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Time-asymmetric initial data for black holes and black-hole collisions

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As a first step in constructing initial data for dynamic black holes and general black-hole collisions, we study nonsingular vacuum Cauchy hypersurfaces with two isometric asymptotically flat ends connected by an Einstein-Rosen-type bridge. These hypersurfaces are assumed to be conformally flat and maximally embedded in spacetime but are neither spherically symmetric nor time symmetric. Three of the four constraints are solved explicitly for suitable extrinsic curvature tensors that possess linear momentum and/or intrinsic angular momentum. The complete initial data are shown to transform invariantly, modulo the sign of the extrinsic curvature tensor, under inversion through a minimal two-surface that represents the "throat" of the geometry. These and other properties show that the data represent a particular epoch in the history of a dynamic black hole. We describe the relation of our data to that of the Schwarzschild and Kerr black holes. Finally, we discuss the generalization to encounters of two or more black holes.

I. INTRODUCTION

In the past two decades, remarkable progress has been made in understanding the physical consequences of general relativity. Nevertheless, it remains true that the two-body problem has not been fully solved. With one exception, which itself is a special case, all previous treatments of this problem have employed perturbation theory or other approximations restricting either the strength of the gravitational field, the speeds of the interacting bodies, or both. Given the growing observational and theoretical importance of high-energy astrophysics, such a situation cannot be considered as entirely satisfactory.

The description of sufficiently general high-speed, strong-field two-body encounters will require the full machinery of the Einstein equations. We shall view the necessary analysis in terms of the associated Cauchy problem. Recent development of theoretical and computational methods[1-4] shows that the problems of initial data, kinematics evolution, and calculation of gravitational radiation and apparent horizons are solvable by coherent, workable methods. In this paper, we begin a study of initial data for many-body encounters. We shall concentrate on black holes because such problems involve only the gravitational field and not the complex behavior of matter in noncollapsed objects. For the greater part of this paper, we shall be dealing with only one black hole which, however, does not possess time-symmetric initial data. This makes the generalization to two or more holes possible in a relatively straightforward manner.

A key idea that we shall generalize originated in the work of Misner, Wheeler, Brill, and Lindquist.[5-8] These authors represented momentarily resting masses without employing a stress-energy tensor by using multiply connected topologies. The initial data they found were originally described as geometric models of "particles" or "wormholes" and later as black holes. The time evolution of such initial data would represent black-hole collisions. In all of this work, the initial data were time symmetric (momentarily static black holes). For the case of two identical black holes colliding head-on from rest, the evolution has been computed numerically in the ground-breaking work of Smarr and Eppley.[9] They found that the two holes merge to form a larger one and that a relatively small amount of energy is emitted in the form of gravitational waves (0.1% of the initial rest-mass energy in order of magnitude). A non–head-on collision would presumably be a more interesting and certainly a more difficult problem. It is expected that significantly more gravitational radiation would be emitted, up to perhaps 10% of the initial rest-mass energy according to some estimates.[10]

In order to study non–head-on collisions or spinning holes, one must drop the condition of time symmetry, that is, the extrinsic curvature tensor K_{ij} must be nonzero initially. The holes must "begin" with some, if only a little, momentum if they are to collide with a nonzero impact parameter. Our purpose here is to take a first step in problems of this kind. We construct data for single moving holes in such a way that the generalization to two or more holes with arbitrary initial motion is not difficult.

We solve explicitly three of the four initial-value equations for a symmetric tensor $\hat{K}_{ij}$ such that $\psi^{-2}\hat{K}_{ij}=K_{ij}$ is the physical extrinsic curvature. The fourth constraint produces $\psi>0$ such that $\psi=1+O(r^{-1})$ as $r\to\infty$. It is convenient that one can compute the linear momentum P^i and intrinsic angular momentum J^i without knowing an explicit solution for ψ. The ψ constraint is nonlinear and we have not obtained an exact solution using analytic methods. However, we present the question of finding ψ in terms of a well-posed boundary-value problem that possesses a unique positive solution. This problem can readily be solved numerically.[11] Furthermore, we show that our boundary-value problem together with the properties of $\hat{K}_{ij}$ ensure that the resulting initial-data surface has two asymptotically flat regions ("ends") that are isometric and are connected by an Einstein-Rosen-type "bridge" containing a closed two-surface of minimal area ("throat"). This throat, although not spherically symmetric, is nevertheless fixed under an inversion that defines the isometry of the two ends. These properties lead us to interpret the data as representing a certain slice through a black-hole spacetime. Because $K_{ij} \neq 0$, the apparent horizon will not coincide with the minimal surface in general. We show in this case that there are two apparent horizons, one on each end, related to each other by the inversion mentioned above. These surfaces can be found numerically by a technique such as that used by Eppley.[12]

The paper is organized as follows. First we review the vacuum constraints and outline a method for solving the momentum constraints. Two of many such solutions are presented explicitly, one corresponding to a prescribed linear momentum P^i, the other to a prescribed intrinsic angular momentum J^i. Then we review the technique of inversion through a sphere using the Kelvin transformation and describe the invariance, possibly up to a sign, of our extrinsic curvature tensors with respect to this mapping. The behavior of ψ under inversion and its geometrical meaning are described in terms of minimal two-surfaces. Next, the constraint equation for ψ is presented as a boundary-value problem. The energy of these solutions is shown to be positive. The equation defining the apparent horizon is presented in detail. Sufficient conditions that the throat has minimal (not maximal) area are derived. The relation of our data to that of the Schwarzschild and Kerr black holes is discussed. In the final section we outline the treatment of general two black-hole encounters.

II. INITIAL-VALUE EQUATIONS AND EXTRINSIC CURVATURE

The initial-value equations are[13]

$$\nabla^j(K_{ij} - g_{ij}\mathrm{tr}K) = 8\pi j_i \,, \tag{1}$$

$$R - g^{im}g^{jn}K_{ij}K_{mn} + (\mathrm{tr}K)^2 = 16\pi\rho, \tag{2}$$

if we use units such that $G=c=1$ and employ the Misner-Thorne-Wheeler[14] (MTW) spacelike conventions. Here ρ is the energy density of sources on the spacelike slice, j_i is the three-momentum one-form of sources, K_{ij} is the extrinsic curvature, $\mathrm{tr}K = g^{ij}K_{ij}$, and R is the scalar curvature of the spacelike three-metric g_{ij}. We shall study these equations using conformal techniques.[13,15] Our basic simplifying assumptions are that (1) the Cauchy slice is conformally flat, $g_{ij} = \psi^4 f_{ij}$, and (2) the Cauchy slice is embedded in spacetime as a maximal hypersurface, $\mathrm{tr}K = 0$. The assumption of conformal flatness is not necessary but it makes the analysis simpler. We discuss this point further in treating the moving Schwarzschild black hole (Sec. VI). The sources and trace-free extrinsic curvature are subjected to the conformal transformations $\rho = \hat{\rho}\psi^{-8}$, $j_i = \hat{j}_i\psi^{-6}$, and $K_{ij} = \hat{K}_{ij}\psi^{-2}$. Then we have identically for all positive ψ that

$$\hat{\nabla}^j\hat{K}_{ij} = 8\pi\hat{j}_i \,, \tag{3}$$

$$\nabla^2\psi = -\tfrac{1}{8}\hat{K}_{ij}\hat{K}^{ij}\psi^{-7} - 2\pi\hat{\rho}\psi^{-3}, \tag{4}$$

where ∇^2 is the flat-space Laplacian, $\nabla^2 = \hat{\nabla}^i\hat{\nabla}_i$, $\hat{\nabla}_i f_{jk} = 0$, and indices on objects with carets are raised and lowered with the flat metric.

Restricting ourselves to the vacuum case, we set $\hat{\rho} = \hat{j}_i = 0$. The momentum constraints (3) are most easily solved by setting[16]

$$\hat{K}_{ij} = (\hat{l}W)_{ij} \equiv \hat{\nabla}_i W_j + \hat{\nabla}_j W_i - \tfrac{2}{3} f_{ij}\hat{\nabla}^k W_k \,. \tag{5}$$

Then $\hat{\nabla}^j\hat{K}_{ij} = 0$ becomes

$$\nabla^2 W_i + \tfrac{1}{3}\hat{\nabla}_i\hat{\nabla}^j W_j = 0. \tag{6}$$

If we set $W_i = V_i - \frac{1}{4}\hat{\nabla}_i\lambda$, then we can solve (6) by solving successively the flat-space equations

$$\nabla^2 V_i = 0, \tag{7}$$

$$\nabla^2\lambda = \hat{\nabla}^i V_i. \tag{8}$$

Equation (7) can easily be solved as three separate ordinary Laplace equations if we use Cartesian coordinates.

We look for solutions of the above equations that vanish as $r \to \infty$ and that are smooth and regular on the space $\mathfrak{M} = R^3 - \{0\}$, i.e., Euclidean space with the origin $r=0$ deleted. We choose from the available solutions for $\hat{K}_{ij}$ the following:

$$\hat{K}^{\pm}_{ij} = \frac{3}{2r^2}[P_i n_j + P_j n_i - (f_{ij} - n_i n_j)P^k n_k] \mp \frac{3a^2}{2r^4}[P_i n_j + P_j n_i + (f_{ij} - 5n_i n_j)P^k n_k], \tag{9}$$

$$\hat{K}_{ij} = \frac{3}{r^3}(\epsilon_{kil}J^l n^k n_j + \epsilon_{kjl}J^l n^k n_i), \tag{10}$$

where P^i and J^i are constant vectors, n^i is the unit normal of a sphere r = constant in flat space

$[n^i = r^{-1}(x,y,z) = (\partial/\partial r)^i]$, ϵ_{ijk} is the unit alternating tensor, and a = constant. Note that (9) comprises two solutions: $\hat{K}^+_{ij}(P)$ and $\hat{K}^-_{ij}(P)$, according to whether the sign of the second term is negative or positive. The fundamental differences between these two cases are described in Sec. III.

These solutions were selected firstly because they transform in the proper way under inversion through a sphere (Sec. III), and secondly because (9) corresponds to a "source" with linear momentum P^i and no intrinsic angular momentum, while (10) corresponds to a source with intrinsic angular momentum J^i and no linear momentum. Actually, each of the two terms of (9) satisfies $\hat{\nabla}^j\hat{K}_{ij} = \mathrm{tr}\hat{K}_{ij} = 0$ and only the first one possesses linear momentum. However, the second term ("multipole moment") is needed for the desired behavior under inversions.[17,18] The statements about P^i and J^i can be checked from the two-surface integrals over the "sphere at infinity" that give the linear and angular momenta,

$$P_i = \frac{1}{8\pi}\oint_\infty K_{ij}d^2S_j\,, \tag{11}$$

$$J_i = \frac{1}{16\pi}\epsilon_{ijk}\oint_\infty (x^jK^{km} - x^kK^{jm})d^2S_m. \tag{12}$$

Here we have written the integrals in terms of Cartesian components. Note that because we are interested in asymptotically flat spaces, we will have $\psi \to 1 + O(r^{-1})$ as $r\to\infty$. Therefore, $\hat{K}_{ij}$ may be used in place of K_{ij} in (11) and (12). In other words, we know the linear and angular momenta corresponding to our initial data *without* having to solve the nonlinear constraint (4) for ψ. However, the total energy E of the solutions does require knowledge of ψ because[19,20]

$$E = -\frac{1}{2\pi}\oint_\infty \hat{\nabla}^i\psi d^2S_i\,. \tag{13}$$

We show in Sec. V that the energy can be written in positive form in these problems.

III. INVERSION TRANSFORMATIONS

In this section we review the well-known technique of inversion through a sphere[21] ("reciprocal radius transformation"; "Kelvin transformation") and its action on the metric and the extrinsic curvature tensor. We use it as a technique to aid in the construction of nontrivial complete initial spacelike hypersurfaces, and associated smooth extrinsic curvature tensors, that can be interpreted as initial data for black holes. In effect, the use of conformal techniques and inversion symmetries in the vacuum initial-value problem removes all apparent incompleteness and/or singularities in the data. For example, the $\hat{K}_{ij}$'s in (9) and (10) diverge as $r\to 0$; however, the physical $K^i_j = \psi^{-6}K^i_j$ tensors that are constructed with the present techniques are regular everywhere and vanish as $r\to 0$.

Inversion through a sphere of radius a is a mapping defined on $\mathfrak{M} = \mathrm{R}^3 - \{0\}$ that is given in spherical polar coordinates by

$$\bar{r} = \frac{a^2}{r}\,,\quad \bar{\theta} = \theta\,,\quad \bar{\phi} = \phi\,, \tag{14}$$

or in Cartesian coordinates by

$$\bar{x} = \frac{a^2}{r^2}x\,,\quad \bar{y} = \frac{a^2}{r^2}y\,,\quad \bar{z} = \frac{a^2}{r^2}z\,. \tag{15}$$

As is well known, this transformation may be viewed as a mapping from the region $r \geq a$ to the region $0 < r \leq a$, with infinity going to the origin and vice versa. We shall denote the inversion mapping by $\bar{x}^i = J^i(x^1,x^2,x^3)$, its Jacobian by $J^i_j = \partial J^i/\partial x^j$, and its inverse Jacobian by $(J^{-1})^i_j$. This J should not be confused with that in (10) and (12) as shall be clear in context.

We are interested in cases where the inversion map defines an isometry of a metric g_{ij}. The isometry condition may be stated as

$$g_{ij}(x) = J^k_iJ^l_jg_{kl}[J(x)] = J^k_iJ^l_jg_{kl}(\bar{x})\,, \tag{16}$$

where $\bar{x}$ denotes the point obtained by inversion from x. For conformally flat matrics, we have $g_{ij} = \psi^4f_{ij}$ and the isometry condition becomes the following condition on the conformal factor:

$$\psi(x) = \frac{a}{r}\psi(\bar{x})\,. \tag{17}$$

Suppose that we have a metric on $\mathfrak{M}$ that is conformally flat and obeys (17). Then we may think of the intrinsic metric geometry of $\mathfrak{M}$ as being composed of two isometric regions ($r\geq a$ and $0<r\leq a$) joined smoothly at $r=a$. An important consequence is found by differentiating (17):

$$\frac{\partial\psi}{\partial r}(x) = -\frac{a}{r^2}\psi(\bar{x}) - \frac{a^3}{r^3}\frac{\partial\psi}{\partial\bar{r}}(\bar{x})\,. \tag{18}$$

Therefore, at $r=\bar{r}=a$, we have

$$\frac{\partial\psi}{\partial r}(a) + \frac{1}{2a}\psi(a) = 0. \tag{19}$$

Later, (19) will be used as a boundary condition. Here, let us examine its geometrical meaning.

Let $\mathcal{A}(r)$ denote the area of a closed two-surface r = constant. In a conformally flat geometry,

$$\mathcal{A}(r) = \int_0^{2\pi}\int_0^{\pi} r^2\psi^4\sin\theta d\theta d\phi\,. \tag{20}$$

Extremizing this area with respect to r, we find $(\partial/\partial r)(r^2\psi^4) = 0$ or

$$\frac{\partial\psi}{\partial r} + \frac{1}{2r}\psi = 0\,. \tag{21}$$

Therefore, from (19) it follows that the area function on surfaces of constant r takes an extreme value at the radius of inversion $r=a$. Therefore, the trace of the extrinsic curvature of the two-surface $r=a$, regarded as a surface embedded in the conformally flat three-space, vanishes. A simple calculation shows that (19) implies a stronger result: The entire extrinsic curvature tensor of $r=a$ vanishes. This is in accord with an observation of Gibbons[22] that a surface fixed under an isometry of the enveloping space must be a totally geodesic surface.

We shall assume, with justification given later, that the surface area of $r=a$ is in fact *minimal*. Moreover, if we also assume that $\psi\to 1$ as $r\to\infty$, then we have two conformally flat, asymptotically flat ends joined at a minimal surface or throat. A simple example of such a throat is found in the time-symmetric initial-data slice of the Schwarzschild-Kruskal black-hole geometry. There, the minimal surface is given by $r=a=M/2$ in isotropic coordinates. Note, however, that $r=a$ in our work will not have the standard metric of a two-sphere; it will be somewhat deformed.

Now we turn to the action of inversion on an extrinsic curvature tensor K_{ij}. Since K_{ij} enters the Hamiltonian constraint quadratically, to obtain isometric geometries on the two ends, it is clear that we should demand that K_{ij} be invariant under inversions except perhaps for a change of sign:

$$K_{ij}(x)=\pm J_i^k J_j^l K_{kl}(\bar{x})\,. \tag{22}$$

A change of sign indicates that if momentum P^i is associated with one end of the three-geometry, then momentum $-P^i$ will be associated with the other end.

The properties of $\hat{K}_{ij}$ follow from those of K_{ij} and ψ if we recall that $K_{ij}=\psi^{-2}\hat{K}_{ij}$. This transformation was chosen so that whenever K_{ij} satisfies vanishing divergence and trace conditions with respect to $g_{ij}=\psi^4 g_{ij}$, identical conditions hold for $\hat{K}_{ij}$ with respect to f_{ij}. Applying this transformation, we see that (22) becomes

$$\psi^{-2}(x)\hat{K}_{ij}(x)=\pm J_i^k J_j^l K_{kl}(\bar{x})\psi^{-2}(\bar{x})\,. \tag{23}$$

Assuming, as will be shown independently in the next section, that ψ satisfies the isometry condition (17), we find the appropriate inversion symmetry for $\hat{K}_{ij}$,

$$\hat{K}_{ij}(x)=\pm\left(\frac{a}{r}\right)^2 J_i^k\, J_j^l \hat{K}_{kl}(\bar{x})\,, \tag{24}$$

or, in Cartesian coordinates,

$$\hat{K}_{ij}(x,y,z)=\pm\left(\frac{a}{r}\right)^6(\delta_i^k-2n^k n_i)(\delta_j^l-2n^l n_j)\hat{K}_{kl}(\bar{x},\bar{y},\bar{z}), \tag{25}$$

where $n^i=x^i/r$. An examination of (9) reveals that $\hat{K}^+_{ij}(\vec{P})$ satisfies (24) with the plus sign and $\hat{K}^-_{ij}(\vec{P})$ with the minus sign, where the a^2 term in these quantities refers to $a=$ radius of inversion. Hence, both of these tensors have appropriate inversion properties for the case in which the physical extrinsic curvature obeys (22) and solves (1) in a metric satisfying the isometry condition (16). A similar examination of (10), the $\hat{K}_{ij}$ with intrinsic angular momentum, shows that it too satisfies (24) or (25) with the negative sign. $\hat{K}_{ij}(\vec{J})$ contains no parameter corresponding to the radius of inversion.

IV. HAMILTONIAN CONSTRAINT

The Hamiltonian constraint is (4) with $\hat{\rho}=0$. We seek a solution on $R^3-\{0\}$ that satisfies $\psi(x)=(a/r)\psi(\bar{x})$. Instead of seeking this solution directly, we pose a boundary-value problem that achieves the same result and also helps in obtaining numerical solutions. Our boundary-value problem is to find ψ for $r\geq a$ such that

$$\nabla^2\psi=-\tfrac{1}{8}\hat{K}_{ij}\hat{K}^{ij}\psi^{-7}\quad\text{for } r\geq a, \tag{26}$$

$$\frac{\partial\psi}{\partial r}+\frac{1}{2a}\psi=0\quad\text{for } r=a, \tag{27}$$

$$\psi>0,\ \lim_{r\to\infty}\psi=1\,. \tag{28}$$

A proof of the uniqueness of a solution of this problem has been obtained.[23] Existence is demonstrated by actually solving (26).[24] Instead of sketching the proof, we shall present an exact solution of the above problem when $\hat{K}_{ij}\hat{K}^{ij}$ closely imitates the exact values obtained from (9). But first we show how the solution of the boundary-value problem is converted into a solution of the original problem.

Let $\psi=F(r,\theta,\phi)$ denote the unique solution of the above boundary-value problem. Define a related function

$$\tilde{F}(r,\theta,\phi)=\frac{a}{r}F\left(\frac{a^2}{r},\theta,\phi\right)=\frac{a}{r}F(\bar{r},\theta,\phi). \tag{29}$$

Note that $\tilde{F}$ is defined for $0<r\leq a$ and that $\tilde{F}(a,\theta,\phi)=F(a,\theta,\phi)$. Moreover, the first derivatives of $\tilde{F}$ and F agree at $r=a$. We need only check the radial derivatives. Equating these at $r=a$ gives [cf. (18) and (19)] $\partial F/\partial r+(2a)^{-1}F=0$, which is guaranteed by the boundary condition.

Likewise, one finds that the second derivatives of $\tilde{F}$ and F also match at $r=a$. In fact, from the well-known inversion theorem of electrostatics,[21] we have

$$\nabla^2\tilde{F}\,|_r=\left(\frac{a}{r}\right)^5\nabla^2 F\,|_{\bar{r}=(a^2/r)}\,. \tag{30}$$

We can use (30) to establish that $\tilde{F}$ is, in fact, a positive solution of the Hamiltonian constraint for $0<r\leq a$. Note from (24) that we have

$$\hat{K}_{ij}(r)\hat{K}^{ij}(r)=\left(\frac{a}{r}\right)^{12}\hat{K}_{ij}(\bar{r})\hat{K}^{ij}(\bar{r})\,. \tag{31}$$

Using (29) shows that

$$\tilde{F}^{-7}(r)=(a/r)^{-7}F^{-7}(\bar{r})\,.$$

Therefore, because F obeys (26) for $r\geq a$, it follows that for $0<r\leq a$,

$$\nabla^2\tilde{F}(r)=-\tfrac{1}{8}\hat{K}_{ij}(r)\hat{K}^{ij}(r)\tilde{F}^{-7}(r)\,, \tag{32}$$

i.e., $\tilde{F}$ is a solution of the Hamiltonian constraint for the interior of the sphere $r=a$ (with the origin deleted). The matching conditions satisfied by F and $\tilde{F}$ show that the complete solution of the Hamiltonian constraint on $R^3-\{0\}$ is given by

$$\begin{aligned}\psi&=F \quad \text{for } r\geq a,\\ \psi&=\tilde{F} \quad \text{for } 0<r\leq a\,.\end{aligned} \tag{33}$$

But the solution inside the sphere was obtained by using the isometry condition (17) that relates the value of the solution at a point to its value at the corresponding inverted point. Therefore, the solution ψ thus obtained automatically satisfies the desired isometry condition under inversion. We obtain two asymptotically flat, conformally flat, isometric ends joined at an extremal surface $r=a$. We see that this occurs primarily because of the boundary conditions and the special inversion properties of $\hat{K}_{ij}$.

To illustrate the above results, we consider a *model* solution. From (9), we have that the actual form of $(\hat{K}_{ij}\hat{K}^{ij})^{\pm}\equiv H^{\pm}$ is

$$H^{\pm}=\frac{9}{2}\frac{P^2}{r^4}\left[\left(1\mp\frac{a^2}{r^2}\right)^2+2\cos^2\theta\,\left(1\pm4\frac{a^2}{r^2}+\frac{a^4}{r^4}\right)\right], \tag{34}$$

where we have chosen P^i in the Z direction. To obtain a problem we can solve exactly, we employ a "model" value (ignoring angular dependence)

$$H_{\text{model}}=6\frac{P^2}{r^4}\left(1-\frac{a^2}{r^2}\right)^2. \tag{35}$$

Note that H_{model} has qualitatively correct asymptotic behavior (terms with r^{-4}, r^{-6}, and r^{-8}) and the correct behavior $H_{\text{model}}(r)=(a/r)^{12}H_{\text{model}}(\bar{r})$ under inversions. Then the solution of $\nabla^2\psi_{\text{model}}=-\frac{1}{8}H_{\text{model}}\psi^{-7}_{\text{model}}$ is

$$\psi_{\text{model}}=\left(1+\frac{2E}{r}+\frac{6a^2}{r^2}+\frac{2a^2E}{r^3}+\frac{a^4}{r^4}\right)^{1/4}, \tag{36}$$

where the total energy is $E=(P^2+4a^2)^{1/2}$. This solution is positive, goes to one at infinity, satisfies the boundary condition at $r=a$, and satisfies the isometry condition (17). Moreover, if P^2/a^2 is negligible, then (36) reduces exactly to the Schwarzschild value $[1+a/r]$, where $E=M=2a$, $M=$ Schwarzschild rest mass. This, together with $E=(P^2+4a^2)^{1/2}$ in the model solution, suggests that $2a$ may be a good estimate of the "rest mass" of the moving object we are studying.

V. ENERGY, MINIMAL SURFACE, AND APPARENT HORIZON

The total energy E can be calculated from (13). We relate this to the Hamiltonian constraint (26) by using Gauss's theorem in the region $r\geq a$ and the boundary conditions (27) and (28). A simple calculation gives

$$E=\frac{1}{16\pi}\int_{r\geq a}\hat{K}_{ij}\hat{K}^{ij}\psi^{-7}d\hat{v}+\frac{a}{4\pi}\int_0^{2\pi}\int_0^{\pi}\psi(a,\theta,\phi)\sin\theta d\theta\,d\phi. \tag{37}$$

Therefore, $\psi>0$ implies $E>0$.

Next we turn to the question of whether the extremal surface $r=a$ is actually minimal. It is easy to find sufficient conditions, relating $\vec{P}$ and a or $\vec{J}$ and a, that a minimum occurs. To do this, we return to the area integral (20) and compute its second derivative with respect to r. Using the Hamiltonian constraint and the boundary condition at $r=a$, and assuming symmetry about the z axis ($\vec{P}$ or $\vec{J}$ in the z direction), we find at $r=a$, after partially integrating angular terms,

$$\begin{aligned}\frac{\partial^2\mathfrak{a}}{\partial r^2}=8\pi a^2\int_0^{\pi}\left[\frac{1}{4a^2}\psi^4(a)-\tfrac{1}{8}\hat{K}_{ij}K^{ij}\psi^{-4}(a)\right]\sin\theta d\theta\\ +24\pi\int_0^{\pi}\psi^2(a)\left(\frac{\partial\psi}{\partial\theta}\right)^2\sin\theta d\theta\,.\end{aligned} \tag{38}$$

The second term on the right is positive. To estimate the first term, note that $\psi(a)\geq1$ follows $\psi>0$, $\psi\to1$ as $r\to\infty$, $\nabla^2\psi\leq0$, and $\partial\psi/\partial r<0$ at $r=a$. Therefore,

$$\frac{1}{4a^2}\psi^4(a)-\tfrac{1}{8}\hat{K}_{ij}\hat{K}^{ij}\psi^{-4}(a)\geq\frac{1}{4a^2}-\tfrac{1}{8}\hat{K}_{ij}\hat{K}^{ij}(a). \tag{39}$$

Hence, sufficient conditions for $\partial^2\mathfrak{a}/\partial r^2>0$ are easily obtained.

For linear momentum, we find from (9) that

$$[\hat{K}_{ij}\hat{K}^{ij}(a)]^{\pm}=\begin{Bmatrix}54\dfrac{P^2}{a^4}\cos^2\theta\\[2ex] 18\dfrac{P^2}{a^4}\sin^2\theta\end{Bmatrix}, \tag{40}$$

while for angular momentum, we find from (10) that

$$\hat{K}_{ij}\hat{K}^{ij}(a)=18\frac{J^2}{a^6}\sin^2\theta. \tag{41}$$

The sufficient conditions that follow from (38) and (39) using these values are, respectively,

$$P \leqslant \frac{a}{3}, \quad P \leqslant \frac{a}{\sqrt{6}}, \quad J \leqslant \frac{a^2}{\sqrt{6}}. \tag{42}$$

These estimates are undoubtedly too conservative. For example, the model solution ψ_{model} has surfaces $r=a$ that are minimal for any value of P/a. The quantity P/a can be taken as a rough measure of how relativistic the characteristic speeds are. For example, if we take $a \sim \frac{1}{2}M$, M = rest mass, then $P/a = 6^{-1/2}$ implies $v \approx 0.2$ ($c=1$) and $(1-v^2)^{-1/2} \approx 1.02$, while $P/a=5$ gives $v \approx 0.93$ and $(1-v^2)^{-1/2} \approx 2.7$. A numerical study involving even higher ratios confirms that all cases have $r=a$ as a minimal surface, even though (42) is violated by a large amount.[11]

An apparent horizon[25] is a closed two-surface on the Cauchy slice whose outward-pointing unit normal $s^i (g_{ij}s^i s^j = +1)$ satisfies

$$\nabla_i s^i - \mathrm{tr}K + K_{ij}s^i s^j = 0. \tag{43}$$

In our case we have $\mathrm{tr}K=0$. If, moreover, one has a time-symmetric slice, or more generally if $K_{ij}s^i s^j=0$, then (43) becomes $\nabla_i s^i = 0$ and the apparent horizon coincides with a minimal surface.

The existence of an apparent horizon means, assuming "cosmic censorship" (more precisely, "future asymptotic predictability"[25]) that the evolution of the data will contain an event horizon. The intersection of the horizon with the initial-data surface will be a closed two-surface that necessarily lies outside of, or coincides with, the apparent horizon.[25]

In the present study we have $\mathrm{tr}K=0$. Let us define the unit outgoing normal of the apparent horizon in the flat metric by $\hat{s}^i = \psi^2 s^i$. Recalling also that $K_{ij} = \psi^{-2}\hat{K}_{ij}$, we find that (43) becomes

$$\hat{\nabla}_j \hat{s}^j + 4\hat{s}^i \hat{\nabla}_i \ln\psi + \psi^{-4}\hat{K}_{ij}\hat{s}^i\hat{s}^j = 0. \tag{44}$$

We expect that an apparent horizon exists for $r \geqslant a$ (call this the "top sheet" of the geometry). Then, from the inversion symmetries, we expect that there be another apparent horizon on the bottom sheet. We may see this as follows. If $s^i(x)$ satisfies (43) with $\mathrm{tr}K=0$, then we define $s^i(\bar{x})$ by

$$s^i(x) = \pm (J^{-1})^i_j s^j(\bar{x}), \tag{45}$$

where $\bar{x}$ is the point obtained from x by inversion. The sign is chosen to be the same as in (22) for K_{ij}. Then we have

$$\hat{s}^i(x) = \pm\left(\frac{a}{r}\right)^2 (J^{-1})^i_j \hat{s}^j(\bar{x}). \tag{46}$$

Substituting (46) into (44) yields again (44), but with each term computed and evaluated at the inverted point. Hence, there will be two solutions of (44). These solutions will, of course, coincide when the apparent horizon is a minimal surface. Such is indeed the case for $\hat{K}_{ij}(\vec{J})$, where one has $\hat{K}_{ij}(\vec{J})\hat{s}^i\hat{s}^j = 0$.

For completeness, we shall exhibit the form taken by (44) when it is solved in practice. We assume that the apparent horizon is defined by the vanishing of the function $\tau(r,\theta) = r - h(\theta)$, when we have axial symmetry: $\vec{P} = P(\partial/\partial z)$. Then we can always write $\hat{s}_i = \lambda \hat{\nabla}_i \tau$, with $\lambda = (1 + h_\theta^2/h^2)^{-1/2}$, $h_\theta = dh/d\theta = -\partial\tau/\partial\theta$. Using either form of $\hat{K}_{ij}(P)$, we find

$$h_{\theta\theta} + h^{-2}h_\theta^3(\cot\theta + 4\psi^{-1}\psi_\theta) + h_\theta^2(-3h^{-1} - 4\psi^{-1}\psi_r) + h_\theta(\cot\theta + 4\psi^{-1}\psi_\theta) + h(-2 - 4h\psi^{-1}\psi_r)$$
$$= \psi^{-4}(1 + h_\theta^2 h^{-2})^{1/2}(h^2\hat{K}_{rr} + h_\theta^2 h^{-2}\hat{K}_{\theta\theta} - 2h_\theta\hat{K}_{r\theta}). \tag{47}$$

Usually this equation will require a numerical solution obtained, for example, by using a technique such as that employed by Eppley[12] or Piran.[11]

VI. SLOWLY MOVING SCHWARZSCHILD BLACK HOLE

To help in the further interpretation of the preceding results, we shall study a slowly moving Schwarzschild vacuum black hole. On the time-symmetric slice of Schwarzschild-Kruskal spacetime, the initial data are

$$g_{ij} = \left(1 + \frac{a}{r}\right)^4 f_{ij}, \quad K_{ij} = 0, \tag{48}$$

where $a = M/2$ and r is the isotropic radial coordinate. We shall make a Lorentz boost of this data, to first order in the boost velocity. This transformation is implemented by using the first-order form of the vacuum Einstein equations, i.e., those giving $\partial_t g_{ij}$ and $\partial_t K_{ij}$, with appropriately chosen lapse function α and shift vector β^i.[13]

The transformation we want to use is not uniquely defined in curved spacetime unless certain conditions are imposed. One such condition is that the lapse function for large r should approach its flat-spacetime value. Another condition we adopt is that the boosted slice should be maximal. Finally, a boundary condition is needed at $r = \frac{1}{2}M$. We have two good choices:

$$\alpha(r,\theta,\phi) = +\alpha\left(\frac{a^2}{r}, \theta, \phi\right), \tag{49}$$

$$\alpha(r,\theta,\phi) = -\alpha\left(\frac{a^2}{r}, \theta, \phi\right). \tag{50}$$

The first condition says that we use a Neumann condition $\partial\alpha/\partial r=0$ at $r=a$ and the second, $\alpha=0$ at $r=a$ (Dirichlet condition). The two distinct boosted maximal slices we obtain are analogous to the two distinct families of spherically symmetric maximal slices of Schwarzschild-Kruskal spacetime (those that reach spatial infinity). Condition (49) gives the family that penetrates the horizon and for which "time" advances "forward" on both sheets[26,27]; condition (50) gives the family that covers only the region exterior to the horizon and for which time runs in opposite directions on the two sheets (i.e., the standard $t_{\rm schw}=\text{const}$ slices in the spherical case).

To select β^i we adopt the "minimal distortion" criterion,[4] which, in this case, gives the trivial result $\beta^i=0$. Hence, from

$$\dot{g}_{ij}=-2\alpha K_{ij}+\nabla_i\beta_j+\nabla_j\beta_i \tag{51}$$

we find that the slowly boosted metric is still given by $g_{ij}=(1+M/2r)^4 f_{ij}$, with $a=M/2$ defining the sphere of inversion symmetry. Through $O(V)$, this sphere is still also an apparent horizon.

The Einstein equations for $\partial_t K_{ij}$ tell us that[13]

$$\partial_t(\mathrm{tr}K)=-\Delta\alpha+K_{ij}K^{ij}\alpha+\beta^i\nabla_i\mathrm{tr}K. \tag{52}$$

With the above demands, this can be rewritten quite simply as

$$\nabla^2(\psi\alpha)=0, \tag{53}$$

$\psi\equiv 1+M/2r$. Our two solutions $\alpha^\pm$, with "+" corresponding to (49) and "−" to (50), are

$$\alpha^\pm=\frac{-Vz}{1+a/r}\left(1\pm\frac{a^3}{r^3}\right), \tag{54}$$

where we have chosen a boost in the negative z direction. [The left-hand side of (54) is really $\alpha^\pm\delta t$, t = standard Schwarzschild time, but we omit δt here.]

The new K_{ij} can be computed from the Einstein equations[13] (here $\beta^i=0$)

$$\dot{K}_{ij}=-\nabla_i\nabla_j\alpha+\alpha[R_{ij}+(\mathrm{tr}K)K_{ij}-2K_{il}K^l_j], \tag{55}$$

where R_{ij} is calculated from g_{ij}. The calculation yields, with $\alpha=\alpha^\pm$, the interesting result

$$K^\pm_{ij}=\psi^{-2}\hat{K}^\pm_{ij}, \tag{56}$$

where $\hat{K}^\pm_{ij}$ is given by (9), with $P_i=MV_i$ and $a=M/2$.

The above discussion shows why $\hat{K}^\pm_{ij}$ takes the form it has for general P^i and a. Of course, our ψ will not be spherically symmetric nor will the simple relationship $M=2a=E$ still hold. We must specify P^i and a, solve for ψ, and then compute E to find $E=E(|\vec{P}|,a)$.

We should also remark that a finitely boosted maximal slice in the Schwarzschild-Kruskal spacetime would not be conformally flat. Calculations similar to the above through $O(V^2)$ give the same extrinsic curvature $K^\pm_{ij}$ but show that the boosted metric is not conformally flat in $O(V^2)$.[28,29] Therefore, the spacetime that is evolved from our data $(\psi^4 f_{ij},\psi^{-2}\hat{K}^\pm_{ij})$ will not be a disguised (i.e., boosted and regauged) Schwarzschild-Kruskal black hole. In this connection, we wish to emphasize that the assumption of conformal flatness was made in our analysis for simplicity and facility in obtaining exact results. It is certainly not necessary as a point of principle.

If we study $\hat{K}_{ij}(\vec{J})$, we may ask about the relation of our data to that of the Kerr metric on its "(t,ϕ)-symmetric" slice ($t\to -t$, $\phi\to-\phi$).[30] This slice is maximal with extrinsic curvature identical to ours in $O(r^{-3})$, the dominant term for large r in this case. Our slice is also maximal and (t,ϕ) symmetric. However, the t = constant slices of Kerr (Boyer-Lindquist time) are not conformally flat. Hence, the spacetime evolved from the data $[\psi^4 f_{ij},\ \psi^{-2}\hat{K}_{ij}(\vec{J})]$ will not be identically the Kerr metric, except in the case that $(J/E^2)^2$ is negligibly small.

Therefore, in the cases of both linear and angular momenta, we conclude that our data will produce "dynamic" black holes involving, as seems likely, the emission of some gravitational radiation in the course of their evolution. However, it is clear from an Arnowitt-Deser-Misner-type Poynting flux vector calculation[9] that the flux of any "wave" energy that may be hidden in the geometry is zero on the initial slice. One expects each to "settle" to standard (boosted) Schwarzschild holes or Kerr holes, respectively. It should be noted that as $|\vec{P}|/E$ becomes small (or J/E^2), the data approach the slowly moving Schwarzschild configuration (or the slowly rotating Kerr configuration) and any "hidden" gravitational energy that is available for radiation will vanish rapidly in this limit. Studies of the area of the apparent horizon (lower limit on irreducible mass) compared to the "initial rest mass" $(E^2-P^2)^{1/2}$ (E and $|P|$ are both conserved because they are calculated at spatial infinity) are under way in order to help settle some of these issues.[11] For example, it seems plausible that $[(E^2-P^2)^{1/2}-\tilde{M}_{\rm irreducible}]$, $\tilde{M}_{\rm irreducible}=[(16\pi)^{-1}\times\text{area of apparent horizon}]^{1/2}$, would give an upper limit on the amount of hidden wave energy that might redistribute itself in the course of evolution.

VII. TWO BLACK HOLES

Two possible methods come to mind when we consider how to extend the previous work to two (or to N) black holes. We first consider the work of Misner,[6] who treated the time-symmetric conformally flat N-body problem. By using the method of images for scalar potentials satisfying Laplace's equation, Misner was able to obtain initial geometries consisting of two asymptotically flat isometric sheets joined by N Einstein-Rosen bridges. This elegant result required the sum of contributions from an infinite number of point sources. Lindquist[8] generalized this work to include electric charge.

If we were to follow an analogous method for the time-asymmetric two-body problem, we would look for extrinsic curvature tensors $\hat{K}_{ij}$ consisting of an appropriate infinite superposition of point-source extrinsic curvature tensors. [Each of these has the form of $\hat{K}_{ij}(\vec{P})$ when we set $a=0$.] The final geometry would consist of two isometric sheets joined by two Einstein-Rosen bridges. Physical equivalence of the upper and lower sheets would require that K_{ij} be completely symmetric (K^{+}_{ij}) or antisymmetric (K^{-}_{ij}) upon inversion through either bridge. However, the potential W^i that generates $\hat{K}_{ij}$ is a vector rather than a scalar and we have not completed work on this approach.

However, there is a second method that can be used. In their analysis of the time-symmetric, conformally flat two-body problem, Brill and Lindquist[7] allowed for two separate and distinct lower sheets. Here the topology is equivalent to $R^3-\{\text{two points}\}$ and no infinite series of images is required. Solution of the momentum constraints in such a case is straightforward: We can simply superpose extrinsic curvature tensors of the type already described. For example, let there be two identical objects (1 and 2) with equal and opposite momenta and (Euclidean) separation vector $\vec{l}$. Then

$$\begin{aligned}\hat{K}^{\pm}_{ij} =& \frac{3}{2r_{(1)}^{2}}[P_i n_{(1)j}+P_j n_{(1)i}-(f_{ij}-n_{(1)i}n_{(1)j})P^k n_{(1)k}]\\ &\mp\frac{3a^2}{2r_{(1)}^{4}}[P_i n_{(1)j}+P_j n_{(1)i}+(f_{ij}-5n_{(1)i}n_{(1)j})P^k n_{(1)k}]\\ &-\{r_{(1)}\to r_{(2)},\, n_{(1)i}\to n_{(2)i}\},\end{aligned} \tag{57}$$

where

$$r_{(1)}=|\vec{r}+\tfrac{1}{2}\vec{l}|_f,\quad n_{(1)i}=(\vec{r}+\tfrac{1}{2}\vec{l})_i r_{(1)}^{-1},$$

$$r_{(2)}=|\vec{r}-\tfrac{1}{2}\vec{l}|_f,\quad n_{(2)i}=(\vec{r}-\tfrac{1}{2}\vec{l})_i r_{(2)}^{-1}.$$

Thus, the momentum constraints are solved exactly. One may then impose boundary conditions at $r_{(1)}=a$ and $r_{(2)}=a$ and solve numerically a boundary-value problem for ψ similar to that for one object. Note that the resulting initial-value problem has insufficient symmetry for it to be reduced to two spatial dimensions, so, for practical reasons, we introduce simplifications.

We can avoid having to solve a new boundary-value problem if we restrict our attention to cases in which $|\vec{P}|$ is sufficiently small, and $|\vec{l}|$ sufficiently large, so that one has, to any desired accuracy, two slowly moving Schwarzschild black holes headed for an encounter. We can create in this way, using the sum of appropriate $\hat{K}_{ij}$'s, initial data for collisions with four parameters (assuming $\vec{P}_1+\vec{P}_2=0$): M_1/M_2, $|\vec{l}|$, V_1, and $b=$impact parameter ($|\vec{l}|$ and b are fixed in the "background" Euclidean metric).

The point of the approximation we desire here is to permit the "kinetic energy" term $\hat{K}_{ij}\hat{K}^{ij}=O(V^2)$ in the constraint that determines ψ to be ignored. However, we wish to retain $\hat{K}_{ij}=O(V)$ in the momentum constraint to preserve time asymmetry and we wish to retain the "geometrostatic interaction energy"[7] between the two black holes.

Let us assume for simplicity that the two holes have equal masses. In (57), we set $P_i=MV_i$ and $a=M/2$. Then ψ satisfies, initially, Laplace's equation, for which the Brill-Lindquist solution[7] is

$$\psi=1+\frac{\lambda}{r_{(1)}}+\frac{\lambda}{r_{(2)}}. \tag{58}$$

The total energy is calculated by a surface integral on the upper sheet and one finds $E=4\lambda$. The mass M of each of the holes is found by performing a surface integration in the asymptotically flat region of one of the two lower sheets. The results are

$$M=2\lambda+2\lambda^2|\vec{l}|^{-1}, \tag{59}$$

$$\lambda=\tfrac{1}{2}|\vec{l}|[(1+2M/|\vec{l}|)^{1/2}-1]. \tag{60}$$

The geometrostatic interaction energy is therefore $\sim M^2/|\vec{l}|$ and our approximation requires that V^2 be sufficiently small relative to $M/|\vec{l}|$. Now obviously the above approximation introduces a small inexactitude in the initial data that could readily be avoided by solving the ψ equation with the $(\hat{K}_{ij}\hat{K}^{ij})\psi^{-7}$ term included. However, the approximation we are making clearly corresponds to a case of astrophysical interest, namely, two black holes that are moving slowly, in a bound state, a relatively large distance apart. There is virtually no gravitational radiation that will be produced by their spiralling, infalling collision that can have been present *initially*. Moreover, from a practical point of view, the evolution will have to be computed numerically, using the *exact* Einstein equations of motion and, therefore, no matter how precise the initial data (and there is no matter of principle whatsoever that prevents the data's being precise), there will inevitably be errors resulting from the numerical computations. It seems very likely, based on experience gained thus far in numerical general relativity, that such errors can be controlled with good precision, but that they will not be smaller than any that would result *solely* from our approximation of the initial data. If this is so, and one can achieve an accuracy of, say N per cent in numerical evolution of the full evolution equations from exact data, then in our approximation the initial separation of the holes would be $l \sim 100\ N^{-1}$ black-hole radii.

Clearly, in the problem of computing the evolution, one must make a compromise in regard to the choice of V^2 versus $M/|\vec{l}|$. Large $|\vec{l}|$ makes the initial data more nearly exact but implies the necessity of a larger grid for computations. Because this collision will involve all three spatial dimensions, a smaller grid is especially desirable. Moreover, larger $|\vec{l}|$ means a longer time before the interesting part of the collision occurs and, hence, a possibly unacceptable increase of numerical inaccuracies as the integration proceeds in time. Therefore, the "trade off" involves the fact that starting the holes off closer together, with less accurate initial data, may well result in a more accurate solution of the evolution problem and computation of gravitational radiation.

The above discussion clearly must be based on assumptions of "stability" in the evolution of initial data. One such assumption is a mathematically rigorously established form of Cauchy stability of the Einstein differential equations.[15,25] (Roughly, small changes in the data result in small changes in the resulting spacetime.) Another kind of stability refers to the finitely differenced Einstein equations and to the way in which these equations are evolved. One method is to solve the constraints and propagate them using the hyperbolic equations of motion ($\dot{g}_{ij}$ and $\dot{K}_{ij}$) together with prescriptions for the choice of lapse and shift functions. Another method periodically resolves the constraints along the way.[31] In the latter case, one will have to solve, in effect, nonlinear ψ equations as the holes near collision. All the answers to the difficult stability problems that arise are simply not known at present; moreover, our discussion is only meant to be a very rough sketch. Up-to-date discussions are available.[1,2] The calculations[1,2] performed thus far give us confidence that, for all practical purposes, the initital-value problem for a bound, spiralling collision is now solved. One expects that the evolution will show, just as in the head-on collision,[9] that the final state will be a single black hole. However, in the present case the deformations of the two holes, as they merge, rotate and settle ("ring down") to a final *Kerr* black hole, should produce a significant amount of gravitational radiation.[10]

Obviously, data for two black holes that will undergo marginally bound or unbound encounters are also of great interest. These cases will be dealt with elsewhere.

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