PHYSICAL REVIEW D VOLUME 21, NUMBER 8 15 APRIL 1980

Propagation equations for test bodies with spin and rotation in theories of gravity with torsion

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(Received 26 November 1979)

We generalize the Papapetrou equations by deriving propagation equations for the energy-momentum and angular momentum of a test body which has both elementary-particle *spin* and macroscopic *rotation* and which is moving in background *metric* and *torsion* fields. Our results show that the torsion couples to spin but not to rotation. Thus a rotating test body with no net spin will ignore the torsion and move according to the usual Papapetrou equations. Hence the standard tests of gravity are insensitive to a torsion field. We propose experiments (although still infeasible) to compare the motion of a spin-polarized body with the motion of a rotating body. If the spin and rotation precess differently, the theory of gravity cannot be a metric theory but may be a torsion theory.

I. INTRODUCTION

In two well-known investigations, Mathisson and Papapetrou[1] found the energy-momentum and angular momentum propagation equations for a rotating test body ("pole-dipole particle") according to Einstein's general relativity. Tulczyjew, Beiglböck, and Madore[2] developed these into laws of motion[3] by adding a definition of a center-of-mass world line. Later Dixon[4] generalized these treatments and made them more rigorous. The results in these papers are actually appropriate to any metric theory of gravity since they depend only on the conservation law, $\nabla_\beta T^{\alpha\beta}=0$, which is valid in any Riemannian spacetime.

Not long ago Trautman and Hehl[5] obtained equations very similar to those of Papapetrou for a particle with *intrinsic* spin in the Einstein-Cartan theory of gravity. Their results are actually valid in any metric and Cartan-connection theory of gravity (a Cartan connection is metric-compatible but may have torsion), since they depend only on the conservation laws (reviewed in Sec. II) appropriate to a Riemann-Cartan spacetime.

From a physical point of view, however, the two sets of propagation equations describe very different situations. Papapetrou's equations were developed to describe the behavior of a test body with internal *orbital* angular momentum about its center of mass (*rotation* or *macroscopic spin*). Trautman's equations describe the behavior of a particle with *microscopic intrinsic elementary-particle spin.*[6] These are two different physical situations involving a test body with angular momentum. They tend to be lumped together in the literature, principally because their laws of motion are the same in any metric theory (as demonstrated in Sec. VA). However, as soon as the torsion is turned on, microscopically and macroscopically spinning test bodies no longer behave in the same way. The torsion couples to the microscopic spin but not to the rotation.[7]

In this paper we further investigate and discuss this difference and its consequences for a broad range of Lagrangian-based, metric and Cartan-connection theories of gravity. The principal focus of our work (the theorem in Sec. IV) is the derivation of the energy-momentum and angular momentum propagation equations in these theories for a test body possessing both spin and rotation. Our treatment is analogous to Papapetrou's. We make no attempt to define a center of mass nor to reach Dixon's level of rigor[8] nor to include higher moments.

In our previous paper,[7] we discussed only the conservation laws obeyed by such a test body. On the basis of those, we were able to conclude that *if* the spin density $S^{\alpha\beta\gamma}$ vanishes, then the conservation laws reduce to the standard general-relativistic form ($\nabla_\beta T^{\alpha\beta}=0$), and so a rotating test body will *not* be affected by the torsion and will move according to the standard Papapetrou equations. However, although a real macroscopic test body usually has no net intrinsic spin, its spin density $S^{\alpha\beta\gamma}$ is always nonzero since it is composed of elementary particles. The more sophisticated treatment of this paper establishes the same result under the weaker assumption that only the integrated spin vanishes.

Our results are of more than theoretical interest. They have a direct bearing on observational questions—specifying which types of experiments might be used to detect a torsion field, and indicating which types have no hope of doing so. For example, our results show that for a broad range of Lagrangian-based classical field theories, a torsion field is detectable only by a test body which possesses a net elementary-particle spin.

Therefore, the Stanford Schiff gyroscope[9] will not feel torsion (since its angular momentum is predominantly rotational) even though it is susceptible to the magnetic, g_{0k}, components of the metric.[10] Furthermore, at the observational level our propagation equations raise some interesting issues about the relationship between the measured energy, momentum, spin, and rotation and the corresponding mathematically defined quantities. (See Sec. VD.)

Since the primary motivation for our theoretical work is observational, it is appropriate to set the experimental context more explicitly. Any proposed theory of gravity must be tested against the classical experiments: gravitational red-shift, perihelion precession of Mercury, and light-ray deflection. These are now supplemented by a number of more recently devised standard tests, such as laser ranging to the moon and the planets, radar echo delay, and gravimeter and tidal measurements. Hopefully, in the near future, these observational tests will be reinforced by the detection and analysis of gravitational waves, by a gyroscope precession experiment, and by increased amounts of data from the binary pulsar. Except for gravity waves and the binary pulsar, the study of the agreement between these experiments and the standard metric theories has been organized into a single framework—the parametrized post-Newtonian (PPN) formalism.[11]

However, outside the realm of metric theories are a large class of metric and Cartan-connection (or metric and torsion) theories of gravity,[12-15] whose torsion is generally immune to the scrutiny of the well-understood and tightly formulated PPN system. As of now these theories are not susceptible to confirmation or falsification with regard to the very characteristic which distinguishes them: their nonzero torsion. How may their viability be tested—or rigorously ruled out—in reasonable experiments? Are there any theoretical problems or ambiguities in such experiments? What we have begun here asks these questions and hazards some preliminary answers.

An example of a particular observational criteria which emerges from our work (Sec. VA) is that, *if* spin and rotation precess differently, then the theory of gravity cannot be a metric theory; in which case it may be a torsion theory. We propose experiments (although still impractical) to compare the precession and propagation of two bodies, one with microscopic spin, the other with rotation (see Sec. VC). If the angular momenta are equal, then the torsion theories predict that the ratio of the spin precession to the rotation precession is essentially the ratio of the Cartan connection $\Gamma^{\alpha}{}_{\beta\gamma}$ to the Christoffel connection, $\{^{\alpha}{}_{\beta\gamma}\}$ (the unique metric-compatible torsion-free connection). There exist theories[16] which predict that the solar system has a $\Gamma^{\alpha}{}_{\beta\gamma}$ which is approximately zero while its $\{^{\alpha}{}_{\beta\gamma}\}$ is approximately the Christoffel connection of the Schwarzschild metric. Thus these theories predict that the rotation should precess much more than the spin.

Finally, it is important to recognize that our results are of practical observational significance only for theories in which the torsion field is dynamic (propagating or existing in the vacuum). The Einstein-Cartan theory[13] and its close relatives[14] have nondynamic torsion, in that the torsion is an algebraic function of the spin density which vanishes in the vacuum. In this paper we study the motion of test bodies in *background* metric and torsion fields. To have a background torsion field in the Einstein-Cartan theory, there must also be a background spin density. In that case, the conservation laws for the test body are more complicated than those considered here, because they must allow for interactions between the spin of the test body and the spin of the background fields. Further, any experiment to detect torsion would also require a spin-polarized background field. Fortunately, however, there are an ever-growing number of dynamic torsion theories[15] to which our conclusions are directly relevant.

We begin, in Sec. II, by giving some formalism and relationships important in discussing metric and Cartan-connection theories of gravity, and then outline the derivation of their conservation laws. Section III introduces the generalized moment description of test bodies. The derivation of the energy-momentum and angular momentum propagation equations for a test body follows in Sec. IV. Included in Sec. V is a treatment of special cases, a discussion of the nondeterministic character of the propagation equations, and an investigation into questions concerning the ambiguity of measurement.

II. TORSION THEORIES AND THEIR CONSERVATION LAWS

Here we give some fundamental relationships important in discussing torsion theories, outline the derivation of the conservation laws, and briefly mention some of their consequences. In any metric and connection theory, there are two connections: the Christoffel connection $\{^{\alpha}{}_{\beta\gamma}\}$ and the full connection $\Gamma^{\alpha}{}_{\beta\gamma}$. Hence, there are also two curvatures: the Christoffel curvature $\tilde{R}^{\alpha}{}_{\beta\gamma\delta}$, denoted with a tilde, and the full curvature $\hat{R}^{\alpha}{}_{\beta\delta\gamma}$ denoted with a caret. It is also necessary to distinguish between the two types of covariant derivatives. We find it convenient to use "mixed" covariant derivatives, such as

$$\nabla_\delta S^{\hat{\alpha}}{}_{\hat{\beta}\tilde{\gamma}} = \partial_\delta S^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\epsilon\delta} S^\epsilon{}_{\beta\gamma} - \Gamma^\epsilon{}_{\beta\delta} S^\alpha{}_{\epsilon\gamma} - \left\{{\epsilon \atop \gamma\delta}\right\} S^\alpha{}_{\beta\epsilon}\,, \tag{1}$$

where some indices (those with a caret) are corrected with the full connection while the others (those with a tilde) are corrected with the Christoffel connection.

In a coordinate basis, the Christoffel connection may be expressed in terms of the metric by the usual formula,

$$\left\{{\alpha \atop \beta\gamma}\right\} = \tfrac{1}{2} g^{\alpha\delta}(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma})\,. \tag{2}$$

The full connection differs from this by the defect tensor,

$$\lambda^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \left\{{\alpha \atop \beta\gamma}\right\}. \tag{3}$$

In a coordinate basis, the torsion $Q^\alpha{}_{\beta\gamma}$ is defined as the antisymmetric part of $\Gamma^\alpha{}_{\beta\gamma}$ or equivalently of $\lambda^\alpha{}_{\beta\gamma}$:

$$Q^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\alpha{}_{\beta\gamma} = \lambda^\alpha{}_{\gamma\beta} - \lambda^\alpha{}_{\beta\gamma}\,. \tag{4}$$

If $\Gamma^\alpha{}_{\beta\gamma}$ is metric compatible (which we assume here), it is called a Cartan connection. In that case the defect is related to the torsion by

$$\lambda^\alpha{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta}(Q_{\beta\delta\gamma} + Q_{\gamma\delta\beta} - Q_{\delta\beta\gamma})\,. \tag{5}$$

The derivation of the conservation laws requires the language of orthonormal frames. In what follows, $\theta^{(\mu)}{}_\alpha$ will denote the components of an orthonormal one-form frame field, $\theta^{(\mu)} = \theta^{(\mu)}{}_\alpha dx^\alpha$, and $e_{(\mu)}{}^\alpha$ the components of its dual vector frame field, $e_{(\mu)} = e_{(\mu)}{}^\alpha \partial_\alpha$. Here frame indices are enclosed in parentheses; coordinate indices are not. Lower-case Greek indices are four-dimensional ($\alpha = 0,1,2,3$); lower-case Latin indices are three-dimensional ($a = 1,2,3$). Upper-case Latin indices denote any general collection of indices.

In any theory of gravity, the conservation laws constitute the basis from which the propagation equations are derived. In turn the conservation laws are derived via Noether's theorem from the fact that the matter Lagrangian L_M is a scalar under coordinate transformations and, when appropriate, also under changes of frame. We here outline the derivation of the conservation laws using two different formalisms (orthonormal frames and coordinate-based tensors) and discuss some of their important consequences.

First using orthonormal frames, the matter Lagrangian,[17]

$$L_M = L_M(\eta_{(\mu)(\nu)}, \theta^{(\mu)}{}_\alpha, \Gamma^{(\mu)}{}_{(\nu)\alpha}, \psi^{(X)}, \partial_\alpha \psi^{(X)})\,, \tag{6}$$

may be regarded as a function of the Minkowski metric $\eta_{(\mu)(\nu)} = \mathrm{diag}(-1,1,1,1)$, the orthonormal-frame components $\theta^{(\mu)}{}_\alpha$, the Cartan-connection coefficients $\Gamma^{(\mu)}{}_{(\nu)\alpha}$, the orthonormal components of the source fields, collectively denoted $\psi^{(X)}$, and their partial derivatives $\partial_\alpha \psi^{(X)}$. We are here neglecting any gauge fields (the photon, weak bosons, and gluons) which must be treated separately (see Sec. V D) from the source fields, since the standard Lagrangian does not have the form (6). The Lagrangian density is then $\mathfrak{L}_M = \theta L_M$ where θ is the determinant of $\theta^{(\mu)}{}_\alpha$.

The (asymmetric) canonical energy-momentum tensor $t_\gamma{}^\alpha$ and its density $\mathrm{t}_\gamma{}^\alpha$, are defined by

$$\theta t_\gamma{}^\alpha = \mathrm{t}_\gamma{}^\alpha = \theta^{(\mu)}{}_\gamma \frac{\delta \mathfrak{L}_M}{\delta \theta^{(\mu)}{}_\alpha}\,, \tag{7}$$

while the canonical spin tensor $S^\gamma{}_\delta{}^\alpha$ and its density $\mathfrak{S}^\gamma{}_\delta{}^\alpha$ are defined by

$$\tfrac{1}{2}\theta S^\gamma{}_\delta{}^\alpha = \tfrac{1}{2}\mathfrak{S}^\gamma{}_\delta{}^\alpha = \theta^{(\mu)}{}_\delta e_{(\nu)}{}^\gamma \frac{\delta \mathfrak{L}_M}{\delta \Gamma^{(\mu)}{}_{(\nu)\alpha}}\,. \tag{8}$$

Applying Noether's theorem to the invariance of the Lagrangian under Lorentz transformations of the orthonormal frames, one obtains the conservation law for angular momentum

$$\nabla_\alpha S^{\hat{\gamma}\hat{\delta}\tilde{\alpha}} = t^{\gamma\delta} - t^{\delta\gamma}\,, \tag{9}$$

and also identifies $S^\gamma{}_\delta{}^\alpha$ with the canonical spin tensor as defined by particle physicists,

$$S_{\gamma\delta}{}^\alpha = \frac{\partial L_M}{\partial \partial_\alpha \psi^{(X)}} R_\psi(\sigma_{\gamma\delta})^{(X)}{}_{(Y)} \psi^{(Y)}\,, \tag{10}$$

where R_ψ is the representation of the Lorentz Lie algebra appropriate to $\psi^{(X)}$. Thus $S^\gamma{}_\delta{}^\alpha$ includes the spin of the quarks and leptons but not that of the gauge fields. Similarly, using the fact that the Lagrangian is a scalar under coordinate transformations, one obtains the conservation law for energy-momentum

$$\nabla_\alpha t_{\tilde{\gamma}}{}^{\tilde{\alpha}} = \tfrac{1}{2} S^\beta{}_\alpha{}^\delta \hat{R}^\alpha{}_{\beta\gamma\delta} + t_\alpha{}^\delta Q^\alpha{}_{\gamma\delta}\,, \tag{11}$$

and identifies $t_\gamma{}^\alpha$ as the canonical energy-momentum tensor as defined by particle physicists,

$$t_\gamma{}^\alpha = L_M \delta^\alpha_\gamma - \frac{\partial L_M}{\partial \partial_\alpha \psi^{(X)}} \nabla_\gamma \psi^{(X)}\,. \tag{12}$$

On the other hand, from a more conventional approach using coordinate bases, the matter Lagrangian[17]

$$L_M = L_M(g_{\alpha\beta}, \partial_\gamma g_{\alpha\beta}, \lambda^\alpha{}_{\beta\gamma}, \psi^X, \partial_\alpha \psi^X) \tag{13}$$

may be regarded as a function of the coordinate components of the metric $g_{\alpha\beta}$, their partial derivatives $\partial_\gamma g_{\alpha\beta}$, the coordinate components of the defect tensor $\lambda^\alpha{}_{\beta\gamma}$, the coordinate components of the source fields ψ^X, and their partial derivatives $\partial_\alpha \psi^X$. The Lagrangian density is now $\mathfrak{L}_M = \sqrt{-g}\, L_M$, where g is the determinant of $g_{\alpha\beta}$.

The (symmetric) metric energy-momentum tensor $T^{\alpha\beta}$, and its density $\mathcal{T}^{\alpha\beta}$, are defined by

$$\tfrac{1}{2}\sqrt{-g}\,T^{\alpha\beta}=\tfrac{1}{2}\mathfrak{T}^{\alpha\beta}=\frac{\delta\mathfrak{L}_M}{\delta g_{\alpha\beta}}\ , \tag{14}$$

while the canonical spin tensor $S^{\gamma}{}_{\delta}{}^{\alpha}$ is now defined by

$$\tfrac{1}{2}\sqrt{-g}\,S^{\gamma}{}_{\delta}{}^{\alpha}=\frac{\delta\mathfrak{L}_M}{\delta\lambda^{\delta}{}_{\gamma\alpha}}\ . \tag{15}$$

Using the chain rule and the relations

$$g_{\alpha\beta}=\eta_{(\mu)(\nu)}\,\theta^{(\mu)}{}_{\alpha}\,\theta^{(\nu)}{}_{\beta}\,, \tag{16}$$

$$\begin{aligned}\lambda^{\delta}{}_{\gamma\alpha}=e_{(\mu)}{}^{\delta}\,\theta^{(\nu)}{}_{\gamma}\,\Gamma^{(\mu)}{}_{(\nu)\alpha}&\\ +g^{\delta\beta}e_{(\mu)}{}^{\epsilon}(g_{\epsilon\gamma}\,\partial_{[\beta}\theta^{(\mu)}{}_{\alpha]}+g_{\epsilon\alpha}\partial_{[\beta}\theta^{(\mu)}{}_{\gamma]}&\\ -g_{\epsilon\beta}\partial_{[\gamma}\theta^{(\mu)}{}_{\alpha]})\,,&\end{aligned} \tag{17}$$

one finds that $S^{\gamma}{}_{\delta}{}^{\alpha}$ defined in (15) coincides with $S^{\gamma}{}_{\delta}{}^{\alpha}$ defined in (8) and also finds that $T^{\alpha\beta}$ may be expressed in terms of $t_{\gamma}{}^{\alpha}$ and $S^{\gamma}{}_{\delta}{}^{\alpha}$ by a generalization of the Belinfante-Rosenfeld symmetrization rule,[18]

$$T^{\alpha\beta}=t^{\alpha\beta}-\tfrac{1}{2}\nabla_{\gamma}(S^{\tilde{\gamma}\tilde{\beta}\tilde{\alpha}}+S^{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}})-\tfrac{1}{2}\nabla_{\gamma}S^{\hat{\alpha}\hat{\beta}\tilde{\gamma}}\,. \tag{18}$$

Again using the scalar nature of L_M under coordinate transformations, one obtains an alternate version of the energy-momentum conservation law,

$$\begin{aligned}\nabla_{\beta}T^{\tilde{\alpha}\tilde{\beta}}=\nabla_{\beta}(\lambda^{[\tilde{\alpha}}{}_{\gamma\delta}\,S^{\tilde{\beta}]\gamma\delta}-\tfrac{1}{2}\,\lambda^{\gamma}{}_{\delta}{}^{\alpha}\,S^{\delta}{}_{\gamma}{}^{\tilde{\beta}})&\\ +\tfrac{1}{2}\,S^{\delta}{}_{\gamma}{}^{\beta}\nabla^{\alpha}\lambda^{\tilde{\gamma}}{}_{\tilde{\delta}\tilde{\beta}}\,.&\end{aligned} \tag{19}$$

This equation may also be derived by differentiating (18) and using (9) and (11). On the other hand since the Lagrangian in the form (13) has no dependence on the frames there is no apparent way to derive a conservation law of angular momentum. Consequently, it appears that Eqs. (9) and (11) together contain more information than just (19). However, in deriving propagation laws for energy-momentum and angular momentum, we found that equivalent equations could be derived either using (19) alone or using both (9) and (11). In the following sections we concentrate on Eqs. (9) and (11) rather than (19) since the former clarifies the distinction between spin and rotation.

One fact is immediately obvious from the conservation laws: If a body has no spin density ($S^{\gamma}{}_{\delta}{}^{\alpha}=0$), then it will not feel the torsion and will propagate its energy-momentum and angular momentum according to the Mathisson-Papapetrou equations of the metric theories. On the one hand setting $S^{\gamma}{}_{\delta}{}^{\alpha}=0$ in (19) yields $\nabla_{\beta}T^{\tilde{\alpha}\tilde{\beta}}=0$. On the other hand setting $S^{\gamma}{}_{\delta}{}^{\alpha}=0$ in (9) and (11) says that $t^{\alpha\beta}$ is symmetric and satisfies $\nabla_{\beta}t^{\tilde{\alpha}\tilde{\beta}}=0$. There is no torsion in these equations. Either equation is the conservation law in a metric theory which is known to lead to the Mathisson-Papapetrou equations. This observation was the subject of our previous paper.[7] However, it is not really sufficiently strong to conclude that a macroscopic gyroscope will not feel a torsion field. Although a real macroscopic gyroscope usually has no net spin, its spin density $S^{\gamma}{}_{\delta}{}^{\alpha}$ is always nonzero since the gyroscope is composed of elementary particles. In the subsequent sections we show that if the elementary-particle spin merely integrates to zero, then the body will not feel the torsion and will propagate according to the Mathisson-Papapetrou equations.

III. GENERAL TEST BODY FORMALISM

Our procedure is essentially that of Papapetrou[1]: integrating the conservation laws and the moment equations based on them over the test body's three-volume to obtain a series of equations which can be marshalled into propagation equations for the energy-momentum and angular momentum.

In describing the test body, we make several reasonable assumptions.

(i) The body is described by a collection of current tensors $J_A{}^{\alpha}$. In our investigation, specifically, these are the body's spin density $S^{\gamma}{}_{\delta}{}^{\alpha}$, and either its canonical energy-momentum tensor $t_{\gamma}{}^{\alpha}$, or its metric energy-momentum tensor $T^{\gamma\alpha}$. In more general systems, they would also include quantities such as the body's electric current. From the current tensors we define the current tensor densities,

$$\mathcal{J}_A{}^{\alpha}=\sqrt{-g}\,J_A{}^{\alpha}=\theta\,J_A{}^{\alpha}\,, \tag{20}$$

and make our second assumption.

(ii) The currents satisfy differential conservations laws which may be written as

$$\partial_{\alpha}\mathcal{J}_A{}^{\alpha}=\mathfrak{F}_A\,, \tag{21}$$

where the $\mathfrak{F}_A$ are generalized forces. Next, we define the world tube of the test body as the support of the currents,

$$\operatorname{supp}\mathcal{J}=\operatorname{cl}\{p\in M:\mathcal{J}_A{}^{\alpha}(p)\neq 0\}\,, \tag{22}$$

(we abbreviate topological closure and interior by "cl" and "int") and make our third assumption.

(iii) There exists a closed set W and a coordinate system x^{α} for which the following apply.

(a) The world tube of the test body is contained in the interior of W, i.e.,

$$\operatorname{supp}\mathcal{J}\subset\operatorname{int}W\,. \tag{23}$$

(b) The interior of W is dense in W, i.e.,

$$W=\operatorname{cl}\operatorname{int}W\,. \tag{24}$$

(c) W is the union of timelike curves.

(d) The coordinate system x^{α} is defined on all of W.

(e) The coordinate basis ∂_{α} is both oriented and time oriented.

(f) The x^0 axis is entirely contained within int$W$.

(g) The x^0 coordinate is an affine parameter along the x^0 axis.

(h) Each level surface of x^0 within W is compact.

In what follows we take W and x^α to be fixed but arbitrary and satisfying assumption (iii).

Assumptions (iii e, f, g) say that the x^0 axis is a future-directed, timelike curve, $X(t)$, which has affine parameter, $t=x^0 \in R$ and $X(t)\in \text{int}W$. The coordinates of $X(t)$ are

$$X^\alpha(t)=x^\alpha(X(t))=t\,\delta^\alpha_0\,, \tag{25}$$

and so its velocity vector is

$$v(t)=\frac{d}{dt}=\partial_0\big|_{X(t)}=v^\alpha\,\partial_\alpha\,, \tag{26}$$

where

$$v^\alpha=\partial_0 X^\alpha=\delta^\alpha_0\,. \tag{27}$$

Since t is an affine parameter, the velocity v is a unit timelike vector, i.e.

$$v^\alpha v_\alpha=-1\,, \tag{28}$$

where the signature of the metric is $(-1,1,1,1)$.

Similarly assumptions (iii e, h) say that each x^0-level surface,

$$\Sigma(t)=\{p\in W{:}x^0(p)=t\}\,, \tag{29}$$

is spacelike and compact, while assumption (iii a) specifies that

$$\mathcal{J}_A{}^\alpha=0 \quad\text{on}\quad \partial\Sigma(t)\,. \tag{30}$$

Further, from definitions (25) and (29), we have that for each t

$$X(t)\in\Sigma(t)\,. \tag{31}$$

In our derivation of the propagation equations, we will want to expand various quantities within $\Sigma(t)$ about $X(t)$. So we introduce a new coordinate system on each surface, $\Sigma(t)$, centered at $X(t)$. At $p\in\Sigma(t)$, we define

$$\delta x^\alpha(p)=x^\alpha(p)-X^\alpha(t) \tag{32}$$

or

$$\delta x^0(p)=0,\quad \delta x^a(p)=x^a(p)\,. \tag{33}$$

Our last assumption, then, follows.

(iv) On each surface $\Sigma(t)$ the coordinate components of the metric $g_{\alpha\beta}$ and the defect $\lambda^\alpha{}_{\beta\gamma}$ may be expanded about $X(t)$ as power series in δx^α.

Hence, the coordinate components of the connections $\{{}^{\alpha}_{\beta\gamma}\}$ and $\Gamma^\alpha{}_{\beta\gamma}$, the curvatures $\tilde{R}^\alpha{}_{\beta\gamma\delta}$ and $\hat{R}^\alpha{}_{\beta\gamma\delta}$, and the torsion $Q^\alpha{}_{\beta\gamma}$ may also be expanded.

Finally, we shall often integrate various quantities over $\Sigma(t)$. So we abbreviate,

$$\int f=\int_{\Sigma(t)} f(x)\,d^3x\,. \tag{34}$$

Such integrals will be regarded as functions of t defined along $X(t)$.

In particular, we introduce the nth integrated moments of the current, which are defined as

$$\begin{aligned}K_A{}^{\beta_1\cdots\beta_n\alpha}&=\int \delta x^{\beta_1}\cdots\delta x^{\beta_n}\,\mathcal{J}_A{}^\alpha\\ &=\int\left(\prod_{j=1}^{n}\delta x^{\beta_j}\right)\mathcal{J}_A{}^\alpha\,,\end{aligned} \tag{35}$$

for $n\geq 0$. These are symmetric in the indices $\beta_1\cdots\beta_n$, and zero if any of these indices are zero. Furthermore, they are to be regarded as tensors defined only along the curve $X(t)$, and specified in the coordinate basis ∂_α. However, it is important to notice that a different set of tensors is defined for each choice of the coordinates x^α.

Certain moments have additional names. The integrated charges are

$$Q_A=K_A{}^0=\int\mathcal{J}_A{}^0\,, \tag{36}$$

and the nth integrated moments of the charge are

$$Q_A{}^{\beta_1\cdots\beta_n}=K_A{}^{\beta_1\cdots\beta_n 0}=\int\left(\prod_{j=1}^{n}\delta x^{\beta_j}\right)\mathcal{J}_A{}^0\,. \tag{37}$$

In deriving the propagation equations, we are essentially finding out how the integrated moments change along $X(t)$. To do this, we integrate over $\Sigma(t)$ all possible moments of the conservation laws (21) and then attempt to solve the resulting equations for $(d/dt)Q_A{}^{\beta_1\cdots\beta_n}$, eliminating or solving for the other (spatial) components of $K_A{}^{\beta_1\cdots\beta_n\alpha}$. The crucial equation follows from the identity,

$$\begin{aligned}\frac{d}{dt}\int\left(\prod_{j=1}^{n}\delta x^{\beta_j}\right)\mathcal{J}_A{}^0&=\sum_{i=1}^{n}\rho^{\beta_i}{}_\alpha\int\left(\prod_{\substack{j=1\\ \neq i}}^{n}\delta x^{\beta_j}\right)\mathcal{J}_A{}^\alpha\\ &\quad+\int\left(\prod_{j=1}^{n}\delta x^{\beta_j}\right)\partial_\alpha\mathcal{J}_A{}^\alpha\,,\end{aligned} \tag{38}$$

which holds for any integer $n\geq 0$. It is proved by integrating by parts in the last integral on the right, dropping a spatial divergence [using Eq. (30)], and using the relation

$$\partial_\alpha\left(\prod_{j=1}^{n}\delta x^{\beta_j}\right)=\sum_{i=1}^{n}\rho^{\beta_i}{}_\alpha\left(\prod_{\substack{j=1\\ \neq i}}^{n}\delta x^{\beta_j}\right). \tag{39}$$

Here, $\rho^\beta{}_\alpha$ is a spatial coordinate projection operator defined by any of the formulas

$$\begin{aligned}\rho^\beta{}_\alpha&=\partial_\alpha\delta x^\beta=\partial_\alpha(x^\beta-X^\beta)=\delta^\beta_\alpha-v^\beta\delta^0_\alpha\\ &=\delta^\beta_\alpha-\delta^\beta_0\delta^0_\alpha=\delta^\beta_a\delta^a_\alpha\,,\end{aligned} \tag{40}$$

where we have used Eq. (27) twice.

Using definitions (35) and (37) and the conservation laws (21), the identity (38) becomes

$$\frac{d}{dt}Q_A{}^{\beta_1\cdots\beta_n}=\sum_{i=1}^{n}\rho^{\beta_i}{}_{\alpha}K_A{}^{\beta_1\cdots\not\beta_i\cdots\beta_n\alpha}+\int\left(\prod_{j=1}^{n}\delta x^{\beta_j}\right)\mathfrak{F}_A. \quad (41)$$

The slash through the index β_i indicates that it is deleted from the list.

For a given theory with a specific set of conservation laws, the procedure for deriving the propagation equations is (a) to assume that only a finite number of the integrated moments are nonnegligible, (b) to expand the generalized forces $\mathfrak{F}_A$ in power series about $X(t)$, (c) to substitute these expansions into Eq. (41), and (d) to solve the resulting equations for $(d/dt)Q_A{}^{\beta_1\cdots\beta_n}$ and $\rho^{\alpha}{}_{\gamma}K_A{}^{\beta_1\cdots\beta_n\gamma}$ in terms of $Q_A{}^{\beta_1\cdots\beta_n}$ and the background geometry.

IV. PROPAGATION EQUATIONS

Using the conservation laws (9) and (11) presented in Sec. II, and the general formalism of Sec. III, we now derive the propagation equations for energy-momentum and angular momentum. Afterwards we recognize that these are equivalent to the corresponding propagation equations derived from the single conservation law (19).

In order to apply Eq. (41) to the conservation laws (9) and (11), we first rewrite the conservation laws in terms of the densities $\mathfrak{t}^{\gamma\delta}$ and $\mathfrak{S}^{\gamma\delta\alpha}$, defined in (7) and (8):

$$\partial_\delta \mathfrak{t}^{\gamma\delta}=(\lambda_{\alpha\delta}{}^{\gamma}-\{^{\gamma}{}_{\alpha\delta}\})\mathfrak{t}^{\alpha\delta}+\tfrac{1}{2}\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}\mathfrak{S}^{\beta\alpha\delta}, \quad (42)$$

$$\partial_\alpha \mathfrak{S}^{\gamma\delta\alpha}=2\mathfrak{t}^{[\gamma\delta]}+2\Gamma^{[\gamma}{}_{\beta\alpha}\mathfrak{S}^{\delta]\beta\alpha}. \quad (43)$$

The generalized forces corresponding to $\mathfrak{F}_A$ in (21) are the force on the right-hand side of (42) and the torque on the right-hand side of (43).

We next define the nth integrated moments of $\mathfrak{t}^{\gamma\delta}$,

$$M^{\beta_1\cdots\beta_n\gamma\delta}=\int\delta x^{\beta_1}\cdots\delta x^{\beta_n}\mathfrak{t}^{\gamma\delta}, \quad (44)$$

and the nth integrated moments of $\mathfrak{S}^{\gamma\delta\alpha}$,

$$N^{\beta_1\cdots\beta_n\gamma\delta\alpha}=\int\delta x^{\beta_1}\cdots\delta x^{\beta_n}\mathfrak{S}^{\gamma\delta\alpha}. \quad (45)$$

Notice that $N^{\beta_1\cdots\beta_n\gamma\delta\alpha}$, like $\mathfrak{S}^{\gamma\delta\alpha}$, is antisymmetric in γ and δ. Certain moments have specific names. The integrated canonical energy-momentum is

$$P^{\gamma}=M^{\gamma 0}=\int\mathfrak{t}^{\gamma 0}. \quad (46)$$

The integrated orbital (rotational) angular momentum is

$$L^{\gamma\delta}=M^{\gamma\delta 0}-M^{\delta\gamma 0}=2\int\delta x^{[\gamma}\mathfrak{t}^{\delta]0}, \quad (47)$$

and the integrated spin angular momentum is

$$S^{\gamma\delta}=N^{\gamma\delta 0}=\int\mathfrak{S}^{\gamma\delta 0}. \quad (48)$$

The indices on these tensors are lowered using the coordinate components of the metric at $X(t)$, e.g.,

$$P_\delta=g_{\delta\gamma}|_{X(t)}P^{\gamma}. \quad (49)$$

Since $g_{\delta\gamma}$ is not constant on $\Sigma(t)$, this yields a different result than computing

$$P'_\delta=\int\mathfrak{t}_\delta{}^{0}=\int g_{\delta\gamma}\mathfrak{t}^{\gamma 0}, \quad (50)$$

which could be used as the definition of the integrated energy-momentum instead of (46). Madore pointed out, as we shall emphasize below, that this ambiguity in the definition of energy-momentum leads to ambiguities in the interpretation of the propagation equations themselves.

Now applying Eq. (41) to the conservation laws (42) and (43), we obtain

$$\frac{d}{dt}M^{\beta_1\cdots\beta_n\gamma 0}=\sum_{i=1}^{n}(M^{\beta_1\cdots\not\beta_i\cdots\beta_n\gamma\beta_i}-v^{\beta_i}M^{\beta_1\cdots\not\beta_i\cdots\beta_n\gamma 0})+\int\delta x^{\beta_1}\cdots\delta x^{\beta_n}[(\lambda_{\alpha\delta}{}^{\gamma}-\{^{\gamma}{}_{\alpha\delta}\})\mathfrak{t}^{\alpha\delta}+\tfrac{1}{2}\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}\mathfrak{S}^{\beta\alpha\delta}], \quad (51)$$

$$\frac{d}{dt}N^{\beta_1\cdots\beta_n\gamma\delta 0}=\sum_{i=1}^{n}(N^{\beta_1\cdots\not\beta_i\cdots\beta_n\gamma\delta\beta_i}-v^{\beta_i}N^{\beta_1\cdots\not\beta_i\cdots\beta_n\gamma\delta 0})+\int\delta x^{\beta_1}\cdots\delta x^{\beta_n}(2\mathfrak{t}^{[\gamma\delta]}+2\Gamma^{[\gamma}{}_{\beta\alpha}\mathfrak{S}^{\delta]\beta\alpha}). \quad (52)$$

We use these to find the propagation equations presented in the following theorem, which is our principal result.

Theorem. In a metric and Cartan-connection theory of gravity, if the integrated moments, $M^{\beta_1\cdots\beta_n\gamma\delta}$ with $n\geq 2$ and $N^{\beta_1\cdots\beta_n\gamma\delta\alpha}$ with $n\geq 1$, are negligible compared to $M^{\gamma\delta}$, $M^{\beta\gamma\delta}$, and $N^{\gamma\delta\alpha}$, then

$$\nabla_v\mathcal{P}^{\tilde{\gamma}}=\tfrac{1}{2}L^{\beta\alpha}\tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}v^{\delta}+\tfrac{1}{2}S^{\beta\alpha}\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}v^{\delta}+\tfrac{1}{2}\lambda_{\alpha\beta}{}^{\gamma}\nabla_v S^{\hat{\alpha}\hat{\beta}}+\tfrac{1}{2}\rho^{\delta}{}_{\nu}N^{\beta\alpha\nu}\nabla^{\epsilon}\lambda_{\tilde{\alpha}\tilde{\beta}\tilde{\delta}}, \quad (53)$$

$$\nabla_v L^{\tilde{\gamma}\tilde{\delta}}+\nabla_v S^{\hat{\gamma}\hat{\delta}}=-2v^{[\gamma}\mathcal{P}^{\delta]}+\lambda_{\alpha\beta}{}^{[\gamma}\rho^{\delta]}{}_{\nu}N^{\alpha\beta\nu}+2\lambda^{[\gamma}{}_{\beta\alpha}N^{\delta]\beta\nu}\rho^{\alpha}{}_{\nu}, \quad (54)$$

$$M^{\gamma\delta}=v^{\delta}P^{\gamma}+\frac{d}{dt}\left(\tfrac{1}{2}L^{\delta\gamma}+v^{(\delta}L^{\gamma)0}-\rho^{(\delta}{}_{\nu}N^{\gamma)0\nu}\right)$$
$$+\tfrac{1}{2}\lambda_{\alpha\beta}{}^{\gamma}\rho^{\delta}{}_{\nu}N^{\alpha\beta\nu}-\{{}^{\gamma}{}_{\alpha\beta}\}\rho^{\delta}{}_{\mu}(v^{\alpha}L^{\beta\mu}-\rho^{\alpha}{}_{\nu}N^{\beta\mu\nu})\,, \tag{55}$$

$$M^{\beta\gamma\delta}=-\rho^{\beta}{}_{\mu}\left(\tfrac{1}{2}N^{\gamma\delta\mu}+v^{(\gamma}L^{\delta)\mu}-\rho^{(\gamma}{}_{\nu}N^{\delta)\mu\nu}\right), \tag{56}$$

$$N^{\gamma\delta\alpha}=v^{\alpha}S^{\gamma\delta}+\rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}\,, \tag{57}$$

where

$$\mathcal{P}^{\gamma}=P^{\gamma}+\{{}^{\gamma}{}_{\alpha\beta}\}(v^{\alpha}L^{\beta 0}-\rho^{\alpha}{}_{\nu}N^{\beta 0\nu})\,. \tag{58}$$

Proof. First notice that Eq. (57) follows immediately from the definitions (48) and (40) of $S^{\gamma\delta}$ and $\rho^{\alpha}{}_{\nu}$. Next expand $(\lambda_{\alpha\delta}{}^{\gamma}-\{{}^{\gamma}{}_{\alpha\delta}\})$, $\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}$, and $\Gamma^{\gamma}{}_{\beta\alpha}$ about $X(t)$ as power series in δx^{μ}:

$$x^{\mu}=X^{\mu}+\delta x^{\mu}\,, \tag{59}$$

$$(\lambda_{\alpha\delta}{}^{\gamma}-\{{}^{\gamma}{}_{\alpha\delta}\})|_{x}=(\lambda_{\alpha\delta}{}^{\gamma}-\{{}^{\gamma}{}_{\alpha\delta}\})|_{X}+\delta x^{\mu}\partial_{\mu}(\lambda_{\alpha\delta}{}^{\gamma}-\{{}^{\gamma}{}_{\alpha\delta}\})|_{X}+\cdots\,, \tag{60}$$

$$\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}|_{x}=\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}|_{X}+\delta x^{\mu}\partial_{\mu}\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}|_{X}+\cdots\,, \tag{61}$$

$$\Gamma^{\gamma}{}_{\beta\alpha}|_{x}=\Gamma^{\gamma}{}_{\beta\alpha}|_{X}+\delta x^{\mu}\partial_{\mu}\Gamma^{\gamma}{}_{\beta\alpha}|_{X}+\cdots\,. \tag{62}$$

When these expansions are substituted into Eqs. (51) and (52), the coefficients may be factored out of the integrals leaving the integrals in the form of integrated moments. Since they are negligible, all integrated moments except $M^{\gamma\delta}$, $M^{\beta\gamma\delta}$, and $N^{\gamma\delta\alpha}$ may be dropped. Also, the evaluations at $X(t)$ may be dropped as implicit. The only nontrivial equations are (51) with $n=0,1,2$ and (52) with $n=0,1$. These are

$$\frac{d}{dt}M^{\gamma 0}=(\lambda_{\alpha\delta}{}^{\gamma}-\{{}^{\gamma}{}_{\alpha\delta}\})M^{\alpha\delta}+\tfrac{1}{2}\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}N^{\beta\alpha\delta}+(\partial_{\mu}\lambda_{\alpha\delta}{}^{\gamma}-\partial_{\mu}\{{}^{\gamma}{}_{\alpha\delta}\})M^{\mu\alpha\delta}\,, \tag{63}$$

$$\frac{d}{dt}M^{\beta\gamma 0}=M^{\gamma\beta}-v^{\beta}M^{\gamma 0}+(\lambda_{\alpha\delta}{}^{\gamma}-\{{}^{\gamma}{}_{\alpha\delta}\})M^{\beta\alpha\delta}\,, \tag{64}$$

$$0=M^{\beta\gamma\delta}+M^{\delta\gamma\beta}-v^{\delta}M^{\beta\gamma 0}-v^{\beta}M^{\delta\gamma 0}\,, \tag{65}$$

$$\frac{d}{dt}N^{\gamma\delta 0}=2M^{[\gamma\delta]}+2\Gamma^{[\gamma}{}_{\beta\alpha}N^{\delta]\beta\alpha}\,, \tag{66}$$

$$0=N^{\gamma\delta\beta}-v^{\beta}N^{\gamma\delta 0}+2M^{\beta[\gamma\delta]}\,. \tag{67}$$

The remainder of the proof consists of analyzing these equations in reverse order. Briefly:

(i) Solve (67) for $M^{\beta[\gamma\delta]}$ in terms of $\rho^{\beta}{}_{\nu}N^{\gamma\delta\nu}$.

(ii) Solve (66) for $M^{[\gamma\delta]}$ in terms of $S^{\gamma\delta}$ and $\rho^{\beta}{}_{\nu}N^{\gamma\delta\nu}$.

(iii) Solve (65) for $M^{\beta(\gamma\delta)}$ in terms of $L^{\gamma\delta}$ and $\rho^{\beta}{}_{\nu}N^{\gamma\delta\nu}$.

(iv) Solve (64) for $M^{\gamma\delta}$ in terms of P^{γ}, $L^{\gamma\delta}$, and $\rho^{\beta}{}_{\nu}N^{\gamma\delta\nu}$, and equate the antisymmetric part to $M^{[\gamma\delta]}$ found in (ii) to obtain the propagation equation for $L^{\gamma\delta}+S^{\gamma\delta}$.

(v) Solve (63) for the propagation equation for P^{γ}.

We now proceed to analyze Eqs. (63) through (67) in more detail. First, using (57), Eq. (67) may be written as

$$M^{\beta[\gamma\delta]}=-\tfrac{1}{2}\rho^{\beta}{}_{\nu}N^{\gamma\delta\nu}\,. \tag{68}$$

Second, using (57) and (26), Eqs. (66) may be written as

$$2M^{[\gamma\delta]}=\frac{d}{dt}S^{\gamma\delta}+v^{\alpha}\Gamma^{\gamma}{}_{\beta\alpha}S^{\beta\delta}+v^{\alpha}\Gamma^{\delta}{}_{\beta\alpha}S^{\gamma\beta}-2\Gamma^{[\gamma}{}_{\beta\alpha}N^{\delta]\beta\nu}\rho^{\alpha}{}_{\nu}$$
$$=\nabla_{v}S^{\hat{\gamma}\hat{\delta}}-2\Gamma^{[\gamma}{}_{\beta\alpha}N^{\delta]\beta\nu}\rho^{\alpha}{}_{\nu}\,. \tag{69}$$

Third, cyclically permute $\beta\gamma\delta$ in (65) twice:

$$0=M^{\gamma\delta\beta}+M^{\beta\delta\gamma}-v^{\beta}M^{\gamma\delta 0}-v^{\gamma}M^{\beta\delta 0}\,, \tag{70}$$

$$0=M^{\delta\beta\gamma}+M^{\gamma\beta\delta}-v^{\gamma}M^{\delta\beta 0}-v^{\delta}M^{\gamma\beta 0}\,. \tag{71}$$

Add (65) and (70) and subtract (71):

$$0=M^{\beta(\gamma\delta)}-M^{\delta[\beta\gamma]}-M^{\gamma[\beta\delta]}-v^{\beta}M^{(\gamma\delta)0}-v^{\delta}M^{[\beta\gamma]0}-v^{\gamma}M^{[\beta\delta]0}\,. \tag{72}$$

Notice that in solving for $M^{\beta(\gamma\delta)}$, all terms are known except $M^{(\gamma\delta)0}$. To find an expression for $M^{(\gamma\delta)0}$, use the identity

$$2M^{(\gamma\delta)0}=2M^{\gamma[\delta 0]}+2M^{\delta[\gamma 0]}+M^{\gamma 0\delta}+M^{\delta 0\gamma}\,, \tag{73}$$

which follows by writing out the symmetrizations. Combine this with the equation obtained from (71) by setting $\beta=0$:

$$2M^{(\gamma\delta)0}=2M^{\gamma[\delta 0]}+2M^{\delta[\gamma 0]}+v^{\gamma}M^{\delta 00}+v^{\delta}M^{\gamma 00}\,. \tag{74}$$

Then using

$$2M^{[\gamma 0]0}=M^{\gamma 00}-M^{0\gamma 0}=M^{\gamma 00}\,, \tag{75}$$

and (68), Eq. (74) becomes

$$M^{(\gamma\delta)0}=M^{\gamma[\delta 0]}+M^{\delta[\gamma 0]}+v^{\gamma}M^{[\delta 0]0}+v^{\delta}M^{[\gamma 0]0}$$
$$=v^{(\gamma}L^{\delta)0}-\rho^{(\gamma}{}_{\nu}N^{\delta)0\nu}\,. \tag{76}$$

Hence, Eq. (72) may be written as

$$M^{\beta(\gamma\delta)}=M^{\gamma[\beta\delta]}+M^{\delta[\beta\gamma]}+v^{\gamma}M^{[\beta\delta]0}+v^{\delta}M^{[\beta\gamma]0}$$
$$+v^{\beta}(M^{\gamma[\delta 0]}+M^{\delta[\gamma 0]}+v^{\gamma}M^{[\delta 0]0}+v^{\delta}M^{[\gamma 0]0})$$
$$=\rho^{\beta}{}_{\mu}(M^{\gamma[\mu\delta]}+M^{\delta[\mu\gamma]}+v^{\gamma}M^{[\mu\delta]0}+v^{\delta}M^{[\mu\gamma]0})$$
$$=\rho^{\beta}{}_{\mu}(-v^{(\gamma}L^{\delta)\mu}+\rho^{(\gamma}{}_{\nu}N^{\delta)\mu\nu})\,. \tag{77}$$

The sum of (68) and (77) is Eq. (56).

Fourth, solve Eq. (64) for $M^{\gamma\delta}$ using (76), (68), and (77):

$$M^{\gamma\delta}=v^{\delta}M^{\gamma 0}+\frac{d}{dt}M^{\delta\gamma 0}-\lambda_{\alpha\beta}{}^{\gamma}M^{\delta[\alpha\beta]}+\{{}^{\gamma}{}_{\alpha\beta}\}M^{\delta(\alpha\beta)}$$
$$=v^{\delta}P^{\gamma}+\frac{d}{dt}\left(\tfrac{1}{2}L^{\delta\gamma}+v^{(\delta}L^{\gamma)0}-\rho^{(\delta}{}_{\nu}N^{\gamma)0\nu}\right)$$
$$+\tfrac{1}{2}\lambda_{\alpha\beta}{}^{\gamma}\rho^{\delta}{}_{\nu}N^{\alpha\beta\nu}+\{{}^{\gamma}{}_{\alpha\beta}\}\rho^{\delta}{}_{\mu}(-v^{\alpha}L^{\beta\mu}+\rho^{\alpha}{}_{\nu}N^{\beta\mu\nu})\,. \tag{78}$$

This is Eq. (55). To find the equation of motion for $L^{\gamma\delta}+S^{\gamma\delta}$, equate Eq. (69) to twice the antisymmetric part of (78):

$$\nabla_{v} S^{\hat{\gamma}\hat{\delta}} - 2\Gamma^{[\gamma}{}_{\beta\alpha} N^{\delta]\beta\nu} \rho^{\alpha}{}_{\nu} = 2v^{[\delta} P^{\gamma]} + \frac{d}{dt} L^{\delta\gamma} + \lambda_{\alpha\beta}{}^{[\gamma} \rho^{\delta]}{}_{\nu} N^{\alpha\beta\nu} - 2v^{[\delta} \{^{\gamma]}{}_{\alpha\beta}\}(-v^{\alpha} L^{\beta 0} + \rho^{\alpha}{}_{\nu} N^{\beta 0\nu})$$
$$+ \{^{\gamma}{}_{\alpha\beta}\}(-v^{\alpha} L^{\beta\delta} + \rho^{\alpha}{}_{\nu} N^{\beta\delta\nu}) - \{^{\delta}{}_{\alpha\beta}\}(-v^{\alpha} L^{\beta\gamma} + \rho^{\alpha}{}_{\nu} N^{\beta\gamma\nu})$$
$$= 2v^{[\delta} \mathcal{P}^{\gamma]} + \nabla_{v} L^{\tilde{\delta}\tilde{\gamma}} + \lambda_{\alpha\beta}{}^{[\gamma} \rho^{\delta]}{}_{\nu} N^{\alpha\beta\nu} - 2\{^{[\gamma}{}_{\alpha\beta}\} N^{\delta]\beta\nu} \rho^{\alpha}{}_{\nu} , \tag{79}$$

where $\mathcal{P}^{\gamma}$ is defined by Eq. (58). Upon rearrangement, Eq. (79) becomes (54).

Finally, the equation of motion for P^{γ} may be found by substituting (69), (78), (57), (68), and (77) into (63):

$$\frac{d}{dt} P^{\gamma} = \lambda_{\alpha\delta}{}^{\gamma}(\tfrac{1}{2} \nabla_{v} S^{\hat{\alpha}\hat{\delta}} - \Gamma^{\alpha}{}_{\mu\nu} N^{\delta\mu\beta} \rho^{\nu}{}_{\beta})$$
$$- \{^{\gamma}{}_{\alpha\delta}\} \left[v^{\alpha} P^{\delta} + \frac{d}{dt} (v^{\alpha} L^{\delta 0} - \rho^{\alpha}{}_{\nu} N^{\delta 0\nu}) + \tfrac{1}{2} \lambda_{\mu\nu}{}^{\alpha} \rho^{\delta}{}_{\beta} N^{\mu\nu\beta} + \{^{\delta}{}_{\mu\nu}\} \rho^{\alpha}{}_{\epsilon}(-v^{\mu} L^{\nu\epsilon} + \rho^{\mu}{}_{\beta} N^{\nu\epsilon\beta}) \right]$$
$$+ \tfrac{1}{2} \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}(v^{\delta} S^{\beta\alpha} + \rho^{\delta}{}_{\nu} N^{\beta\alpha\nu}) + (\partial_{\mu} \lambda_{\alpha\delta}{}^{\gamma})(-\tfrac{1}{2} \rho^{\mu}{}_{\nu} N^{\alpha\delta\nu}) - (\partial_{\mu} \{^{\gamma}{}_{\alpha\delta}\}) \rho^{\mu}{}_{\epsilon}(-v^{\alpha} L^{\delta\epsilon} + \rho^{\alpha}{}_{\nu} N^{\delta\epsilon\nu}) . \tag{80}$$

This equation may be manipulated into (53) using the definition of $\tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}$, the identity

$$\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} - \tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} = \nabla^{\gamma} \lambda_{\tilde{\alpha}\tilde{\beta}\tilde{\delta}} - \nabla_{\delta} \lambda_{\hat{\alpha}\hat{\beta}}{}^{\tilde{\gamma}} , \tag{81}$$

and Eqs. (58) and (26), completing the proof of the theorem.

Notice that the theorem was stated in terms of the moments of $\mathfrak{t}^{\gamma\alpha}$ and $\mathcal{S}^{\gamma\delta\alpha}$ and proved using both of the conservation laws (9) and (11). Recall that the alternate conservation law (19) could be derived from these, and so apparently contains less information. However, in spite of this, we have shown that one can still derive both propagation equations using only the single conservation law (19). That demonstration is outlined here.

We first define the nth integrated moments of the density $\mathcal{T}^{\gamma\delta}$, defined in (14),

$$m^{\beta_1 \cdots \beta_n \gamma\delta} = \int \delta x^{\beta_1} \cdots \delta x^{\beta_n} \mathcal{T}^{\gamma\delta} , \tag{82}$$

the integrated metric energy-momentum,

$$p^{\gamma} = m^{\gamma 0} = \int \mathcal{T}^{\gamma 0} , \tag{83}$$

and the integrated total angular momentum,

$$J^{\gamma\delta} = m^{\gamma\delta 0} - m^{\delta\gamma 0} = 2 \int \delta x^{[\gamma} \mathcal{T}^{\delta]0} . \tag{84}$$

The nth integrated moments of $S^{\gamma\delta\alpha}$, namely $N^{\beta_1 \cdots \beta_n \gamma\delta\alpha}$, and the integrated spin angular momentum $S^{\gamma\delta}$ are still defined as in (45) and (48). Applying the procedure of Sec. III to the single conservation law (19), we obtained the propagation equations,

$$\nabla_{v} \mathfrak{p}^{\tilde{\gamma}} = \tfrac{1}{2} J^{\beta\alpha} \tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^{\delta} + \tfrac{1}{2} N^{\beta\alpha\delta} \nabla^{\gamma} \lambda_{\tilde{\alpha}\tilde{\beta}\tilde{\delta}} , \tag{85}$$

$$\nabla_{v} J^{\tilde{\gamma}\tilde{\delta}} = -2v^{[\gamma} \mathfrak{p}^{\delta]} + 2\lambda^{[\gamma}{}_{\beta\alpha} N^{\delta]\beta\alpha} + N^{\alpha\beta[\delta} \lambda_{\alpha\beta}{}^{\gamma]} , \tag{86}$$

where

$$\mathfrak{p}^{\gamma} = p^{\gamma} + \{^{\gamma}{}_{\alpha\beta}\} v^{\alpha} J^{\beta 0} + \tfrac{1}{2} \lambda^{0}{}_{\alpha\beta} N^{\gamma\alpha\beta}$$
$$- \tfrac{1}{2} \lambda^{\gamma}{}_{\alpha\beta} N^{0\alpha\beta} + \tfrac{1}{2} S^{\beta\alpha} \lambda_{\alpha\beta}{}^{\gamma} . \tag{87}$$

To see that Eqs. (85) and (86) are equivalent to Eqs. (53) and (54), it is necessary to express p^{γ} and $J^{\gamma\delta}$ in terms of P^{γ}, $L^{\gamma\delta}$, $S^{\gamma\delta}$, and $\rho^{\alpha}{}_{\nu} N^{\gamma\delta\nu}$. This is done using the Belinfante-Rosenfeld symmetrization rule (18) or its density form,

$$\mathcal{T}^{\gamma\delta} = \mathfrak{t}^{\gamma\delta} - \tfrac{1}{2} \partial_{\beta}(\mathcal{S}^{\beta\delta\gamma} + \mathcal{S}^{\beta\gamma\delta} + \mathcal{S}^{\gamma\delta\beta})$$
$$- \{^{\gamma}{}_{\alpha\beta}\} \mathcal{S}^{\beta\delta\alpha} + \lambda^{[\gamma}{}_{\alpha\beta} \mathcal{S}^{\delta]\alpha\beta} . \tag{88}$$

Keeping only first moments of $\mathfrak{t}^{\gamma\delta}$ and zeroth moments of $\mathcal{S}^{\gamma\delta\alpha}$, one finds

$$p^{\gamma} = P^{\gamma} - \{^{\gamma}{}_{\alpha\beta}\} N^{\beta 0\alpha} + \tfrac{1}{2} \lambda^{\gamma}{}_{\alpha\beta} N^{0\alpha\beta} - \tfrac{1}{2} \lambda^{0}{}_{\alpha\beta} N^{\gamma\alpha\beta} , \tag{89}$$

$$J^{\gamma\delta} = L^{\gamma\delta} + S^{\gamma\delta} , \tag{90}$$

and hence

$$\mathfrak{p}^{\gamma} = P^{\gamma} + \{^{\gamma}{}_{\alpha\beta}\}(v^{\alpha} L^{\beta 0} - \rho^{\alpha}{}_{\nu} N^{\beta 0\nu}) + \tfrac{1}{2} S^{\beta\alpha} \lambda_{\alpha\beta}{}^{\gamma}$$
$$= \mathcal{P}^{\gamma} + \tfrac{1}{2} S^{\beta\alpha} \lambda_{\alpha\beta}{}^{\gamma} . \tag{91}$$

[Equation (90) justifies the name "total angular momentum" for $J^{\gamma\delta}$.] Substituting Eqs. (90) and (91) into (85) and (86) one obtains Eqs. (53) and (54) written with all Christoffel covariant derivatives and Christoffel curvatures. Hence the propagation equations are equivalent. In the following section we restrict our attention to the form of the propagation equations appearing in the theorem because it clarifies the distinction between spin $S^{\gamma\delta}$ and rotation $L^{\gamma\delta}$.

V. DISCUSSION AND SPECIAL CASES

Careful examination of our principal result, as expressed in the theorem of Sec. IV, yields a number of interesting conclusions pertinent to gravitational experiments and their sensitivity to torsion fields. Our discussion of these focuses on special cases of the propagation equations (53) and (54). First in Sec. VA, we look at the metric theory case, for which the torsion is zero. After digressing briefly, in Sec. VB, on the nondeterministic character of the propagation equations, we proceed, in Sec. VC, to investigate two important special situations involving nonzero torsion: the cases in which the test body possesses only rotational angular momentum or only intrinsic spin angular momentum. These two cases lead us to suggest experiments to detect a torsion field. In Sec. VD, we discuss certain problems and possible ambiguities in interpreting and measuring the variables in the propagation equations. Finally in Sec. VE, we indicate several directions in which the results presented here might be profitably generalized.

A. Metric theories

The energy-momentum and angular momentum propagation equations for metric theories are obtained simply by setting the defect $\lambda^{\alpha}{}_{\beta\gamma}$ to zero in Eqs. (53) through (58). They read

$$\nabla_v \mathcal{P}^{\tilde{\gamma}} = \tfrac{1}{2}(L^{\beta\alpha} + S^{\beta\alpha})\tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^{\delta}, \qquad (92)$$

$$\nabla_v (L^{\tilde{\gamma}\tilde{\delta}} + S^{\tilde{\gamma}\tilde{\delta}}) = -2v^{[\gamma}\mathcal{P}^{\delta]}, \qquad (93)$$

$$M^{\gamma\delta} = v^{\delta}P^{\gamma} + \frac{d}{dt}\left(\tfrac{1}{2}L^{\delta\gamma} + v^{(\gamma}L^{\delta)0} - \rho^{(\gamma}{}_{\nu}N^{\delta)0\nu}\right)$$

$$- \left\{{}^{\gamma}{}_{\alpha\beta}\right\}\rho^{\delta}{}_{\mu}(v^{\alpha}L^{\beta\mu} - \rho^{\alpha}{}_{\nu}N^{\beta\mu\nu}), \qquad (94)$$

$$M^{\beta\gamma\delta} = -\rho^{\beta}{}_{\mu}\left(\tfrac{1}{2}N^{\gamma\delta\mu} + v^{(\gamma}L^{\delta)\mu} - \rho^{(\gamma}{}_{\nu}N^{\delta)\mu\nu}\right), \qquad (95)$$

$$N^{\gamma\delta\alpha} = v^{\alpha}S^{\gamma\delta} + \rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}, \qquad (96)$$

where

$$\mathcal{P}^{\gamma} = P^{\gamma} + \left\{{}^{\gamma}{}_{\alpha\beta}\right\}(v^{\alpha}L^{\beta 0} - \rho^{\alpha}{}_{\nu}N^{\beta 0\nu}). \qquad (97)$$

The propagation equations (92) and (93) are essentially the Mathisson-Papapetrou equations. It is important to notice, however, that when Mathisson, Papapetrou,[1] and the other authors extending[2] their work and making it more rigorous,[4] refer to "spin angular momentum," they do not mean the intrinsic elementary-particle spin. Rather, they mean the total angular momentum, $J^{\gamma\delta} = L^{\gamma\delta} + S^{\gamma\delta} = 2\int \delta x^{[\gamma}\mathcal{T}^{\delta]0}$, computed using the metric energy-momentum tensor. They denote $J^{\gamma\delta}$ by $S^{\gamma\delta}$, whereas we use $S^{\gamma\delta}$ to denote the integrated elementary-particle spin only. Furthermore, they talk about $J^{\gamma\delta}$ as though it were only the rotational angular momentum, $L^{\gamma\delta}$, computed about the center of mass.

Our version of the Papapetrou equations (92) and (93) clearly demonstrate that in a metric theory, $L^{\gamma\delta}$, $S^{\gamma\delta}$, and $J^{\gamma\delta}$ all behave identically. Or, put another way, the propagation of energy-momentum and total angular momentum is independent of how $J^{\gamma\delta}$ is divided between $L^{\gamma\delta}$ and $S^{\gamma\delta}$. This result provides us with an experimental criteria of some value: If there is any measured discrepancy between the behavior of a spin-polarized test body and of a rotating test body, then the theory of gravity cannot be a metric theory. As discussed below, a metric and Cartan-connection theory may predict such a discrepancy.

B. Nondeterminism of the propagation equations

On a theoretical level, the propagation equations for both metric theories [Eqs. (92) to (97)] and metric and torsion theories [Eqs. (53) to (58)] are nondeterministic to a different degree. That is, from a knowledge of the values of the variables at a given time, the equations are not sufficient to predict the values at a later time.

The nondeterministic character of the equations stems from three different sources: (1) the lack of knowledge about the spin currents $\rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}$ in the body; (2) the lack of information about how the total angular momentum $J^{\gamma\delta}$ is divided between spin $S^{\gamma\delta}$ and rotation $L^{\gamma\delta}$; and (3) the lack of a definite specification of a center-of-mass world line.

The last source of nondeterminism can be eliminated as was done in the metric theory case by Tulczyjew, Madore, Beiglböck,[2] and Dixon.[4] Using Christoffel geodesics and a generalized Fermi coordinate system defined in terms of the Christoffel connection, they constructed a center-of-mass world line along which the condition

$$J^{\gamma\delta}P_{\delta} = 0 \qquad (98)$$

is satisfied. The same construction also works in the case of a metric and Cartan-connection theory, but it is no longer in any sense unique. One could, for instance, use Cartan geodesics or a generalized Fermi coordinate system based on the Cartan connection or both. Further, one could replace condition (98) by the condition

$$L^{\gamma\delta}P_{\delta} = 0, \qquad (99)$$

or some other condition. Any of these constructions eliminates the third source of nondeterminism from the propagation equations, but it is far from obvious which construction is theoretically preferred and which yields a center of mass which can be experimentally measured.

The first two sources of nondeterminism arise from our ignorance about the internal dynamics of the test body. To produce a determinisitic system one would have to introduce additional equations to describe the evolution of the internal variables or to relate the internal variables to each other. More specifically, for the metric theory Eqs. (92) through (97), it is not possible to determine completely $M^{\gamma\delta}$, $M^{\gamma\delta\alpha}$, and $N^{\gamma\delta\alpha}$ because the quantities $L^{\gamma\delta}$ and $\rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}$ are neither expressed in terms of P^{γ} and $L^{\gamma\delta}+S^{\gamma\delta}$ nor given their own evolution equations. For the metric and torsion theory equations (53) through (58), not only are $M^{\gamma\delta}$, $M^{\gamma\delta\alpha}$, and $N^{\gamma\delta\alpha}$ left undetermined, but even the propagation equations (53) and (54) themselves are undetermined. This rather surprising result stems from the fact that the torsion does not couple to $L^{\gamma\delta}$; it only couples to $S^{\gamma\delta}$ and $\rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}$. One solution may lie in finding a new combination of variables for which the propagation equations are deterministic. We have not been able to find such variables and do not know if such variables exist.

One way to improve the predictability of the equations is to set $\rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}=0$. This can be justified on the grounds that one normally expects the spin currents in a macroscopic body to be negligible, even when the net spin is large. With this assumption, Eqs. (53) through (58) for metric and torsion theories reduce to

$$\nabla_v \mathcal{P}^{\tilde{\gamma}} = \tfrac{1}{2}L^{\beta\alpha}\tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}v^{\delta} + \tfrac{1}{2}S^{\beta\alpha}\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}v^{\delta} + \tfrac{1}{2}\lambda_{\alpha\beta}{}^{\gamma}\nabla_v S^{\hat{\alpha}\hat{\beta}}, \tag{100}$$

$$\nabla_v L^{\tilde{\gamma}\tilde{\delta}} + \nabla_v S^{\hat{\gamma}\hat{\delta}} = -2v^{[\gamma}\mathcal{P}^{\delta]}, \tag{101}$$

$$M^{\gamma\delta} = v^{\delta}P^{\gamma} + \frac{d}{dt}\left(\tfrac{1}{2}L^{\delta\gamma} + v^{(\delta}L^{\gamma)0}\right) - \left\{{}^{\gamma}{}_{\alpha\beta}\right\}\rho^{\delta}{}_{\mu}v^{\alpha}L^{\beta\mu}, \tag{102}$$

$$M^{\beta\gamma\delta} = -\rho^{\beta}{}_{\mu}v^{(\gamma}L^{\delta)\mu}, \tag{103}$$

$$N^{\gamma\delta\alpha} = v^{\alpha}S^{\gamma\delta}, \tag{104}$$

where

$$\mathcal{P}^{\gamma} = P^{\gamma} + \left\{{}^{\gamma}{}_{\alpha\beta}\right\}v^{\alpha}L^{\beta0}. \tag{105}$$

Although predictability has been improved, Eqs. (100) and (101) are still nondeterministic. The torsion couples to $S^{\gamma\delta}$ but not to $L^{\gamma\delta}$, and there is no way of predicting the evolutions of $L^{\gamma\delta}$ and $S^{\gamma\delta}$ separately.

C. Propagation equations for $L^{\gamma\delta}$ alone and $S^{\gamma\delta}$ alone

Equations (100) through (105) can, in fact, be made deterministic by specifying how the total angular momentum $J^{\gamma\delta}$ is split between spin $S^{\gamma\delta}$ and rotation $L^{\gamma\delta}$. Two interesting special cases are those in which the test body possesses (1) only rotation and (2) only spin.

For the situation in which a rotating test body possesses no net spin ($S^{\gamma\delta}=0$) and no net spin currents ($\rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}=0$), Eqs. (100) through (105) reduce to

$$\nabla_v \mathcal{P}^{\tilde{\gamma}} = \tfrac{1}{2}L^{\beta\alpha}\tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}v^{\delta}, \tag{106}$$

$$\nabla_v L^{\tilde{\gamma}\tilde{\delta}} = -2v^{[\gamma}\mathcal{P}^{\delta]}, \tag{107}$$

$$M^{\gamma\delta} = v^{(\gamma}P^{\delta)} + \frac{d}{dt}\left(v^{(\gamma}L^{\delta)0}\right) - \left\{{}^{(\gamma}{}_{\alpha\beta}\right\}\rho^{\delta)}{}_{\mu}v^{\alpha}L^{\beta\mu}, \tag{108}$$

$$M^{\beta\gamma\delta} = -\rho^{\beta}{}_{\mu}v^{(\gamma}L^{\delta)\mu}, \tag{109}$$

where

$$\mathcal{P}^{\gamma} = P^{\gamma} + \left\{{}^{\gamma}{}_{\alpha\beta}\right\}v^{\alpha}L^{\beta0}. \tag{110}$$

Notice that the torsion has completely dropped out of these equations and that these equations coincide with the Papapetrou equations (92) through (97) with $S^{\gamma\delta}=0$ and $\rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}=0$. This result strengthens the conclusion of our earlier paper[7]: A test body with no net spin is insensitive to torsion and behaves according to the usual Papapetrou equations.

In the case in which a spin-polarized test body has no rotation ($L^{\gamma\delta}=0$) and no spin currents ($\rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}=0$), Eqs. (100) through (105) become

$$\nabla_v P^{\tilde{\gamma}} = \tfrac{1}{2}S^{\beta\alpha}\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}v^{\delta} + P^{\beta}\lambda_{\beta\delta}{}^{\gamma}v^{\delta}, \tag{111}$$

$$\nabla_v S^{\hat{\gamma}\hat{\delta}} = -2v^{[\gamma}P^{\delta]}, \tag{112}$$

$$M^{\gamma\delta} = v^{\delta}P^{\gamma}, \tag{113}$$

$$N^{\gamma\delta\alpha} = v^{\alpha}S^{\gamma\delta}. \tag{114}$$

In form, these propagation equations coincide with those found by Trautman and Hehl.[5] However, our equations describe a macroscopic body, whereas theirs describe a single spinning point particle. Also notice that the assumptions $L^{\gamma\delta}=0$ and $\rho^{\alpha}{}_{\nu}N^{\gamma\delta\nu}=0$ are equivalent to the assumption $M^{\beta\gamma\delta}=0$, (i.e., the test body has no integrated first moments of $t^{\gamma\delta}$) as can be seen from Eq. (56).

Both systems of equations, (106) through (110) and (111) through (114), are deterministic once a background geometry (metric and torsion) and a world line $X(t)$ have been specified. From the values of $\mathcal{P}^{\gamma}$ and $L^{\gamma\delta}$ at one time, their values at all other times may be found from (106) and (107), and then the values of $M^{\gamma\delta}$, $M^{\beta\gamma\delta}$, and P^{γ} may be found from (108), (109), and (110). Similarly, from the values of P^{γ} and $S^{\gamma\delta}$ at one time, their values at all other times may be found from (111) and (112), and then the values of $M^{\gamma\delta}$ and $N^{\gamma\delta\alpha}$ may be found from (113) and (114).

We emphasize that in the presence of torsion, the two sets of propagation equations are quite different. Both energy-momentum propagation equations (106) and (111) contain a force coupling the angular momentum to a curvature. However, in (106) the *rotational* angular momentum couples to the *Christoffel* curvature, whereas in (111) the *spin* angular momentum couples to the *Cartan* curvature. Further, Eq. (111) contains an additional force coupling the energy-momentum to the torsion. Similarly, both angular momentum propagation equations (107) and (112) contain a

$v \times P$ torque (accounting for the orbital angular momentum of the test body as it orbits the bodies which have produced the gravitational field), but Eq. (107) contains a *Christoffel* covariant derivative of the *rotation*, whereas (112) contains a *Cartan* covariant derivative of the *spin*. This can be seen more clearly by expanding the covariant derivative in (107) and (112):

$$v^\alpha \partial_\alpha L^{\gamma\delta} = -2v^{[\gamma}\mathcal{P}^{\delta]} + 2v^\alpha \{^{[\gamma}{}_{\beta\alpha}\} L^{\delta]\beta}, \qquad (107')$$

$$v^\alpha \partial_\alpha S^{\gamma\delta} = -2v^{[\gamma}P^{\delta]} + 2v^\alpha \Gamma^{[\gamma}{}_{\beta\alpha} S^{\delta]\beta}. \qquad (112')$$

If $L^{\gamma\delta}$ and $S^{\gamma\delta}$ are equal, then the extra torques are in the ratio of the Christoffel connection $\{^{\gamma}{}_{\beta\alpha}\}$ in (107′), to the Cartan connection $\Gamma^{\gamma}{}_{\beta\alpha}$ in (112′).

Unfortunately for the prospect of measuring torsion, many theories with dynamic torsion predict that the difference between $\{^{\gamma}{}_{\beta\alpha}\}$ and $\Gamma^{\gamma}{}_{\beta\alpha}$ is negligible in the solar system. Specifically, the Birkhoff theorem proved by Ramaswamy and Yasskin[15] and the generalizations by Neville[15] show that for many $\hat{R}+\hat{R}^2$ theories the unique spherical solution is the Schwarzschild metric and zero torsion. Hence $\{^{\gamma}{}_{\beta\alpha}\}$ and $\Gamma^{\gamma}{}_{\beta\alpha}$ coincide in the solar system except for nonspherical perturbations.

Fortunately however, there are some dynamic torsion theories for which the difference between $\{^{\gamma}{}_{\beta\alpha}\}$ and $\Gamma^{\gamma}{}_{\beta\alpha}$ is expected to be large. Specifically, Møller's[15] teleparallel theories constrain the Cartan curvature to vanish, $\hat{R}^{\alpha}{}_{\beta\gamma\delta}=0$. Hence, there is a global, everywhere-parallel, frame field relative to which the Cartan connection vanishes, $\Gamma^{\alpha}{}_{\beta\gamma}=0$. On the other hand, $\{^{\alpha}{}_{\beta\gamma}\}$ is the Christoffel connection for the metric computed from his field equations. Hence, relative to the teleparallel frame, the defect $\lambda^{\alpha}{}_{\beta\gamma}=\Gamma^{\alpha}{}_{\beta\gamma}-\{^{\alpha}{}_{\beta\gamma}\} = -\{^{\alpha}{}_{\beta\gamma}\}$, is as large as the Christoffel connection. Similarly, Hehl, Ne'eman, Nitsch, and von der Heyde[15] claim that the post-Newtonian limit of their theory coincides with the teleparallelism limit. For such theories the spin-connection torque in (112′) vanishes but the rotation-connection torque in (107′) does not. Similarly, the spin-curvature force in (111) vanishes but the rotation-curvature force in (106) does not.

This leads us to propose an experiment to compare (a) the precession of the axis of rotation of an ordinary (unmagnetized) gyroscope such as that being designed for the Stanford Schiff gyroscope experiment, with (b) the precession of the axis of polarization of a spin-polarized body such as an iron magnet, a polarized beam of elementary particles, or a polarized He^3 superfluid. It seems reasonable to require the two systems to have equal amounts of angular momentum. To obtain an order-of-magnitude estimate we answer the question: How large of an iron magnet is required in order for it to have a spin angular momentum equal to the orbital angular momentum of the Stanford gyroscope?

The Stanford gyroscope is a quartz sphere of radius $r=1$ cm, density $\rho=2.2$ g/cm^3, and angular velocity $\omega=4\pi\times10^2$ rad/sec. Hence its mass is $m=\frac{4}{3}\pi r^3\rho=9.2$ g, and its rotational angular momentum is $L=\frac{2}{5}mr^2\omega=4.6\times10^3$ g cm^2/sec. On the other hand, the spin angular momentum of the iron magnet is $S=\frac{1}{2}\hbar N_e$ where N_e is the number of polarized electrons. Assuming that one electron per atom is polarized, we have $N_e=N_AM/A$, where M is the mass of the iron magnet, $N_A=6\times10^{23}$ particles/mole is Avogadro's number, and $A=56$ g/mole is the atomic weight of iron. Equating L and S, we find $M=2AS/(N_A\hbar)=8.5\times10^8$ g. Unfortunately, it is beyond present technology to put such a large mass of iron into orbit around the earth; so this experiment is not feasible in orbit. Perhaps other polarized systems or an experimental setup on the earth, may turn out to be more promising. We emphasize that only experiments involving test bodies or test systems with a net intrinsic spin are capable of directly detecting torsion and measuring its effects.

D. Ambiguities in interpretation and measurement

There are several ambiguities in the interpretation of our propagation equations which we discuss in this subsection. These include the ambiguity in the definition of a center-of-mass world line, and ambiguities in the definition and interpretation of certain variables, especially the spin density and the integrated energy-momentum. Each of these theoretical difficulties is accompanied by the experimental problem of measuring the corresponding quantity. We are not sure how to resolve these ambiguities, but feel that it is important to point them out in this paper.

First, as discussed in Sec. V B, a center-of-mass world line can be defined in the context of a metric and Cartan-connection theory by analogy with the metric theory constructions of Tulczyjew, Madore, Beiglböck, and Dixon. However, the construction is nonunique in that one must choose between the Christoffel and Cartan connections and choose to impose condition (98) or (99) or some other condition. Further, it is unclear how an experimenter determines that he is measuring the center-of-mass world line of a particular definition. However, we emphasize that our propagation equations and their derivation are valid for any choice of center-of-mass world line and any choice of coordinate system centered on that world line.

Second, as seen in Sec. IV, there are many quantities which could be called the total energy-momentum of the test body: There are the integrated canonical energy-momentum, $P^\gamma = \int t^{\gamma 0}$, and the integrated metric energy-momentum $p^\gamma = \int \mathcal{T}^0$. Then there are the quantities which appear in the propagation equations: $\mathcal{P}^\gamma$ defined in (58) appears in (53) and (54), while $\mathfrak{p}^\gamma$ defined in (87) appears in (85) and (86). The same quantity, written as $\mathfrak{p}^\gamma = \mathcal{P}^\gamma + \frac{1}{2}S^{\beta\alpha}\lambda_{\alpha\beta}{}^\gamma$ also appears in (53) and (54) when they are written using all Christoffel covariant derivatives and Christoffel curvatures. Similarly the quantity, $\bar{\mathcal{P}}^\gamma = \mathcal{P}^\gamma - \frac{1}{2}L^{\beta\alpha}\lambda_{\alpha\beta}{}^\gamma$, appears in (53) and (54) when they are written using all Cartan covariant derivatives and Cartan curvatures. Madore pointed out that one could also define a total energy-momentum $p'_\gamma = \int \mathcal{T}_\gamma{}^0$, using the covariant coordinate components of the metric energy-momentum. Similarly, there is $P'_\gamma = \int t_\gamma{}^0$. One could also take the free index as orthonormal, defining $\hat{P}^{(\mu)} = \int t^{(\mu)0}$ and $\hat{p}^{(\mu)} = \int \mathcal{T}^{(\mu)0}$. The list goes on and on.

But why are we allowed to make so many definitions of energy-momentum? Simply because all we know about the general-relativistic definition is that it must have the correct special-relativistic and Newtonian limits. All of the definitions mentioned above satisfy this criterion. Each is a mathematically well-defined, and in general distinct, quantity. The real questions are experimental: When an experimenter purports to measure the energy-momentum of a test body, which mathematically defined quantity is he really measuring? Is the difference between the various methematically defined quantities significant compared to the experimental accuracy of any forseeable experiment? How does one measure the energy-momentum in the first place? We have not attempted to answer these questions, but merely make some comments on the last one—the difficulty in measuring the energy-momentum.

It is well known in special relativity that a test body with mass m, charge q, and four-velocity v^μ, moving in a background electromagnetic field with vector potential A_μ, has energy-momentum $P_\mu = mv_\mu + qA_\mu$, which is not necessarily parallel to v^μ. Similarly, in a metric theory, the Papapetrou equations imply that the energy-momentum $\mathcal{P}^\gamma$ of a test body with angular momentum is not parallel to v^γ. In a metric and Cartan-connection theory, there are additional spin-torsion terms. To find these terms, we contract v_δ into Eq. (54) yielding

$$\mathcal{P}^\gamma = Ev^\gamma - v_\delta(\nabla_v L^{\tilde{\gamma}\tilde{\delta}} + \nabla_v S^{\hat{\gamma}\hat{\delta}} - \lambda_{\alpha\beta}{}^{[\gamma}\rho^{\delta]}{}_\nu N^{\alpha\beta\nu} - 2\lambda^{[\gamma}{}_{\alpha\beta}N^{\delta]\beta\nu}\rho^\alpha{}_\nu)\,, \qquad (115)$$

where $E = -\mathcal{P}^\delta v_\delta$ is the energy in a comoving frame. Even if we could measure E, v^γ, $L^{\gamma\delta}$, $S^{\gamma\delta}$, and $\rho^\gamma{}_\nu N^{\alpha\beta\nu}$, we could not use Eq. (115) to compute $\mathcal{P}^\gamma$ because we do not know the values of the background metric and torsion fields, which are what we really want to measure.

Finally, we discuss the ambiguities in the definitions of the angular momenta, $L^{\gamma\delta}$, $S^{\gamma\delta}$, and $J^{\gamma\delta}$. Presumably, if one were to keep higher-order moments in the propagation equations, there would be multiple definitions of the spin, rotational, and total angular momenta as there are for the energy-momentum. However, there are more fundamental ambiguities in the interpretation of the variable $S^{\gamma\delta}$ that we wish to discuss.

The quantity $S^{\gamma\delta} = \int \sqrt{-g}S^{\gamma\delta 0}$ is that property of a test body which couples to torsion. Is it really the total spin of all elementary particles except gauge fields? This is actually three questions: Is $S^{\gamma\delta}$ really spin, why are gauge fields excluded, and may other fields be excluded?

As described in Sec. II, if there is a Lagrangian L_M, then $S^{\gamma\delta\alpha}$ is defined by either Eq. (8) or Eq. (15), and one proves, via Noether's theorem, that $S^{\gamma\delta\alpha}$ satisfies the appropriate conservation laws. Further, one proves that L_M is minimally coupled to the Cartan connection $\Gamma^\alpha{}_{\beta\gamma}$ if and only if $S^{\gamma\delta\alpha}$ is the sum of the canonical spin densities of all the fields in the Lagrangian. However, even if L_M is nonminimally coupled (so that the nonminimal contributions to $S^{\gamma\delta\alpha}$ are not in the form of canonical spin densities), the quantity $S^{\gamma\delta\alpha}$ would still contribute to the total angular momentum appearing in the angular momentum conservation law, and so perhaps deserve the name spin. But the usual procedure is to avoid nonminimal couplings.

However, one could still minimally couple some fields to the Cartan connection, $\Gamma^\alpha{}_{\beta\gamma}$, and other fields to the Christoffel connection $\{^\alpha{}_{\beta\gamma}\}$. In that case, $S^{\gamma\delta\alpha}$ would include the spins of only those particles coupled to $\Gamma^\alpha{}_{\beta\gamma}$. At the other extreme, one could minimally couple all fields to $\{^\alpha{}_{\beta\gamma}\}$, so that $S^{\gamma\delta\alpha} = 0$, while keeping torsion terms in the gravitational Lagrangian. But in that case there would be no way to measure the torsion field except through its effects on the metric. So the usual procedure is to minimally couple as many fields as possible to $\Gamma^\alpha{}_{\beta\gamma}$.

However, according to the standard dogma (see Hehl, von der Heyde, Kerlick, and Nester[13]), there is no way to minimally couple a gauge field to a Cartan connection without breaking gauge invariance. This dogma is not absolute. Recently Hojman, Rosenbaum, Ryan, and Shepley[19] constructed a Lagrangian in which the electromagnetic field was, in a certain sense, minimally coupled to a connection with torsion without breaking gauge invariance. However, Ni[19] showed that their theory disagrees with solar system experiments. Perhaps there is some other gauge-invariant way to couple the torsion to gauge fields, but the standard procedure is to use a Lagrangian in which the gauge fields are not coupled to the torsion but all other fields are. Hence, $S^{\gamma\delta\alpha}$ will not include the spins of the gauge fields which must therefore appear as part of the rotational (or orbital) angular momentum.

Finally, one may not want to use a Lagrangian at all. One could define $S^{\gamma\delta\alpha}$ and $t^{\gamma\alpha}$ phenomenologically and simply require them to satisfy the conservation laws *ad hoc*. (This is analogous to the usual treatment of a perfect fluid in general relativity.) Then it might seem that $S^{\gamma\delta\alpha}$ is quite arbitrary. However, there are strong constraints placed on $S^{\gamma\delta\alpha}$ by the fact that, at least in the special-relativistic limit, the quantity $S^{\gamma\delta\alpha}$ must be added to the density of orbital angular momentum to produce a conserved quantity. Thus, $S^{\gamma\delta\alpha}$ must be interpreted as some form of angular momentum. What is more natural than spin?

From the above considerations, it seems reasonable to accept the standard interpretation that $S^{\gamma\delta\alpha}$ is the net spin of all elementary particles other than gauge fields. We have adopted this interpretation throughout this paper.

E. Directions for future research

Our treatment of the energy-momentum and angular momentum propagation equations could be extended and made more rigorous in a number of worthwhile directions. First, on the formal level, a more detailed investigation into the nondeterminacy and ambiguities in the propagation equations is needed. One could generalize the work of Tulczyjew, Beiglböck, and Madore to specify a center of mass in the case of metric and Cartan-connection theories. This would require a detailed study of the choice between Christoffel and Cartan connections at all stages of their construction, or an attempt to avoid the use of either. One could also make our computations more rigorous along the lines of Dixon's work, again carefully distinguishing between Christoffel and Cartan connections. It would also be useful to study the internal balance between spin and rotational angular momentum in a body possessing both, and to study more carefully the definitions of spin and energy-momentum.

Our propagation equations could be generalized to the metric and connection theories in which the connection is non-metric-compatible, and/or extended to include nongravitational interactions such as electromagnetism and other gauge interactions. The electromagnetic generalization seems to merit high priority since every elementary particle with spin (except the neutrinos) also has a magnetic moment. Hence a macroscopic body which is spin-polarized also has a magnetic field. This magnetic field may be useful in constructing experiments, but that would require isolating the system from extraneous magnetic fields. Further, the assumption of vanishing spin currents $\rho^{\alpha}{}_{\nu} N^{\gamma\delta\nu}=0$ may be inappropriate in the presence of magnetic fields.

This brings us to experiments and their analysis. Much work is needed in the development of practical experiments which are sensitive to torsion. As shown here, such experiments must involve spin-polarized systems. The analysis of such experiments would be simplified by an extension of the PPN framework to the metric and torsion theories. Even more important is a detailed study of the ambiguities in the relationship between the measured values of the dynamic variables and the corresponding mathematically defined quantities, as discussed in Sec. V D. In all of these extensions and improvements on our work, we do not expect any fundamental modification to our basic result that torsion couples to spin but not to rotation.

ACKNOWLEDGMENTS

We would especially like to thank Professor Charles Misner for first suggesting this problem to us and for his support and encouragement. Our thanks also go to Dr. Lee Lindblom, Dr. Robert Gowdy, Dr. James Nester, Dr. James Isenberg, Dr. Jeeva Anandan, and Dr. C. W. F. Everitt for helpful and stimulating discussions. The research was supported by NASA under Grant No. NGR-21-002-010 and by NSF under Grant No. PHY 77-18762. One of us (P.Y.) would like to thank the Chaim Weizmann Foundation for fellowship support.

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footnote 1. They distinguish between equations of motion and laws of motion. They call " 'laws of motion' the expressions relating the variation of some particle variables to the (unspecified) force, and 'equations of motion' these same expressions but with the force specified either in terms of fields or in terms of the variables describing the particles exerting the force." Thus the equations of motion describe the behavior of a body as part of a self-consistent solution of the gravitational field equations, while the laws of motion are independent of the specific solution of gravitational field equations. We further divide the laws of motion into an equation for the center-of-mass world line and propagation equations along that world line for the energy-momentum, angular momentum, and other charges of the body.

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[17] If the matter Lagrangian is allowed to depend on higher derivatives of $\theta^{(\mu)}{}_{\alpha}$, $\Gamma^{(\mu)}{}_{(\nu)\alpha}$, and $\psi^{(X)}$ or of $g_{\alpha\beta}$, $\lambda^{\alpha}{}_{\beta\gamma}$, and ψ^{X}, then the conservation laws (9), (11), and (19) remain unchanged although it may be necessary to modify Eqs. (10) and (12) which express $S^{\gamma}{}_{\delta}{}^{\alpha}$ and $t_{\gamma}{}^{\alpha}$ in terms of $\psi^{(X)}$.

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