hbar\omega}{2kT}\right). \quad (70)$$

We proceed in a style analogous to that used above. We insert the expressions (48) and (49), average over the random phases, eliminate the δ functions, sum over the polarizations, and insert the spectral function (70) to obtain for $i,j=1,2,3$

$$\langle E_{Ti}(0,s-t/2)E_{Tj}(0,s+t/2)\rangle=\langle B_{Ti}(0,s-t/2)B_{Tj}(0,s+t/2)\rangle=\int\frac{d^3k}{4\pi^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\hbar\omega\coth\left(\frac{\hbar\omega}{2kT}\right)\cos\omega t, \quad (71)$$

$$\langle E_{Ti}(0,s-t/2)B_{Tj}(0,s+t/2)\rangle=\int\frac{d^3k}{4\pi^2}\epsilon_{ijl}\frac{k_l}{k}\hbar\omega\coth\left(\frac{\hbar\omega}{2kT}\right)\cos\omega t. \quad (72)$$

Since the spectral function is isotropic depending on the frequency ω alone, the angular integrations are easily carried out and give a vanishing value for (72) while (71) becomes

$$\langle E_{Ti}(0,s-t/2)E_{Tj}(0,s+t/2)\rangle=\delta_{ij}\tfrac{2}{3}\frac{\hbar}{\pi c^3}\int_0^\infty d\omega\,\omega^3\coth\left(\frac{\hbar\omega}{2kT}\right)\cos\omega t. \quad (73)$$

Here just as in Eq. (27), we must break up the integral into divergent and convergent parts,

$$\int_0^\infty d\omega\,\omega^3\coth(\hbar\omega/2kT)\cos\omega t=\int_0^\infty d\omega\,\omega^3\cos\omega t+\int_0^\infty d\omega\frac{2\omega^3}{\exp(\hbar\omega/kT)-1}\cos\omega t$$
$$=\frac{6}{t^4}+\left\{2\left(\frac{\pi kT}{\hbar}\right)^4\operatorname{csch}^2\left(\frac{\pi kTt}{\hbar}\right)\left[3\operatorname{csch}^2\left(\frac{\pi kTt}{\hbar}\right)+2\right]-\frac{6}{t^4}\right\}. \quad (74)$$

The divergent integral is of the form (61) while the convergent integral can be obtained from an integral table.[23] Thus we find

$$\langle E_{Ti}(0,s-t/2)E_{Tj}(0,s+t/2)\rangle=\langle B_{Ti}(0,s-t/2)B_{Tj}(0,s+t/2)\rangle=\delta_{ij}\frac{4\hbar}{\pi c^3}\left(\frac{\pi kT}{\hbar}\right)^4\left[\operatorname{csch}^4\left(\frac{\pi kTt}{\hbar}\right)+\tfrac{2}{3}\operatorname{csch}^2\left(\frac{\pi kTt}{\hbar}\right)\right], \quad (75)$$

$$\langle E_{Ti}(0,s-t/2)B_{Tj}(0,s+t/2)\rangle=0,\quad i,j=1,2,3. \quad (76)$$

D. Comparison of the correlation functions

If we compare the correlation functions (68) and (69) with (75) and (76), we conclude that a point detector accelerating through classical electromagnetic zero-point radiation indeeds finds a stationary fluctuation pattern as though it were located in an isotropic distribution of random electromagnetic radiation. However, it does not find Planck's spectrum for the random radiation. The expression (75) differs in functional form from (68) by the additional term

$$(4\hbar/\pi c^3)(\pi kT/\hbar)^4\tfrac{2}{3}\operatorname{csch}^2(\pi kTt/\hbar).$$

If we turn back to the correlation function (28) for the scalar field, we see that the functional form is very similar to this additional term. Indeed, comparing the integrals in (27) and (74) with the expressions (68) and (75), one finds immediately that the spectrum seen at the detector accelerating through zero-point radiation corresponds to

$$\pi^2\mathfrak{h}^2(\omega)=\tfrac{1}{2}\hbar[\omega+(a/c)^2/\omega]\coth(\pi c\omega/a). \quad (77)$$

If we write $T=\hbar a/(2\pi ck)$ as in Eq. (29), then the accelerating detector experiences a spectrum $\mathfrak{h}_A$.

$$\pi^2\mathfrak{h}_A{}^2(\omega)=\tfrac{1}{2}\hbar[\omega+(\pi 2kT/\hbar)^2/\omega]\coth(\hbar\omega/2kT), \quad (78)$$

rather than the Planck spectrum $\mathfrak{h}_T{}^2$ given in (70). The two spectra (70) and (78) clearly agree at high frequency where $\hbar\omega\gg kT$. However, at low frequency or high temperature (large acceleration), $kT\gg\hbar\omega$, we can expand $\coth x\sim 1/x+x/3$

$-x^3/45+\cdots$, and find the acceleration-related spectrum

$$\pi^2 \mathfrak{h}_A{}^2(\omega)\sim \tfrac{1}{2}\hbar\omega\left[\pi^2\left(\frac{2kT}{\hbar\omega}\right)^3+\left(\frac{\pi^2}{3}+1\right)\left(\frac{2kT}{\hbar\omega}\right)+O\left(\frac{\hbar\omega}{kT}\right)\right], \tag{79}$$

whereas Planck's thermal spectrum is

$$\pi^2 \mathfrak{h}_T{}^2(\omega)\sim kT+O\left(\frac{\hbar\omega}{kT}\right). \tag{80}$$

Thus the acceleration-related spectrum does not go over to energy equipartition at low frequency. We conclude that the perfect connection between acceleration through the zero-point radiation and the Planck spectrum which arose for a scalar field does not continue to the case of the vector electromagnetic field.

E. Connections with quantum electrodynamics

Now all previous work on the thermal effects of acceleration have been in terms of quantum theory, and one might expect to connect the disagreement found for the electromagnetic case with the use of a classical analysis. However, this will not do. The close connection between quantum free fields and random classical fields was pointed out previously. Just as we saw above for the scalar fields, there is precise agreement between the correlation functions of the random electromagnetic fields and the expectation values of the symmetrized products of quantum electromagnetic fields at any finite temperature. It has been shown[14] that

$$\langle E_{Ti}(\vec{r},t)E_{Tj}(\vec{r}\,',t')\rangle=\tfrac{1}{2}\langle|\{\underline{E}_i(\vec{r},t)\underline{E}_j(\vec{r}\,',t')\}|\rangle_T, \tag{81}$$

$$\langle B_{Ti}(\vec{r},t)B_{Tj}(\vec{r}\,',t')\rangle=\tfrac{1}{2}\langle|\{\underline{B}_i(\vec{r},t)\underline{B}_j(\vec{r}\,',t')\}|\rangle_T, \tag{82}$$

$$\langle E_{Ti}(\vec{r},t)B_{Tj}(\vec{r}\,',t')\rangle=\tfrac{1}{2}\langle|\{\underline{E}_i(\vec{r},t)\underline{B}_j(\vec{r}\,',t')\}|\rangle_T, \tag{83}$$

where $\underline{E}_i(\vec{r},t)$ and $\underline{B}_j(\vec{r},t)$, $i,j=1,2,3$, stand for the electric and magnetic fields in quantum field theory. Hence our more complex results for the electromagnetic field carry over directly into quantum theory.

Indeed, the spectrum (78) appears in related though far more complicated work by Candelas and Deutsch[27] involving the acceleration of an infinite plane barrier through the quantum vacuum. They term the spectrum (78) a "thermal" spectrum. Apparently what is involved is the following: The distribution of eigenvalues for the electromagnetic case is different from that for the scalar case, and this distribution depends upon properties of the reference frame. The frame of reference associated with an accelerating observer has strange properties which are not found in familiar Minkowskian space-time. For example, there is an event horizon in the sense that in certain directions, events occurring beyond a certain distance from the observer can never be reported to the observer by light signals. The observer is running away with ever-increasing speed from these space-time events, and light signals never catch up. It is the long-wavelength waves which are cut off by the event horizon and hence account for the failure of energy equipartition noted in Sec. IV D. A more sophisticated treatment of these intriguing questions has been given by Candelas and Dowker.[28]

V. SUMMARY

Astrophysics sometimes deals with conditions which are widely different from those encountered in laboratory physics. While exploring an astrophysical problem involving black holes, Davies[1] and Unruh[2] came upon the idea that a detector accelerating relative to the vacuum should react as though it were in a bath of thermal radiation. This result has been explored by several authors.

The previous work to date on the apparent thermal effects of acceleration has all been from the quantum point of view, discussing transitions between quantum energy levels or else discussing Feynman particle propagators. However, what is involved is no more than the free-field aspect of the quantum fields, and this aspect at least can be just as well seen in a classical free field with stochastic initial conditions. Thus as a way of broadening the perspective on the problem and sharpening our understanding of just what is crucial, we have here given the analysis from a purely classical point of view while showing how to immediately convert our expressions to the quantum theory.

For a scalar field we confirm the connection that an observer accelerating through the Lorentz-invariant zero-point radiation interprets the field fluctuations as those of isotropic thermal radiation corresponding to the Planck spectrum with zero-point radiation. For the vector electromagnetic field the situation is more complicated. The observer accelerating through classical electromagnetic zero-point radiation finds an isotropic stationary field of radiation which corresponds to thermal radiation in his non-Minkowskian space-time. He does not find the Planck spectrum.

Although all calculations reported here involve purely classical fields, the proven close connections between the free quantum theories and classical theories with random radiation make it possible to apply our results directly to quantum theory.

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[1]P. C. W. Davies, J. Phys. A 8, 609 (1975).

[2]W. G. Unruh, Phys. Rev. D 14, 870 (1975).

[3]G. B. Gibbons and M. J. Perry, Phys. Rev. Lett. 36, 985 (1976).

[4]P. Candelas and D. Deutsch, Proc. R. Soc. London A354, 79 (1977).

[5]P. Candelas and D. W. Sciama, Phys. Rev. Lett. 38, 1372 (1977).

[6]W. Troost and H. Van Dam, Phys. Lett. 71B, 149 (1977).

[7]S. M. Christensen and M. J. Duff, Nucl. Phys. B146, 11 (1978).

[8]J. S. Dowker, J. Phys. A 10, 115 (1977).

[9]W. Troost and H. Van Dam, Nucl. Phys. B152, 442 (1979).

[10]D. Sciama, in *Proceedings of the New York Einstein Centennial* (New York Academy of Sciences, New York, to be published). The use of correlation-function techniques along lines similar to those used in the present manuscript was proposed by Sciama in Ann. N. Y. Acad. Sci. 302, 161 (1977).

[11]T. W. Marshall, Proc. R. Soc. London A276, 475 (1963).

[12]T. W. Marshall, Proc. Cambridge Philos. Soc. 61, 537 (1965).

[13]T. H. Boyer, Phys. Rev. 182, 1374 (1969).

[14]T. H. Boyer, Phys. Rev. D 11, 809 (1975).

[15]T. H. Boyer, in *Foundations of Radiation Theory and Quantum Electrodynamics*, edited by A. O. Barut (Plenum, New York, to be published).

[16]T. H. Boyer, Phys. Rev. A 21, 66 (1980).

[17]H. B. G. Casimir, K. Ned. Akad. Wet. Proc. B 51, 793 (1948); J. Chim. Phys. 46, 407 (1949).

[18]L. L. Henry and T. W. Marshall, Nuovo Cimento 41, 188 (1966).

[19]T. H. Boyer, Phys. Rev. A 9, 2078 (1974), and earlier references given there.

[20]See, for example, J. D. Jackson, *Classical Electrodynamics*, 2nd edition (Wiley, New York, 1975), p. 563, problem 11.3.

[21]See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 13, Eq. (11.27). In the present article, unrationalized units are used corresponding to our choice of units later in the electromagnetic case. Thus both scalar and vector energy densities involve a factor of $(8\pi)^{-1}$ rather than the factor of $\frac{1}{2}$ appearing in rationalized units.

[22]See, for example, Ref. 20, p. 521, Eq. (11.29).

[23]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965), p. 494, No. 13:

$$\int_0^\infty dx\,\frac{x^{2m+1}\cos bx}{e^x-1} = (-1)^m\,\frac{\partial^{2m+1}}{\partial b^{2m+1}}\left(\frac{\pi}{2}\coth\pi b-\frac{1}{2b}\right),\quad b>0.$$

[24]In Ref. 14 the connection is given for the electromagnetic field.

[25]See Ref. 21, p. 27, Eq. (12.7). There is a change of a factor of $(4\pi)^{1/2}$ from the expression (34) of the present article because of the change to rationalized units referred to earlier.

[26]See Ref. 20, p. 552, Eq. (11.148).

[27]See Ref. 4, Eq. (34).

[28]P. Candelas and J. S. Dowker, Phys. Rev. D 19, 2902 (1979).