Multiparticle production by bubbling flux tubes

A. Casher, H. Neuberger,* and S. Nussinov
Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel
(Received 15 June 1979)

The chromoelectric-flux-tube model is realized in 1 + 1-dimensional field theory. Multiparticle production is represented by a cascade of bubbles of true vacuum created by tunneling within flux tubes.

I. INTRODUCTION

A widely accepted conjecture is that quarks interact at large distance through linearly rising potentials. This assumption leads to reasonable spectroscopic predictions and is very appealing as a means of quark confinement. In the framework of quantum chromodynamics (QCD) linear potentials may be acheived as a result of a nonperturbative behavior of chromodynamics at large distances: The field lines are assumed to get collimated in tubes of constant width. Thus, free quarks must carry infinitely long tubes which cost an infinite amount of energy. This picture is supported by lattice formulations of QCD.

These tubes, if they exist, can in fact be the major nonperturbative factor which establishes the gross features of hadronic reactions. The constant field within the tubes can create new pairs of quarks, thus changing the structure of the interacting hadronic matter.[1]

We now proceed to a very brief outline of the chromoelectric-flux-tube model set up in Ref. 1. The basic assumptions and approximations of the model are listed below:

(1) The relevant scale of the processes under consideration is such that quarks may be treated as massive Dirac particles. The relevant masses are the "constituent" masses ($m_u = m_d = 350$ MeV, $m_s = 500$ MeV).

(2) In a $q\bar{q}$ system confinement is implemented through the generation of a chromoelectric flux tube of universal thickness for which the quark and antiquark act as source and sink. If g is the strong coupling constant and Λ the radius of the tube, one finds with the aid of Gauss's law that the field E is given by $E = g/(2\pi\Lambda^2)$. Constant forces imply linear Regge trajectories. If the Regge slope is α', then

$$\tfrac{1}{2}gE = \frac{1}{\pi\alpha'}\,. \tag{1}$$

(3) The unique process that is treated quantum mechanically is the creation of a pair by the chromoelectric field within the tube. This field is treated as a c-number external source in Dirac's equation. Interactions between particles created in this way are neglected until a point is reached where the members of the pair are subjected in turn to the confinement hypothesis. Then they screen the field which created them.

Clearly the model is well suited to deal with e^+e^- annihilation. The basic information needed in order to be able to make quantitative predictions is the probability of pair creation at a given transverse momentum. For a uniform field which fills all space, it is known that[1,2]

$$\lim_{T\to\infty} |\langle \text{vac}|e^{-iHT}|\text{vac}\rangle|^2 \sim \exp\left\{VT\frac{gE}{8\pi^3}\int d^2p_T \ln\left[1-\exp\left(-\frac{2\pi(m^2+p_T{}^2)}{gE}\right)\right]\right\}. \tag{2}$$

V is the three-dimensional volume of the world and m is the quark mass. The probability for pair creation may be read off as

$$P(p_T)\,d^2p_T = -\frac{gE}{8\pi^3}\ln\left[1-\exp\left(-\frac{2\pi}{gE}(m^2+p_T{}^2)\right)\right]. \tag{3}$$

The integrated probability (per unit time and per unit volume) to create a pair of quarks of mass m is

$$P_m = \frac{g^2E^2}{16\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-(2\pi m^2/gE)n}\,. \tag{4}$$

This model has been used successfully to predict various features of e^+e^- hadronic annihilation.[1] These include the jet structure and suppression of strangeness and baryon production. Also, the model gave a reasonable estimate for the lifetime of excited mesons. We still lack, however, a complete picture of the cascade space-time development. The aim of this work is to provide such a description.

We will focus our investigation on the time development of the system in the longitudinal direction of the flux tube. Although the model has provided a phenomenologically valid picture for the transverse structure of the cascade, we will forget, for simplicity, about the directions perpendicular to the tube axis. We believe that within our approximations one may treat the flux-tube model as if the transverse and longitudinal directions were decoupled. Specifically we present a $1+1$-dimensional field-theoretical model. This model manifests both correct relativistic kinematics as developed for stringlike models[3] and an underlying dynamic instability due to pair production analogous to that of the genuine flux-tube model.

The program of the paper is as follows. In Sec. II we present the model using the basic field variables. Section III deals with the bosonized version of the decay of the tube by pair production. The cascade of particles is described in Sec. IV and a short concluding discussion is given in the last section.

II. CHOICE OF THE MODEL

In our picture for particle creation the dynamics is essentially one dimensional. The influence of the transverse degrees of freedom may be incorporated in an effective mass term. We are thus led to search for a one-dimensional relativistic field-theoretical model which realizes the chromoelectric-flux-tube picture. The fact that we are interested in one space dimension has a great advantage; the Coulomb force now automatically confines and there is no need for an *ad hoc* assumption about the existence of flux tubes. Obviously, the screening mechanism also operates in an automatic manner. However, we should remark that a one-dimensional model confines at all scales, and in this respect it is very different from the real world.

From what we said above, it is clear that what we have in mind is the massive Schwinger model: QED in $1+1$ dimensions. The model is defined by the Lagrangian[4]

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{5}$$

We are going to use semiclassical approximations which are reliable when the barrier penetration factor is small, that is when the mass of the fermion is large compared to the force constant of the flux tube. This means that we are interested in the extremely massive case:

$$m/e \gg 1\,. \tag{6}$$

Equation (6) may have a deeper significance. We present the following conjecture: Assume that the functional average of the Wilson loop operator in the gluonic vacuum is exactly given by the exponent of the minimal area enclosed by the curve (linear confinement at all scales):

$$\left\langle \exp\left(i \oint A_\mu dx^\mu\right)\right\rangle = \exp(-\alpha A_{\min})\,. \tag{7}$$

Suppose now that we have an initial state which consists of two very energetic quarks receding from a given point in space. The conjectured claim is that the subsequent development of the system, in the limit of infinite quark mass ($m_q^2/\alpha \to \infty$), will be effectively governed by the massive Schwinger model [Eq. (5)]. This means that in this limit we may neglect the effects of the transverse directions and the non-Abelian character of QCD. This is not unreasonable because we expect the transverse momentum to be cut off by a finite number (which depends on α only) and therefore the transverse velocity of the quarks goes to zero. Production of quarks of "perverse" color, i.e., the production of baryons, will be suppressed by a factor that vanishes exponentially when $m/e \to \infty$. Thus the other color degrees of freedom may decouple and we are left with an essentially Abelian theory. Note also that gluon-bound-state production is forbidden *a priori* by hypothesis since Eq. (7) neglects area fluctuations.

Before entering the main subject we would like to discuss the question why we did not choose the exactly soluble Schwinger model[5] (massless QED in $1+1$ dimensions) as a prototype of multiparticle production. In fact it is well known that some encouraging results may be obtained by employing this model, and it has been claimed that these results may be understood in a semiclassical framework.[3,6] The reasons (all of which are somehow connected to the well-known infrared problems in one space dimension) for not considering this model as a candidate for our picture are as follows:

(a) In massless QED in $1+1$ dimensions the Higgs mechanism operates, and therefore the system reacts to stationary external charges like a plasma. External charges will be shielded irrespectively of their value. If something analogous happened in QCD we would have free particles carrying fractional electromagnetic charges.

(b) The vacuum of the massless Schwinger model is not "empty," it is filled with pairs as signalled by the nonvanishing vacuum expectation value of $\bar{\psi}\psi$. In fact these pairs are responsible for the plasmalike reaction to external charges as mentioned in (a). The description of the time development of a system in terms of the coordinates of the bare particles is very complicated.

(c) Massless fermions bound by linear forces may lead to the spontaneous breakdown of chiral symmetry.[7] Indeed, as explained in Ref. 7, if we tried to describe a boson by a pair of massless quarks bound by a constant force, we would run into trouble because at the classical turning points local chiral invariance would have to be broken. But then the meson one-particle state is to be described in the massless fermions' language as a quark pair superimposed on the filled vacuum. In $3+1$ dimensions we expect this state to be described in an approximate but very economic manner by a pair of massive "constituent" quarks.

Hence, the treatment of massive $1+1$ QED seems more appropriate.

III. BOSONIZED VERSION OF QUARK PAIR PRODUCTION

A technical advantage of working in one dimension is the fact that we will not need to do semiclassical approximations involving fermions. In one space dimension there exists a transformation ("bosonization") which turns any fermion field theory into an equivalent boson field theory.[8] The bosonized form of the renormalized Hamiltonian corresponding to Eq. (5) is

$$\mathcal{H}=N_\mu[\tfrac{1}{2}(\partial_0\phi)^2+\tfrac{1}{2}(\partial_1\phi)^2+\tfrac{1}{2}\mu^2\phi^2 -cm\mu\cos(2\sqrt{\pi}\phi-\theta)]\,, \tag{8}$$

where N_μ denotes normal ordering with respect to a mass μ ($\mu^2=e^2/\pi$), c is a numerical constant, and θ is an angle which defines the electric field at infinity. The following relations hold:

$$\begin{aligned} :\bar\psi\psi: &= -cmN_m\cos 2\sqrt{\pi}\phi = -c\mu N_\mu \cos 2\sqrt{\pi}\phi\,,\\ :\bar\psi\gamma^\mu\psi: &= \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi = j^\mu\,, \end{aligned} \tag{9}$$

where :: denotes Fermi normal ordering. The electric field F_{01} is given by

$$F_{01}=\frac{e}{\sqrt{\pi}}\left(\phi+\frac{1}{2\sqrt{\pi}}\theta\right). \tag{10}$$

In what follows we will always consider the C- and P-invariant case $\theta=0$. If e were zero, the quarks would be free fermions. The mass of the fermions should be, according to Eq. (5), given by m. These free fermions appear also in the bosonized version [Eq. (8)] of the model when the quadratic term is neglected. These are the well-known solitons of the sine-Gordon equation. Their mass, in the zero-order semiclassical approximation depends on our renormalization convention. The Hamiltonian in Eq. (8) defines the properly subtracted theory which, if solved exactly, would give for the soliton mass the value m. In the WKB method employed by DHN,[9] one cannot calculate the soliton mass exactly. This WKB method, as any other perturbative scheme starts off with the unrenormalized form of the Lagrangian from which one gets the zero-loop value for the fermion mass. One may define the regularization and substraction procedures perturbatively in the loop expansion by the requirement that the soliton's mass should stay at the correct value, m, order by order. We therefore take our unrenormalized boson theory to be such that the classical mass of the soliton is equal to the exact value m. With this definition of renormalization, we are sure that the results of the, practically impossible, summation of the perturbation series will be the same as those obtained from the Hamiltonian in Eq. (8).

The unrenormalized Hamiltonian of the boson theory is given by ($\theta=0$)

$$\mathcal{H}_u=\tfrac{1}{2}(\partial_0\phi)^2+\tfrac{1}{2}(\partial_1\phi)^2+\tfrac{1}{2}\mu^2\phi^2-\tfrac{1}{16}\pi m^2\cos(2\sqrt{\pi}\phi)\,. \tag{11}$$

This gives the solitons a mass

$$M_{\rm sol}=m\,. \tag{12}$$

The fact that we have chosen our renormalization convention in such a way that the zero-loop term gives the exact answer does not mean that the WKB evaluation of the soliton mass is a good approximation. In fact it is a bad approximation, and this is indeed expected since the WKB expansion parameter is very large for the case of free fermions ($2\sqrt{\pi}$).[9] However, for the mechanism of pair production we will be dealing with a WKB expansion parameter which will be μ/m, which we take to be small.

The potential associated with the unrenormalized version of Eq. (8) is, for $\theta=0$, given by

$$\begin{aligned} V(\phi)&=\tfrac{1}{2}\mu^2\phi^2-\alpha^2\cos(2\sqrt{\pi}\phi)+C\,,\\ \alpha^2&=\tfrac{1}{16}\pi m^2 \end{aligned} \tag{13}$$

(see Fig. 1). The minimum of the potential is lo-

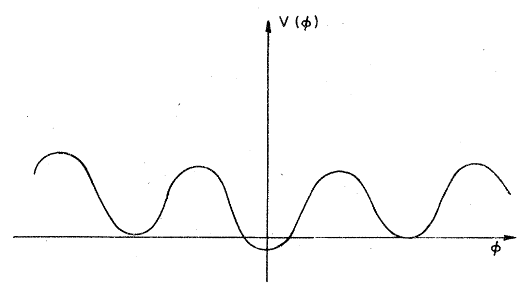

FIG. 1. The potential of the boson model.

cated at $\phi=0$. The introduction of an external point charge Q forces ϕ to change by a discontinuous jump of $\sqrt{\pi}Q$ [see Eq. (9)]. If the charges are not external, but the true intrinsic particles of the theory, we have in the zero-loop approximation a field configuration given by the stationary solution which represents the soliton of the noninteracting theory. Thus, our initial state of two receding quarks corresponds in the Bose language to a fast-growing segment of space where ϕ has the value $\sqrt{\pi}$ (see Fig. 2). In this region $V(\phi)$ is approximately at a local minimum. The constant C in Eq. (13) has been chosen in such a way that $V=0$ in this segment of space. This minimum is not absolute because the discrete symmetry $\phi \to \phi + n\sqrt{\pi}$, which was present in the noninteracting theory, has been softly and weakly broken by the mass term.

In the region between the two receding quarks our one-dimensional world is in a false vacuum. This state is characterized by the existence of a constant electric field [see Eq. (10)]:

$$F_{01} \equiv E = e = \mu\sqrt{\pi}\ . \qquad (14)$$

The flux tube thus generated has, in our approximation [Eq. (6)], a very low-energy density, and therefore it describes a narrow metastable state. The width of the state may be calculated by WKB methods. In four dimensions one may consider an analogous boson problem. Of course, this problem has little to do with quarks and flux tubes, but it bears some resemblance to a baglike picture of hadrons. In the zero-loop order, Coleman[10] has explained what the fate of such a false vacuum will be. How to calculate the first loop correction has been shown in a subsequent paper.[11] The false vacuum decays by producing bubbles of true vacuum (see Fig. 3). For the case when the false vacuum has been manufactured by a very weak breaking of a discrete symmetry originally present, the wall which separates the inside of the bubble from its outside is very thin. In this "thin-wall approximation" all we need in order to perform the zero-loop computation is the stationary solution which connects the two vacua of the unbroken theory. In one dimension the bubble is a segment and the walls are its ends. The stationary solution represents the soliton, and therefore the bubbles are quark pairs.

We now reduce Coleman's treatment of the thin-wall case to one dimension. Within our approximation the relevant contributions to the Euclidean path integral are dilute configurations of bubbles. The shape of each bubble in Euclidean space-time is spherically symmetric[12] around its center. The radial dependence of the field is governed by the Euclidean equations of motion (of our field-theory model) applied to spherically symmetric functions. In our case, if ρ [$\rho=(x^2+t_E{}^2)^{1/2}$] is the radial Euclidean distance from the center of the bubble, we have

$$\frac{d^2\phi}{d\rho^2}+\frac{1}{\rho}\frac{d\phi}{d\rho}=V'(\phi)\,, \qquad \phi(\infty)=\phi_+\,, \qquad (15)$$

where ϕ_+ is the value of ϕ in the false vacuum [$\phi_+=\sqrt{\pi}+O(\mu/m)$]. Defining $x=\rho\alpha$ and $\epsilon=\mu/\alpha$, we get

$$\frac{d^2\phi}{dx^2}+\frac{1}{x}\frac{d\phi}{dx}=\epsilon^2\phi+2\sqrt{\pi}\,\sin 2\sqrt{\pi}\,\phi\ . \qquad (16)$$

For $\epsilon \ll 1$ we have the following approximate solution:

$$\phi(x)=\begin{cases}0, & x\ll r\\ f_s(x-r), & x\sim r\\ \phi_+, & x\gg r\,.\end{cases} \qquad (17)$$

The function f_s is just the sine-Gordon soliton

$$f_s(x)=\frac{2}{\sqrt{\pi}}\tan^{-1}[\exp(2\sqrt{\pi}x)]\,. \qquad (18)$$

The parameter r in Eq. (17) may be fixed with the help of the variational principle for the Euclidean action,

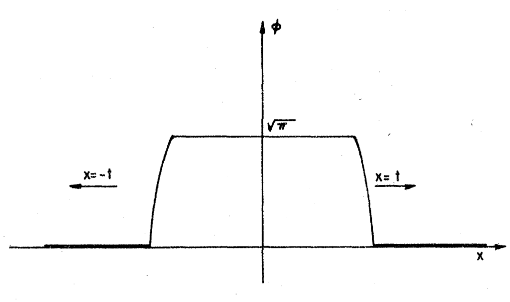

FIG. 2. The boson-field configuration of the initial state.

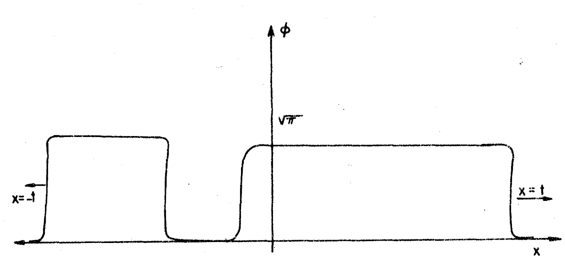

FIG. 3. The initial state plus a bubble of true vacuum.

$$S_E = 2\pi \int_0^\infty \rho\left[\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + V(\rho)\right] d\rho$$
$$\approx -\frac{\pi^2 r^2 \epsilon^2}{2} + 8\sqrt{\pi} r\,,$$
$$\frac{dS_E}{dr} = 0 \Rightarrow r = \frac{8}{\pi\sqrt{\pi}}\epsilon^{-2}\,. \qquad (19)$$

The radius of the bubble, R, is thus

$$R = \frac{8}{\pi\sqrt{\pi}\mu}\epsilon^{-1} = \frac{2m}{\pi\mu^2}\,. \qquad (20)$$

The value of the classical action at extremum is

$$S_0 = \frac{32}{\pi\epsilon^2}\,. \qquad (21)$$

It is very easy to see that these results are essentially equivalent to our treatment of the pair-production problem in Ref. 1. The variational equation for r ensures that the amount of volume energy gained by the bubble is equal to the amount of surface energy lost by it. Since the walls are solitons and they are at rest at $\rho = R$ (the turning points of the classically forbidden trajectory), we know that the walls cost an amount equal to $2M_{sol}$. In the zero-loop approximation we therefore get

$$2M_{sol} = 2R(\tfrac{1}{2}E^2)\,. \qquad (22)$$

Using Eqs. (11) and (14) we find

$$R = \frac{8}{\pi\sqrt{\pi}\mu}\epsilon^{-1}\,, \qquad (23)$$

which agrees with Eq. (20). Clearly $\frac{1}{2}E^2$ is the energy density in the flux tube denoted by k in Ref. 1. The analog of Eq. (18) in Ref. 1 would be

$$2R = \frac{M_{sol}}{k}\,. \qquad (24)$$

The discrepancy between Eqs. (24) and (22) consists of a factor of 2. This is the reflection of the fact that we are dealing with an interacting theory now. The electric field is not external; half of it is contributed by each end quark. Therefore, only half of it acts like an external field. Equation (21) may be rewritten in terms of k ($k = \frac{1}{2}E^2$) and M_{sol},

$$S_0 = \frac{\pi M_{sol}^2}{k}\,. \qquad (25)$$

The width per unit length of the metastable state (Γ/L) has been shown by Coleman[10] to be given by

$$\Gamma/L = Ae^{-S_0/\hbar}[1 + O(\hbar)]\,. \qquad (26)$$

This means that the probability for a region of length L to stay in the false vacuum for a time T is given by

$$P = \exp(-LTAp')\,,$$
$$p' = \exp(-S_0) \quad (\hbar = 1)\,. \qquad (27)$$

The expression for p' should be compared to Eq. (2) with zero transverse momentum. There is a discrepancy of a multiplicative factor of 2 in S_0. The origin of this factor is the same as in the case of the evaluation of the bubble radius.

The computation of A and the related problem of renormalization are more complicated matters which will not be undertaken in this work. We are able to guess what A should be by a line of reasoning similar to that employed in Ref. 1 for pair production in an external electric field. We should correct for the dimensionality of space (absence of transverse degrees of freedom and spin) and by a factor of $\frac{1}{2}$ in the field strength to take into account that we are treating an interacting case. Our renormalization procedure was designed in such a way that it seems reasonable that no changes will occur in the exponent in Eq. (26),

$$A = \frac{1}{2}\frac{k}{\pi} = \frac{1}{4}\mu^2\,. \qquad (28)$$

For an external field E the result would be just $A = eE/2\pi$. This is indeed true as may be checked by solving the problem exactly in the Fermi language. We should remark that in this exactly soluble fermion problem the WKB evaluation of A in the boson language looks very similar to our problem and seems rather complicated.

We may summarize our findings until now. The false vacuum is an unstable state which decays by producing quark pairs (bubbles). The width density is given by

$$\gamma = \frac{\Gamma}{L} = \frac{k}{2\pi}\exp\left(-\frac{\pi m^2}{k}\right). \qquad (29)$$

The subsequent time development of the bubble is given in the classical approximation by the analytic continuation to Minkowski space of the solution presented in Eq. (17). This means that the walls will expand on the hyperbola

$$x^2 - t^2 = R^2\,. \qquad (30)$$

The velocity, $v = dx/dt$, of the quark satisfies

$$\frac{d}{dt}\frac{V}{(1 - V^2)^{1/2}} = \frac{1}{R}\,. \qquad (31)$$

Formula (31) is the relativistic equation of motion under the influence of a constant force of strength $M/R = \frac{1}{2}E^2 = \frac{1}{2}e^2$. Indeed, this is the force which acts on each end quark. The motion of the pair is described by the world lines given by the branches of the hyperbola $x^2 - t^2 = R^2$.

Our semiclassical calculations have estimated the rate for the observable transition false vacuum → expanding bubbles. The semiclassical trajectory should not be considered to give a good approximation for the time development of the system right through the barrier. The system does (in some sense) behave according to classical laws only far from the turning points of the classical motion. Thus, we should not claim that an observer will see a bubble appearing at some place with a length (diameter) $2R$ and zero velocity walls which start to expand. This clearly cannot be true: A boosted observer would claim to see something different, in spite of the fact that the initial state is boost invariant and therefore looks the same. Since the observer will see a pair of particles moving on the hyperbola $x^2 - t^2 = R^2$, he will be able to state where the center of the bubble was. The relative simultaneous velocity of the members of the pair is something we have no prediction for, and we should not, because this is frame-dependent, irrelevant information.

IV. A CASCADE OF BUBBLING FLUX TUBES

Using the results of the previous section we will now set up a complete semiclassical picture for the time development of the system of two very energetic fermions that recede from a given point. This will result in a one-dimensional model for particle production. We believe that something rather similar happens in the real world. Our picture is very similar to one-dimensional string models which have been treated in the literature.[3] The main new aspect is that our model contains the dynamics of string fission, whereas the string-model treatments we refer to are essentially kinematic and rest on *ad hoc* assumptions. It is paradoxical to note that what was considered to be a model of a string with massless ends is in fact, in the central region, a model of a field theory with extremely massive fermions.

Let us now turn to our case. We will work within the adiabatic approximation (see Ref. 1). We know that the influence of finite times, though not negligible, has no dramatic effects.[1] From Sec. III we may conclude that the probability of a region of space-time having a total 2-volume V_2 to stay in the false vacuum (that is, to be occupied by flux) is given by $\exp(-\gamma V_2)$, where γ is evaluated in Eq. (29). When the false vacuum decays, a bubble of zero electric field is formed. The center of the bubble is a well-defined observable quantity. We are now dealing with a pair of highly energetic quarks (how large their energy is assumed to be will be seen later on) which depart with the velocity of light (almost) from the origin. In between a flux tube is spanned. In the c.m. frame of these quarks (like in any other frame) there will be a bubble which is produced first (what we mean is that the center of this bubble has the lowest value for its time coordinate). We can easily calculate when this will happen on the average[1]:

$$\langle t\rangle = \int_0^\infty \exp(-\gamma t^2)dt = (\pi/4)^{1/2}\gamma^{-1/2}$$
$$= 0.88\gamma^{-1/2}. \qquad (32)$$

Were the end quarks stationary we would know that in the limit of weak fields the dilute-gas approximation holds and we will have a dilute set of bubbles produced. Their centers will be uniformly distributed in Euclidean space-time and their density will be given by γ. In Minkowski space-time their centers should be spacelike with respect to each other located at an average spacelike distance $\gamma^{-1/2}$. The fact that the borders of the false vacuum region are nonstationary in our case will be taken into account as influencing directly only the total two space-time volume spanned by the flux tube. Therefore, in Minkowski space we have a time-ordered (this is a frame-dependent ordering) set of points each avoiding the future light cone opened by the previous ones. All points are confined to be within the future light cone of the origin. The pairs of quarks which are the walls of the bubbles centered at these points move on hyperbolas which are asymptotic to the above mentioned light cones. This averaged picture of the process is presented in Fig. 4. Clearly, the various points are situated on the hyperbola

$$t^2 - x^2 = \langle t\rangle^2. \qquad (33)$$

The end quarks will leave their straight world line when their velocity drops below 1. This would happen when all but a finite part of their kinetic energy has been turned into field energy. If $s^{1/2}$ is the total c.m. energy of the system and we assume that we are in the c.m. frame, the end quarks will be reached by the chasing bubble walls (neutralization) after a time T,

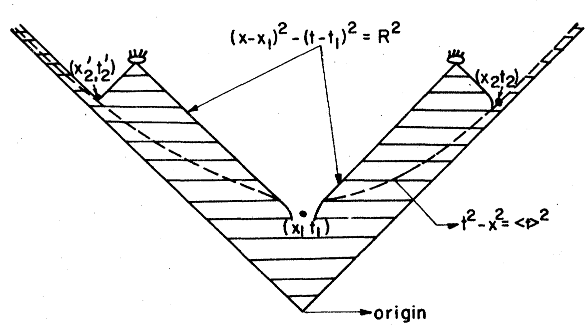

FIG. 4. Space-time picture of the cascade. The shaded area is the space-time region occupied by flux.

$$T \approx \frac{1}{2k} s^{1/2} , \quad (34)$$

where k is the energy density in the tube. In order to have many bubbles produced we assume

$$T^2 \gg \gamma^{-1} ,$$
$$s \gg k^2 \gamma^{-1} \sim e^{1/\epsilon} , \quad (35)$$

where ϵ is our small parameter.

Consider now two adjacent bubble centers. Their locations, parametrized by the hyperbolic angles $\theta_{1,2}$ are

$$x_{1,2} = \langle t \rangle \sinh\theta_{1,2} ,$$
$$t_{1,2} = \langle t \rangle \cosh\theta_{1,2} . \quad (36)$$

The spacelike distance of these points is

$$d = [(\Delta x)^2 - (\Delta t)^2]^{1/2} = 2\langle t \rangle \sinh\tfrac{1}{2}\Delta\theta . \quad (37)$$

In Euclidean space d should be analytically continued into the average spacing between bubbles, and therefore $d = \gamma^{-1/2}$. We therefore find that the bubble centers on the hyperbola are uniformly distributed in the hyperbolic angle. We have one bubble per angle interval $\Delta\theta$:

$$\Delta\theta = 2 \sinh^{-1}(\pi^{-1/2}) = 1.08 . \quad (38)$$

Since the total θ interval is $2 \ln\sqrt{s}$, we find that the total number of bubbles is

$$\langle N \rangle \underset{s\to\infty}{\simeq} 0.93 \ln s . \quad (39)$$

As a consistency check we now calculate the amount of space-time volume occupied by field per bubble. We consider still the two adjacent points from Eq. (36) and assume that the bubble walls move on straight lines (this approximation is very good in the central region if $\epsilon \ll 1$). The total shaded area in Fig. 5, S, is given by

$$S = \tfrac{1}{2}\langle t \rangle^2 (e^{\Delta\theta} - 1) = 0.97\langle t \rangle^2 = 0.76\gamma^{-1/2} . \quad (40)$$

The fact that our result is somewhat less than $\gamma^{-1/2}$ may be understood as a result of calculating only the area actually filled by field, that is, we took into account the influence of the bubbles already present. Nevertheless, we may guess that the coefficient of $\ln s$ in Eq. (39) is not overestimated.

Up to now we avoided the discussion of what happens when the walls of adjacent bubbles collide. In one-space dimension, and especially in our case, this is much easier than in the analogous three-dimensional problem. The weak interaction between the sine-Gordon solitons may be neglected during the collision, and semiclassically we know that the solitons will emerge from the scattering region with unchanged velocities. No new particles will be produced. In fact, this result of the infinite set of conservation laws of the sine-Gordon system survives quantization. The subsequent motion of the pair of solitons is that of an oscillating bound system. This system does not interact with the rest of the world. The space-time picture of this pair is shown in Fig. 6. This mode of motion has been dubbed the "yo-yo" mode.[3]

The pair has created a cluster. The cluster may decay but all its decay products are causally related to the point where the cluster was formed. The decay products will carry observable information which relates them back to the parent cluster. This is very different from the relations between different clusters. Their formation points are relatively spacelike and their distribution is therefore uncorrelated. We remark that the clusters resemble bound states for the following reason: The space-time area they span within one oscillation is easily seen to be less than γ^{-1}. Therefore, the system will oscillate more than once, on the average, before a decay can occur.

We want now to calculate the mass and rapidity of the clusters. This is a kinematic problem which has been solved in the work of Andersson *et al.*[3] The mass of the cluster M_c is given by its energy E_c and its momentum P_c,

$$M_c^2 = E_c^2 - P_c^2 . \quad (41)$$

E_c and P_c may be evaluated by essentially geometrical considerations,

$$E_c = k\langle t \rangle (\sinh\theta_2 - \sinh\theta_1) ,$$
$$P_c = k\langle t \rangle (\cosh\theta_2 - \cosh\theta_1) . \quad (42)$$

Using Eq. (38) we find

$$M_c = k\gamma^{-1/2} \quad (43)$$

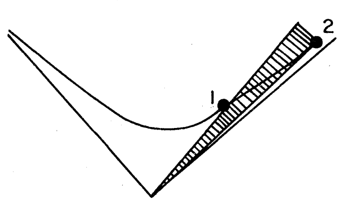

FIG. 5. The space-time region of field associated with two adjacent bubbles.

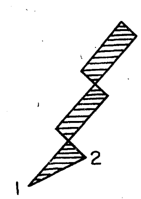

FIG. 6. The space-time picture of oscillating pairs. The "yo-yo" mode.

and

$$P_c = k\gamma^{-1/2}\sinh\bar{\theta}\,,$$
$$\bar{\theta} = \tfrac{1}{2}(\theta_1 + \theta_2)\,. \qquad (44)$$

We see that the hyperbolic angles $\theta_{1,2}$ linearly define the rapidity of the clusters. Since the number of clusters is equal to the number of bubbles and the latter were produced uniformly in the hyperbolic angle, we see that the clusters are uniformly spaced in rapidity. From Eq. (38) we find the height of the plateau,

$$\frac{dN_c}{dy} = \frac{1}{1.08} = 0.93\,. \qquad (45)$$

We also see from Eq. (44) that the plateau is filled, and this is true for any moderately boosted frame, starting from the middle. Thus our picture gives an "inside-out" cascade.

This is as far as we would like to go with our description of the cascade. We would like to stress again that most of the observable features were derived on kinematical grounds only and they appeared in previous treatments of string models (with massless ends) in one space dimension. Our contribution is in setting up a dynamical scheme, within relativistic quantum field theory, where, with suitable approximations, the string-model picture for particle production may be shown to hold.

V. DISCUSSION

Our treatment of the two-dimensional model was incomplete. First of all, we did not actually compute the one-loop correction. Furthermore, the relations between the semiclassical computation and observable quantities were established on the basis of somewhat heuristic considerations. The standard and much better way to extract observable information in such cases is to express everything of interest in terms of Green's functions and to use the semiclassical methods to approximate those. We should remark that, in addition to the more complex technicalities with which such calculations are involved, one may encounter some difficulties in the way of interpreting the results which are related to the analytic continuation that connects Euclidean to Minkowski space.

Despite the rough nature of our treatment it is hard to resist the temptation to extrapolate blindly the results of the previous section to the real world. The fact that in reality the (constituent) mass of the quarks is not very large with respect to the energy density in the flux tube should have little effect. This is supported by the observation that the numerical value of the infinite sum in Eq. (4) is well approximated by the first term. From Eqs. (43) and (32) we find that the mass of the clusters may be written as

$$M_c = 2\pi^{-1/2}k\langle t\rangle\,. \qquad (46)$$

This is mainly a result of the kinematics of the model. Using the numbers from Table I of Ref. 1 for $\langle t\rangle$ and the value $k = 0.177$ GeV^2,[1] we get

$$M_c \simeq 1.0 - 1.3 \text{ GeV}\,. \qquad (47)$$

If we tried to take into account the transverse motion of the quarks, we should consider Eq. (47) as the value of the "transverse" mass of the cluster. Therefore, the actual value of the cluster mass is lower by 10% approximately (taking $\langle p_T^2\rangle_{\text{quark}} \sim 0.1$ GeV^2 [1]).

From Eq. (45) we may infer that on the average 0.9 clusters are produced per unit rapidity interval. Our results about the clusters are in good agreement with CERN ISR results.[13] For e^+e^- annihilation, on the other hand, these numbers would imply that

$$\langle N_{\text{ch}}\rangle_{e^+e^-} \underset{s\to\infty}{\simeq} c \ln s\,,$$
$$c \sim 1.5\,, \qquad (48)$$

where we assume that 1.5 charged particles emerge per cluster on the average. The coefficient of the logarithm is larger by a factor of about 1.5 to 2 than the fits to data up to 8 GeV approximatively.[14] More recent results from Pluto and Tasso[15] seem to indicate a much steeper slope, which might result in c being even larger than 1.5.

ACKNOWLEDGMENTS

We would like to thank Andre Neveu for a very important and helpful comment. We are indebted to T. Banks and Y. Dothan for many discussions. This work has been supported in part by the United States–Israel Binational Science Foundation (BSF) Jerusalem, Israel.

---

*Present address: University of California, Berkeley, Department of Physics, Berkeley, Calif. 94720.

[1]A. Casher, H. Neuberger, and S. Nussinov, Phys. Rev. D 20, 179 (1979); H. Neuberger, *ibid.* 20, 2936 (1979). The second paper contains the tables referred to in the text.

[2]W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936); J. Schwinger, Phys. Rev. 82, 664 (1951); E. Brezin and C. Itzykson, Phys. Rev. D 2, 1191 (1970).

[3]B. Andersson, G. Gustafson, and C. Peterson, Z. Phys.

C1, 105 (1979); B. Andersson and G. Gustafson, University of Lund Report No. LU TP 79-2, 1979 (unpublished); X. Artru and G. Mennessier, Nucl. Phys. B70, 93 (1974); T. Toriu, Prog. Theor. Phys. 61, 218 (1979).

[4]S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. (N.Y.) 93, 267 (1975); S. Coleman, *ibid.* 101, 233 (1976).

[5]J. Schwinger, Phys. Rev. 128, 2425 (1962); J. Lowenstein and A. Swieca, Ann. Phys. (N.Y.) 68, 172 (1971).

[6]A. Casher, J. Kogut, and L. Susskind, Phys. Rev. D 10, 732 (1974); J. Kogut and D. K. Sinclair, *ibid.* 10, 4181 (1974).

[7]A. Casher, Phys. Lett. 83B, 395 (1979).

[8]S. Coleman, Phys. Rev. D 11, 2088 (1978); S. Mandelstam, *ibid.* 11, 3026 (1975); J. Kogut and L. Susskind, *ibid.* 11, 3594 (1975).

[9]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4130 (1974).

[10]S. Coleman, Phys. Rev. D 15, 2924 (1977).

[11]C. Callan and S. Coleman, D 16, 1762 (1977).

[12]S. Coleman, V. Glaser, and A. Martin, Commun. Math. Phys. 58, 211 (1978).

[13]R. E. Diebold, in *Proceedings of the 19th International Conference on High Energy Physics, Tokyo, 1978*, edited by S. Homma, M. Kawaguchi, and H. Miyazawa (Phys. Soc. of Japan, Tokyo, 1979), p. 666.

[14]G. G. Hanson, SLAC Report No. SLAC-PUB-2118, 1978 (unpublished).

[15]Data presented at the Summer Institute on Particle Physics, SLAC, 1979 (unpublished).