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Comment on the recoupled wave functions of four-quark mesons

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A phase inconsistency in Jaffe's recoupled wave functions for certain S-wave four-quark mesons is corrected.

We would like to point out a phase inconsistency in the wave functions of the $J=0$ four-quark bag states studied by Jaffe.[1,2] This problem arises because these states were constructed after two transformations of the representation, one of which was the result of dynamical mixing. Although the calculation appeared to have been done correctly at each step, the combined results contained errors due to a phase inconsistency between the two steps, as we shall show in this note. Fortunately, the problem does not affect the dynamical results of the paper concerning the masses of these four-quark states.

In Refs. 1 and 2, Jaffe studied S-wave $Q^2\bar{Q}^2$ mesons in terms of states of the SU(6) color-spin group. The SU(6) states of a pair of quarks

$$[6]\times[6]=[21]+[15] \tag{1}$$

have the following $\mathrm{SU}(3)_c\times \mathrm{SU}(2)_s$ contents:

$$[21]=(6,3)+(\bar{3},1)\,,\quad [15]=(6,1)+(\bar{3},3)\,. \tag{2}$$

The pair state will be totally antisymmetric if the symmetric color-spin multiplet [21] appears only with the antisymmetric flavor multiplet $\underline{\bar{3}}$, and if the antisymmetric [15] is associated with the symmetric flavor multiplet $\underline{6}$.

There are four S-wave $Q^2\bar{Q}^2$ states with spin $J=0$:

$$\left.\begin{array}{l}\left.\begin{array}{l}|1\rangle=|(6,3)\underline{\bar{3}};\ (\bar{6},3)\underline{3};\ (1,1)\rangle\\ |2\rangle=|(\bar{3},1)\underline{\bar{3}};\ (3,1)\underline{3};\ (1,1)\rangle\end{array}\right\}\text{ in }[21]\times\overline{[21]}\\ \left.\begin{array}{l}|3\rangle=|(6,1)\underline{6};\ (\bar{6},1)\underline{\bar{6}};\ (1,1)\rangle\\ |4\rangle=|(\bar{3},3)\underline{6};\ (3,3)\underline{\bar{6}};\ (1,1)\rangle\end{array}\right\}\text{ in }[15]\times\overline{[15]}\end{array}\right.\quad \text{of SU(6)}\,, \tag{3}$$

in Jaffe's notation. The spectroscopy of these states is determined by the gluon magnetic interaction. This interaction is proportional to the color-spin operator

$$A=-\sum_{i<j}\vec{\sigma}_i\cdot\vec{\sigma}_j\vec{\lambda}_i\cdot\vec{\lambda}_j \tag{4}$$

which is diagonal in SU(6) representations, but mixes the $\mathrm{SU}(3)_c\times\mathrm{SU}(2)_s$ states in each SU(6) multiplet. As a result, the 4×4 matrix A in the space of Eq. (3) has the form

$$A=\begin{pmatrix}B & O\\ O & C\end{pmatrix},\quad B=-16\begin{pmatrix}\frac{11}{6} & (\frac{3}{2})^{1/2}\\ (\frac{3}{2})^{1/2} & 1\end{pmatrix},\quad C=-16\begin{pmatrix}-\frac{1}{2} & (\frac{3}{2})^{1/2}\\ (\frac{3}{2})^{1/2} & \frac{1}{3}\end{pmatrix}. \tag{5}$$

Its eigenvalues and eigenvectors are

$$\begin{array}{l}\begin{array}{ll}a(0^+,\underline{9})=-43.37\,, & |0^+,\underline{9}\rangle\\ a(0^+,\underline{9}^*)=-1.97\,, & |0^+,\underline{9}^*\rangle\end{array}\Bigg\}=\begin{Bmatrix}0.8130\\ -0.5822\end{Bmatrix}|1\rangle+\begin{Bmatrix}0.5822\\ 0.8130\end{Bmatrix}|2\rangle\,;\\ \begin{array}{ll}a(0^+,\underline{36})=-19.37\,, & |0^+,\underline{36}\rangle\\ a(0^+,\underline{36}^*)=22.03\,, & |0^+,\underline{36}^*\rangle\end{array}\Bigg\}=\begin{Bmatrix}0.5822\\ 0.8130\end{Bmatrix}|3\rangle+\begin{Bmatrix}0.8130\\ -0.5822\end{Bmatrix}|4\rangle\,.\end{array} \tag{6}$$

Thus all states are 66/34 mixtures of the SU(3) $\times$ SU(2) states. The eigenvalues and the mixture probabilities are of course independent of the phase convention used for the states.

The question of phase appears when the eigenstates in Eq. (6) are expressed in the recoupled representation $(Q\bar{Q})^2$. The $SU(2)_s$ recoupling coefficients in the Condon-Shortley phase convention[3] are given by the linear coefficients of the following equations:

$$\begin{aligned} |(Q^2\bar{Q}^2)11, 1\rangle &= \tfrac{1}{2}|(Q\bar{Q})^2 11, 1\rangle + (\tfrac{3}{4})^{1/2}|(Q\bar{Q})^2 33, 1\rangle \\ |(Q^2\bar{Q}^2)33, 1\rangle &= (\tfrac{3}{4})^{1/2}|(Q\bar{Q})^2 11, 1\rangle - \tfrac{1}{2}|(Q\bar{Q})^2 33, 1\rangle . \end{aligned} \tag{7}$$

To get the correct signs of the off-diagonal matrix elements of A shown in Eq. (5), the $SU(3)_c$ recoupling coefficients are so defined that in each of the three SU(2) subsets of $SU(3)_c$ the Condon-Shortley phase convention is followed. The results are

$$\begin{aligned} |(Q^2\bar{Q}^2)\bar{3}3, 1\rangle &= (\tfrac{1}{3})^{1/2}|(Q\bar{Q})^2 11, 1\rangle + (\tfrac{2}{3})^{1/2}|(Q\bar{Q})^2 8\bar{8}, 1\rangle \\ |(Q^2\bar{Q}^2)6\bar{6}, 1\rangle &= (\tfrac{2}{3})^{1/2}|(Q\bar{Q})^2 11, 1\rangle - (\tfrac{1}{3})^{1/2}|(Q\bar{Q})^2 8\bar{8}, 1\rangle . \end{aligned} \tag{8}$$

The eigenstates in Eq. (6) can now be expressed in the recoupled $(Q\bar{Q})^2$ representation, with the results shown in Table I. They disagree with the results obtained in Ref. 1. Fortunately, there is no major qualitative change in these wave functions.

It is useful to point out how Jaffe's phase problem might have occurred. In both Refs. 1 and 2, 98% of the state $|0^+, \underline{9}\rangle$ is state $|1\rangle$ of Eq. (3), while 54% of $|0^+, \underline{36}\rangle$ is state $|4\rangle$. The correct 66% admixtures can be obtained by changing the phase in either the dynamical mixing problem in SU(6), or the algebraic SU(6) $\to$ SU(3)$\times$SU(2) transformation. Only the latter change will give the correct recoupled wave functions shown in Table I when used with the color-spin $Q^2\bar{Q}^2 \to (Q\bar{Q})^2$ recoupling coefficients of Ref. 2. We therefore conclude that Jaffe's recoupling coefficients are not phase-consistent with his SU(3)$\times$SU(2) decomposition of the SU(6) states.

TABLE I. Recoupled wave functions in the $(Q\bar{Q})^2$ representation of S-wave, $J^P = 0^+$ four-quark mesons. The symbols P, V, $\underline{P}$, and $\underline{V}$ denote a color-singlet pseudoscalar, a color-singlet vector, a color-octet pseudoscalar, and a color-octet vector $Q\bar{Q}$ meson, respectively.

Method	Flavor	PP	VV	$\underline{P}\cdot\underline{P}$	$\underline{V}\cdot\underline{V}$
Here	$\underline{9}$	0.743	−0.041	−0.169	0.646
	$\underline{36}$	0.644	0.177	0.407	−0.623
	$\underline{9}^*$	−0.177	0.644	0.623	0.407
	$\underline{36}^*$	0.041	0.743	−0.646	−0.169
Jaffe (Ref. 1)	$\underline{9}$	0.743	0.328	−0.432	−0.393
	$\underline{36}$	−0.644	0.269	−0.322	−0.639
	$\underline{9}^*$	0.178	−0.556	0.479	−0.655
	$\underline{36}^*$	0.041	0.715	0.692	−0.089

Dynamical mixing occurs also for the four $J=1$ states in the $[21]\times[\overline{15}]$ and $[15]\times[\overline{21}]$ multiplets of SU(6). For these states our method gives the same recoupled wave functions in the $(Q\bar{Q})^2$ representation as those shown in Table V of Ref. 1. It is nevertheless interesting to note that the original states in the $Q^2\bar{Q}^2$ representation are 67/33 admixtures, not the 7/93 admixtures which might have been deduced from Ref. 2. In addition, there are probably two sign misprints in the second crossing matrix for spins shown in Table V of Ref. 2.

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[1]R. L. Jaffe, Phys. Rev. D 15, 267 (1977).
[2]R. L. Jaffe, Phys. Rev. D 15, 281 (1977).
[3]E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1959), p. 48.