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SU(5) × U(1) phenomenology: Theorems on neutral-current analysis

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We embed the SU(5) unified theory of Georgi and Glashow in a U(5) theory. This may result from the breaking of an SU(N), $N>5$, theory or of a GL(5,c) theory. At low energy this leads to an SU(2) × U(1) × U(1) electroweak theory. We show that, with a suitable choice of Higgs representations, the predictions of this theory for neutral-current experiments are characterized by three parameters. For appropriate values of these parameters, the predictions are practically indistinguishable from the standard SU(2) × U(1) theory. Certain theorems on the analysis of neutral-current interactions are proved. (Section V is independent for readers who are interested only in the theorems.) More accurate neutral-current measurement might answer the question of whether SU(5) × U(1) is relevant. Possible verification of the present electroweak theory can result from (roughly) an order suppression relative to the standard prediction on the asymmetries in $e^+e^-\to\mu^+\mu^-$ and discovery of two Z bosons around 90–100 GeV. GL(n,c) gauge theories are formulated in the Appendix.

I. INTRODUCTION

Much of modern physics is concerned with the search for ever larger symmetries. In the last decade or so we have learned that even badly broken symmetries are still useful provided that the breaking is soft. Thus, for instance, electromagnetism has been found to be part of an SU(2) × U(1) electroweak theory[1] which in turn may be only part of an SU(5) unified theory[2] of strong, weak, and electromagnetic interactions. However, it would surprise us if the SU(5) theory, in fact, describes the real world in spite of its numerous successes. There are indications that SU(5) may not be the whole story. There may be a larger theory which breaks down to SU(5)× $\tilde{G}$. If $\tilde{G}$ is not trivial on the known elementary fermions, there is hope of probing the factor group $\tilde{G}$. In this spirit, we would like to consider the simplest extension of SU(5) theory to SU(5)× U(1). We list two motivations which lead us to consider SU(5)× U(1) and which fix the U(1) quantum numbers of ordinary fermions.

(A) SU(N) gauge theory. Suppose the true supergrand unified theory is an SU(N) gauge theory and it is spontaneously broken to the SU(5) gauge theory. An adjoint representation of Higgs can break SU(N) symmetry down to SU(5)× SU($N-5$)× U(1) where SU($N-5$) is neutral to ordinary fermions. Since SU(5)× U(1) descended from an SU(N) theory, their coupling constants are equal at the grand unified mass scale. We suppose the $\underline{5}$ and $\underline{\overline{10}}$ of fermions come from the ψ^α and $\psi_{\alpha\beta}$ of $\underline{\mathrm{SU}(\overline{N})}$, respectively. We adopt the heavy-color philosophy[3]: Identifying SU($N-5$) as heavy color we postulate that fields carrying heavy-color indices ($\alpha,\beta=6, 7,\ldots,N$) are confined. Then U(1) counts -2 and +1 as SU(5) indices for $\underline{\overline{10}}$ and $\underline{5}$, respectively. Since we might have started from an appropriate set[4] of fermion representations in the original SU(N) to avoid the Adler-Bell-Jackiw anomaly, renormalizability and freedom from anomalies should be maintained in the broken SU(5)× U(1) ×⋯ theory also.

(B) GL(5, c) theory. Instead of considering ever larger simple unitary groups, we wish to examine theories based on general linear groups.[5,6] The usual restriction to simple unitary groups comes about because the usual kinetic energy terms are invariant only under unitary transformations. For example, the kinetic energy term for an n-component complex scalar field ϕ, $\partial_\mu\phi^\dagger\partial^\mu\phi$, is invariant under transformation $\phi\to z\phi$ only if z is an $n\times n$ unitary matrix. Let us relax the restriction to unitary transformation by requiring the theory be invariant under the transformation

$$\phi\to z\phi\,, \tag{1.1}$$

where z is an arbitrary $n\times n$ complex matrix whose inverse z^{-1} exists. The set of such matrices clearly forms a group over the complex numbers GL(n, c). The theory[5] of GL(n, c) is spelled out in more detail in the Appendix. Here we wish to examine the possibility of embedding the SU(5) theory of Georgi and Glashow in GL(n, c). The gauge fields A_μ contain 50 real fields and can be written as $A_\mu=B_\mu+iC_\mu$. The gauge fields B_μ are associated with the unitary subgroup U(5) of GL(5, c). We assign the known fermions in exactly the same way as Georgi and Glashow. The left-handed $\bar{d},\nu_e,e^-$ are fitted into the five-dimensional $\underline{5}$ of GL(5, c), which transforms as

$$\psi^\alpha\to z^{\alpha\beta}\psi^\beta\,, \tag{1.2a}$$

where $\alpha, \beta = 1, \ldots, 5$ and z denotes a complex 5×5 matrix whose inverse exists. The left-handed $\bar{u}$, u, d, and e^+ are fitted into the ten-dimensional representation $\underline{10}$, which transforms as

$$\psi_{\alpha\beta} \to z^{*\alpha\gamma} z^{*\beta\delta} \psi_{\gamma\delta} , \tag{1.2b}$$

where $\psi_{\alpha\beta} = -\psi_{\beta\alpha}$ is antisymmetric. (Here the asterisk denotes complex conjugation.) As in the SU(5) theory the observed sequential families of fermions are simply incorporated by repeating the $\underline{5} + \underline{10}$ structure. Introducing a metric field P transforming as

$$P \to z^{\dagger -1} P z^{-1} , \tag{1.3}$$

we can construct kinetic-energy terms

$$\tfrac{1}{2} \bar{\psi}^\alpha P^{\alpha\beta} (i\gamma^\mu D_\mu \psi)^\beta + \text{H.c.} \tag{1.4}$$

and

$$\tfrac{1}{2} \bar{\psi}_{\alpha\beta} P^{*\alpha\gamma} P^{*\beta\delta} (i\gamma^\mu D_\mu \psi)_{\gamma\delta} + \text{H.c.} \tag{1.5}$$

As explained in the Appendix, after the "metric" field P associated with GL(5, c) has been gauged away and the gauge fields C_μ become massive [at a mass scale which we imagine to be larger than the unification scale of ~10^{16} GeV obtained[7] for SU(5)], we are left with an effective U(5) = SU(5) × U(1) theory. The coupling constants are in principle different, but the "hypercharge" $\tilde{Y}$ associated with U(1) is definitely fixed. Indeed, $\tilde{Y}$ simply counts the number of SU(5) indices assigning the value +1 to each covariant index and −1 to each contravariant index. Thus, $\tilde{Y} = +1$ on $\underline{5}$, and $\tilde{Y} = -2$ on $\underline{10}$. We would like to refer to this quantum number as *quinticity*, in analogy to the well-known triality of SU(3). It is noteworthy that the U(1) quantum numbers in this case are the same as in case (A).

The preceding remarks motivate us to undertake a phenomenological analysis of a U(5) = SU(5) × U(1) theory.

The U(5) gauge theory may be broken by the Higgs mechanism into an $SU(3)_c \times SU(2)_L \times U(1) \times U(1)$ theory. Thus, we end up not with the standard $SU(2)_L \times U(1)$ electroweak theory,[1] but with an SU(2) × U(1) × U(1) electroweak theory with the gauge coupling[8]

$$g\vec{W} \cdot \vec{T} + g'B \frac{Y}{2} + \tilde{g}\tilde{B} \frac{\tilde{Y}}{2} . \tag{1.6}$$

The first two terms are familiar from the standard theory. $\tilde{B}$ is the gauge field coupling to the quinticity with coupling constant $\tilde{g}$.

SU(2) × U(1) × U(1) electroweak theories have been discussed previously in the literature.[9] However, in these dicussions, typically the "hypercharge" $\tilde{Y}/2$ is assigned various values not fixed from any general principle. In contrast, in our theory, $\tilde{Y}/2$ for a given fermion is fixed according to what SU(5) representation it belongs to. Thus, for the electron-up-down family the quinticity assignment reads

$$\begin{array}{cccccc} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L , & e_R , & \begin{pmatrix} u \\ d \end{pmatrix}_L , & u_R , & d_R ; \\ \frac{\tilde{Y}}{2} = \frac{1}{2}, & 1, & -1, & 1, & -\frac{1}{2}. \end{array} \tag{1.7}$$

Furthermore, in the discussions in the literature the coupling constant $\tilde{g}$ is totally unknown while here we have some information about $\tilde{g}$ (see below).

Because the theory descended from an SU(5) × U(1), the electric charge operator Q has to be identified in the usual way: $Q = T_3 + Y/2$.

II. NEUTRAL-CURRENT ANALYSIS

In order to confront theory with the experimental data on neutral currents, we have to work out how SU(2) × U(1) × U(1) is broken down into $U(1)_{em}$. As usual, the experimental predictions of the theory depend on the pattern of symmetry breaking. We will assume that dynamical symmetry breaking in GL(5, c) or in U(5) produces two effective Higgs fields, one transforming as ϕ^α and the other transforming as the totally antisymmetric tensor $\phi^{\alpha\beta\gamma\delta\epsilon}$ under GL(5, c) and U(5). The field ϕ^α transforms under SU(2) as a doublet. The second field $\phi^{\alpha\beta\gamma\delta\epsilon}$ is needed in order to leave only the photon massless. Our choice to have it transform as a singlet under the usual SU(2) × U(1) is motivated by the consideration that the standard normalization of neutral-current data relative to charged-current data be maintained (see below).

The relevant mass term in the Lagrangian then reads

$$\tfrac{1}{4} [v^2 g^2 (W_1{}^2 + W_2{}^2) + v^2 (gW_3 - g'B + \tilde{g}\tilde{B})^2 + w^2 (\tilde{g}\tilde{B})^2] . \tag{2.1}$$

The ratio w/v is related to the effective vacuum expectation values by[10,11]

$$\frac{w}{v} = \frac{5 \langle \phi^{12345} \rangle}{\langle \phi^4 \rangle} . \tag{2.2}$$

Diagonalizing Eq. (2.1), we find that the masses of the two Z bosons Z_1 and Z_2 are given by

$$M_{Z_i}{}^2 = \tfrac{1}{4} (g^2 + g'^2) v^2 (1 + x_i r^2) , \tag{2.3}$$

where

$$x_i = \frac{1}{2r^2} \{ (\rho r^2 - 1) \mp [(\rho r^2 - 1)^2 + 4r^2]^{1/2} \} , \tag{2.4}$$

$$\rho = 1 + w^2/v^2 , \tag{2.5}$$

$$r^2 = \frac{\tilde{g}^2}{g^2+g'^2} . \quad (2.6)$$

[In Eq. (2.4) we will choose the conventions that x_1 and x_2 correspond to the − and + sign, respectively.]

We note from Eqs. (2.1) and (2.4) that one of the Z bosons, namely Z_1, is always lighter than the mass of the Z boson in the standard SU(2)×U(1) model for the same value of g'/g.

Proceeding in a straightforward fashion, we determine that Z_i couples to

$$\frac{g^2 T_3 - g'^2 \frac{Y}{2} + x_i \tilde{g}^2 \frac{\tilde{Y}}{2}}{(g^2+g'^2+x_i^2\tilde{g}^2)^{1/2}} . \quad (2.7)$$

The photon, of course, couples to

$$\frac{gg'}{(g^2+g'^2)^{1/2}} Q \equiv eQ . \quad (2.8)$$

III. NEUTRINO SCATTERING

The available neutral-current data can be classified as $\nu, \bar{\nu}$-hadron, $\nu, \bar{\nu}$-electron, and electron-hadron. Let us first examine neutrino scattering. According to Eq. (2.7) and the quinticity assignment of the neutrino given in Eq. (1.7), the relevant Lagrangian for neutrino scattering reads

$$\mathcal{L} = -\sum_i \frac{1}{2} \frac{g^2+g'^2+x_i\tilde{g}^2}{(g^2+g'^2+x_i^2\tilde{g}^2)^{1/2}} \bar{\nu}_L \gamma^\mu \nu_L \times \frac{1}{M_{Z_i}{}^2}\left[\frac{g^2T_3 - g'^2 Y/2 + x_i \tilde{g}^2 \tilde{Y}/2}{(g^2+g'^2+x_i^2\tilde{g}^2)^{1/2}}\right]_\mu , \quad (3.1)$$

where the current $[\ ]_\mu$ is interpreted to be summed over electron and quark representations. An examination of Eq. (2.3) shows that this expression can be simplified to read

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}} \bar{\nu}_L \gamma^\mu \nu_L \sum_i \left[\frac{g^2T_3 - g'^2Y/2 + x_i\tilde{g}^2\tilde{Y}/2}{(g^2+g'^2)(1+x_i^2r^2)}\right]_\mu . \quad (3.2)$$

Remarkably enough, the following two identities can be readily verified which is a consequence of $r^2x_i^2 - Bx_i - 1 = 0$,

$$\sum_i \frac{1}{1+x_i^2r^2} = 1 , \quad (3.3)$$

$$\sum_i \frac{x_i}{1+x_i^2r^2} = 0 . \quad (3.4)$$

Thus, Eq. (3.2) reduces to exactly the same expression as in the standard SU(2)×U(1) model:

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}} \bar{\nu}_L \gamma_\mu \nu_L \left[\frac{g^2T_3 - g'^2Y/2}{g^2+g'^2}\right]^\mu . \quad (3.5)$$

With the choice $g'^2/(g^2+g'^2) \equiv \sin^2\theta_W \sim 0.23$, all neutral-current data involving neutrino scattering could be fit.

IV. ELECTRON SCATTERING

The parity-violating part of electron-quark scattering is usually parametrized as[12]

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} [\bar{e}\gamma_\mu\gamma_5 e(C_1^u \bar{u}\gamma^\mu u + C_1^d \bar{d}\gamma^\mu d) + \bar{e}\gamma_\mu e(C_2^u \bar{u}\gamma^\mu\gamma_5 u + C_2^d \bar{d}\gamma^\mu\gamma_5 d)] . \quad (4.1)$$

Using Eq. (2.7), we easily obtain

$$C_1^u = \sum_i \frac{(-\frac{1}{2}+\frac{4}{3}\sin^2\theta_W)}{(1+x_i^2r^2)} = -\tfrac{1}{2}+\tfrac{4}{3}\sin^2\theta_W , \quad (4.2)$$

$$C_1^d = \sum_i \frac{(\frac{1}{2}-\frac{2}{3}\sin^2\theta_W + \frac{3}{2}x_i r^2)}{(1+x_i^2r^2)} = \tfrac{1}{2}-\tfrac{2}{3}\sin^2\theta_W , \quad (4.3)$$

$$C_2^u = \sum_i \frac{(-\frac{1}{2}+2\sin^2\theta_W + \frac{3}{2}x_i r^2)(1-4x_ir^2)}{(1+x_i^2r^2)(1+x_ir^2)} , \quad (4.4)$$

$$C_2^d = \sum_i \frac{(\frac{1}{2}-2\sin^2\theta_W - \frac{3}{2}x_ir^2)}{(1+x_i^2r^2)} = \tfrac{1}{2}-2\sin^2\theta_W . \quad (4.5)$$

Again, remarkably enough, by using the identities (3.3), and (3.4), we see that C_1^u, C_1^d, and C_2^d have exactly the same form as in the standard SU(2)×U(1) model. Thus, the present theory differs from the standard one in only one neutral-current parameter: C_2^u.

We remark that the parameter C_2^u does not contribute to the dominant parity-violating effects in heavy atoms (Bi and Tl). Thus, the present theory can be distinguished from the standard SU(2)×U(1) theory only by the polarized-electron–deuterium scattering experiment[13] and by the observation of parity violation in hydrogen atoms.

It is useful to rewrite C_2^u, by repeated use of the identities (3.3) and (3.4), in the form

$$C_2^u = -\tfrac{1}{2}+2\sin^2\theta_W - 10(1-\sin^2\theta_W)\left(\sum_i \frac{1}{(1+x_i^2r^2)(1+x_ir^2)} - 1\right) . \quad (4.6)$$

We recognize the expression $(-\frac{1}{2}+2\sin^2\theta_W)$ as the value of C_2^u in the standard SU(2)×U(1) model. Remarkably, we find the deviation of C_2^u from its value in the standard model

$$-10(1-\sin^2\theta_W)\left(\sum_i \frac{1}{(1+x_i^2r^2)(1+x_ir^2)} - 1\right) = \frac{-10(1-\sin^2\theta_W)}{\rho - 1} \quad (4.7)$$

to be independent of $r^2 = \tilde{g}^2/(g^2+g'^2)$. Thus, the neutral-current predictions at zero momentum transfer in this model are independet of the coupling constant $\tilde{g}^2$.

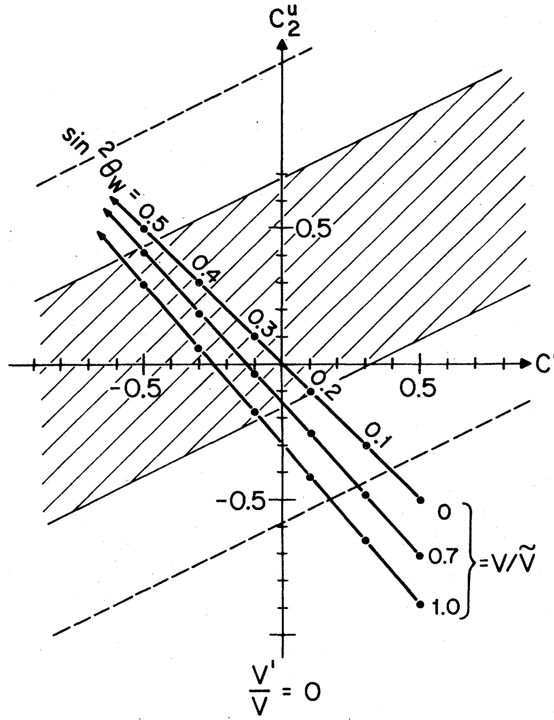

FIG. 1. C_2^u vs C_2^d with two Higgs bosons ϕ^μ and $\phi^{\alpha\beta\gamma\delta\epsilon}$ for several values of $\sin^2\theta_W$ and $v/\tilde{v} = |\langle\phi^\mu\rangle/\langle\phi^{\alpha\beta\gamma\delta\epsilon}\rangle|$. For given $\sin^2\theta_W$, C_2^d does not depend on $v/\tilde{v}$.

The deviation of C_2^u from that of the SU(2)×U(1) theory calls for a plot of C_2^u vs C_2^d, which is given in Fig. 1. This single figure is sufficient to confront the neutral-current data, since other confrontations are identical to those of the SU(2)×U(1) theory. In Fig. 1, the experimentally allowed region (1σ) is shown as a band within two solid lines and the 2σ limit is shown as dotted lines. The line $v/\tilde{v}=0$ corresponds to the limit to the Weinberg-Salam model.[4] From Fig. 1, we read off a reasonable limit (1σ) on $v/\tilde{v}$

$$\frac{v}{\tilde{v}} \lesssim 0.6\,, \tag{4.8}$$

where

$$\tilde{v} \equiv \tfrac{1}{5} w \tag{4.9}$$

and $\sin^2\theta_W = 0.23$. For a 2σ allowance, the limit on $v/\tilde{v}$ is not very restrictive. For other values of $\sin^2\theta_W$, $v/\tilde{v}$ can be varied to a wide range, but then the other neutral-current data are not well fitted. The mass of the neutral boson Z_1 is lower than that of the standard Z boson, but is close to it:

$$M_{Z_1} \gtrsim 0.99 M_Z\,, \tag{4.10}$$

for $\sin^2\theta_W = 0.23$.

V. TWO THEOREMS ON NEUTRAL-CURRENT ANALYSIS

The remarkable identities (3.3), (3.4), and (4.7) which we just encountered call for some theorems. The appropriate theorems turn out not to be too difficult to prove. First of all, we need a general formula for neutral-current interactions. Such a formula has in fact been given by Georgi and Weinberg.[14] We briefly review their analysis here for the sake of completeness.

For a general gauge theory, let the electrically neutral gauge bosons A_μ^α couple to $g_\alpha T_\alpha$. T_α are electrically neutral generators and the electric charge operator is $Q=\sum_\alpha c_\alpha T_\alpha$. The photon field is given by $A_\mu = \sum_\alpha p_\alpha A_\mu^\alpha$, where p_α is the normalized eigenvector of the mass matrix $\mu^2{}_{\alpha\beta}$ with eigenvalue zero, i.e.,

$$\sum_\beta \mu^2{}_{\alpha\beta} p_\beta = 0\,, \quad \sum_\alpha p_\alpha{}^2 = 1\,.$$

It is easy to see that $p_\alpha = N c_\alpha / g_\alpha$ in order that A_μ couple to the electric charge. We check that A_μ couples to

$$\sum_\alpha p_\alpha\, g_\alpha T_\alpha = N \sum_\alpha c_\alpha\, T_\alpha = NQ\,.$$

Thus the unit of charge e is equal to N which is fixed by the normalization of p_α, yielding the relation

$$\frac{1}{e^2} = \sum_\alpha \left(\frac{c_\alpha}{g_\alpha}\right)^2 . \tag{5.1}$$

The massive Z-bosons are given by $Z_i^\mu = \sum_\alpha u_{i\alpha} A_\alpha^\mu$, where $u_{i\alpha}$ are normalized eigenvectors of $\mu^2{}_{\alpha\beta}$:

$$\sum_\beta \mu^2{}_{\alpha\beta} u_{i\beta} = m_i^2 u_{i\alpha}\,.$$

The boson Z_i^μ couples to $\sum_\alpha u_{i\alpha} g_\alpha T_\alpha$ and so the effective neutral-current Hamiltonian at zero momentum transfer is given by

$$\mathcal{H}_N = \tfrac{1}{2} \sum_{\alpha\beta} [g_\alpha T_\alpha]^\mu \Delta_{\alpha\beta} [g_\beta T_\beta]_\mu\,, \tag{5.2}$$

where

$$\Delta_{\alpha\beta} = \sum_i \frac{u_{i\alpha} u_{i\beta}}{m_i^2}$$

and $[\;]^\mu$ is interpreted as a sum over the fermion currents. Now let us pick out one generator contributing to the charge operator Q out of the set T_α, call it T_0, and refer to the rest as T_i's. Denote the inverse of the submatrix $\mu^2{}_{ij}$ of the mass matrix $\mu^2{}_{\alpha\beta}$ by $(\mu^{-2})_{ij}$ and introduce the photon annihilation operator $\zeta_{\alpha\beta} = \delta_{\alpha\beta} - p_\alpha p_\beta$. It can be shown[14] easily that

$$\Delta_{\alpha\beta} = \sum_{ij} \zeta_{\alpha i} (\mu^{-2})_{ij} \zeta_{\beta j}\,.$$

Inserting this identity into Eq. (5.2), one finds that the neutral-current interaction Hamiltonian at zero momentum transfer is given by

$$\mathcal{H}_N=\tfrac{1}{2}\sum_{ij}[n_i]^{\mu}(\mu^{-2})_{ij}[n_j]_{\mu}\ , \tag{5.3}$$

with

$$n_i=g_i\left(T_i-\frac{e^2}{g_i^2}c_iQ\right). \tag{5.4}$$

Starting with Eq. (5.3), which is quite general, Georgi and Weinberg[14] were able to prove a theorem[15] stating that, for a $G_1\times G_2\times \mathrm{U}(1)$ theory, if (a) all Higgs fields are either singlet under G_1 or singlet under G_2, and if (b) the neutrino is neutral under G_2, then the neutral current interaction of the neutrino will be precisely the same as if the gauge group were just $G_1\times\mathrm{U}(1)$.

However, our result in Sec. III that the neutral-current interaction of the neutrino in the SU(2)× U(1)× U(1) theory is precisely the same as in the standard SU(2)× U(1) theory, is *not* covered by this theorem. Neither the $\underline{5}$ Higgs field nor the neutrino is a singlet under $\mathrm{U}(1)_{\tilde{Y}}$. Thus, we must prove a generalization of this theorem. But first, we will prove the following theorem.

Theorem 1. Suppose that, among the c_i's, $c_p=0$. In other words, the electric-charge operator Q does not depend on T_p. Then, the neutral-current interaction (at zero momentum transfer) does not depend on the coupling constant g_p.

Our observation [in Eq. (4.7)] that C_2^u does not depend on $\tilde{g}$ is a consequence of this theorem, since Q does not depend on $\tilde{Y}$. To prove the theorem, let us denote the symmetry-breaking vacuum expectation values by v_a, where the index a runs over some set. Also, denote by t_{ia} the eigenvalue of that Higgs component, which has vacuum expectation value v_a, under the generator T_i of the gauge group. Then the mass submatrix $\mu^2{}_{ij}$ is given by

$$\mu^2{}_{ij}=g_ig_j\sum_a v_a^2t_{ia}t_{ja}\ . \tag{5.5}$$

It is then clear that the (ij)th entry of the inverse matrix $(\mu^{-2})_{ij}$ has the form $g_i^{-1}g_j^{-1}$ (factor independent of coupling constants). Now notice from Eq. (5.4) that $c_p=0$ implies $n_p=g_pT_p$. Referring to Eq. (5.3) we see that the theorem as stated is proved.

We next restrict ourselves to theories with gauge groups of the form $G_1\times G_2\times\mathrm{U}(1)$ and ask under what circumstances the neutral-current interaction is the same as if the gauge group were of the form $G_1\times\mathrm{U}(1)$. To avoid being inundated by indices, we content ourselves first with the case in which G_1 and G_2 each has only one neutral generator (denoted as T_1 and T_2). Thus, for practical purposes, the discussion which follows covers the case SU(2) × SU(2)× U(1) and SU(2)× U(1)× U(1). The generator T_0 will denote that of $\mathrm{U}(1)_Y$. Generalizations will be mentioned later.

Assuming that there are only two T_i's, we can readily write $(\mu^{-2})_{ij}$:

$$(\mu^{-2})_{ij}=\frac{1}{\det}\sum_a v_a^2\begin{pmatrix} g_2^2t_{2a}{}^a, & -g_1g_2t_{1a}t_{2a}\\ -g_1g_2t_{1a}t_{2a}, & g_1^2t_{1a}^2\end{pmatrix}, \tag{5.6}$$

where

$$\begin{aligned}\det=\det\mu^2{}_{ij}&=g_1^2g_2^2\sum_{a<b}v_a^2v_b^2(t_{1a}t_{2b}-t_{2a}t_{1b})^2\\ &=g_1^2g_2^2\sum_{ab}v_a^2v_b^2(t_{1a}t_{2b}-t_{2a}t_{1b}).\end{aligned} \tag{5.7}$$

Now extract from $\mathcal{H}_N$ in Eq. (5.3) the interaction of neutrino (or more generally any electrically neutral fermion) with any other fermion [note the factor of 2 difference between (5.3) and (5.8)],

$$\mathcal{H}_N^{\nu}=\bar{\nu}_L\gamma_\mu\nu_L\sum_{ij}g_it_{i\nu}(\mu^{-2})_{ij}[g_j(T_j-\frac{e^2}{g_j^2}c_jQ)]^{\mu}\ , \tag{5.8}$$

where $t_{i\nu}$ is the eigenvalue of the neutrino field under T_i. Let us focus on the quantity

$$K_j\equiv\sum_i g_it_{i\nu}(\mu^{-2})_{ij}g_j\quad (j=\text{not summed}) \tag{5.9}$$

appearing in Eq. (5.8). Inserting Eq. (5.6), we find

$$K_1=\frac{1}{\det}(g_1^2g_2^2)\sum_a v_a^2t_{2a}(t_{1\nu}t_{2a}-t_{2\nu}t_{1a}), \tag{5.10}$$

$$K_2=\frac{1}{\det}(-g_1^2g_2^2)\sum_a v_a^2t_{1a}(t_{1\nu}t_{2a}-t_{2\nu}t_{1a}). \tag{5.11}$$

An effective limit to $G_1\times\mathrm{U}(1)$ can be obtained if for finite g_1 and g_2,

$$(\mu^{-2})_{ij}\to\begin{pmatrix}x&0\\0&0\end{pmatrix},$$

where x is a nonzero entry. This limit can be achieved in a variety of ways. One way of achieving this is $g_2\to 0$. Another formal limit is $|t_{2a}|\to\infty$ for some a. Then the (11)th element of $(\mu^{-1})_{ij}$ is

$$\frac{1}{g_1^2}\ \frac{\sum_a v_a^2t_{2a}^2}{\sum_{ab}v_a^2v_b^2t_{1a}t_{2b}(t_{1a}t_{2b}-t_{2a}t_{1b})}\ .$$

Further, $(\mu^{-2})_{ij}$ will be exactly the same as that of $G_1\times\mathrm{U}(1)$ without consideration of t_{2a} (i.e., neglecting the contribution of the group G_2) if $t_{1a}=0$ for the a which makes $|t_{2a}|\to\infty$. In this case the (11)th element is

$$\frac{1}{g_1^2\sum_a v_a^2t_{1a}^2}\ . \tag{5.12}$$

In this limit, the gauge theory effectively becomes a $G_1 \times U(1)$ theory and the neutrino neutral-current couplings become

$$K_1 \to K_1(|t_{2a}| \to \infty,\ t_{1a}=0,\ \text{for some } a) = t_{1\nu}\left(\sum_a v_a{}^2 t_{1a}{}^2\right)^{-1}, \quad (5.13)$$

$$K_2 \to K_2(|t_{2a}| \to \infty,\ t_{1a}=0,\ \text{for some } a) = 0\,. \quad (5.14)$$

Thus, the conditions that the neutral-current interaction of ν be the same as if the gauge group were $G_1 \times U(1)$, with the same set of vacuum expectation value of course, simply read

$$K_i = K_i(|t_{2a}| \to \infty,\ t_{1a}=0,\ \text{for some } a)\,. \quad (5.15)$$

We can now state a number of theorems.

Theorem 2. Suppose that the vacuum expectation values v_a can be divided into two classes: (i) v_r's which belong to the same representation as the neutrino, and (ii) v_s's which belong to representations such that $t_{1s}=0$. (The index set $\{a\}$ is equal to $\{r\}+\{s\}$.) Then the neutral-current interaction of the neutrino is the same as if the gauge group were $G_1 \times U(1)$ except for an overall normalization.

We should remark that here and in what follows the word neutrino could always be replaced by the phrase "any electrically neutral fermions (not necessarily neutral under G_2)." Noting that the condition (i) implies that $t_{1\nu}t_{2r}=t_{2\nu}t_{1r}$, we see that the sum in K_2 vanishes term by term and that K_1 reduces to

$$K_1 = \frac{1}{\det} g_1{}^2 g_2{}^2 \left(\sum_s v_s{}^2 t_{2s}{}^2\right) t_{1\nu} \quad (5.16)$$

which is proportional to $K_1(|t_{2a}| \to \infty,\ t_{1a}=0$ for some a). Note also

$$\det = g_1{}^2 g_2{}^2 \left[\left(\sum_r v_r{}^2 t_{1r}{}^2\right)\left(\sum_a v_a{}^2 t_{2a}{}^2\right) - \left(\sum_r v_r{}^2 t_{1r} t_{2r}\right)\left(\sum_{r'} v_{r'}{}^2 t_{1r'} t_{2r'}\right)\right]. \quad (5.17)$$

Theorem 3. Let the conditions on the vacuum expectation values be the same as in Theorem 2. In addition, suppose that $t_{1\nu}=t_{2\nu}$. Then the neutral-current interaction of the neutrino is precisely the same as if the gauge group were $G_1 \times U(1)$.

To prove this, we note that $t_{1\nu}=t_{2\nu}$ implies $t_{1r}=t_{2r}$ and thus, from Eq. (5.17),

$$\det = g_1{}^2 g_2{}^2 \left(\sum_r v_r{}^2 t_{1r}{}^2\right)\left(\sum_s v_s{}^2 t_{2s}{}^2\right). \quad (5.18)$$

Comparing with Eqs. (5.16) and (5.13) we see that

$$K_1 = \left(\sum_r v_r{}^2 t_{1r}{}^2\right)^{-1} t_{1\nu} = K_1(|t_{2a}| \to \infty,\ t_{1a}=0,\ \text{for some } a) \quad (5.19)$$

as desired.

The gauge theory described in Sec. II satisfies the condition of Theorem 3. In particular we have $t_{1\nu}=t_{2\nu}=\frac{1}{2}$, a Higgs ϕ^α with $t_{1x}=t_{2x}=\frac{1}{2}$ and another Higgs $\phi^{\alpha\beta\gamma\delta\epsilon}$ with $t_{1y}=0$, $t_{2y}=\frac{5}{2}$, respectively.

For the sake of completeness let us also state the theorem[16] in Ref. 14 in our present context.

Theorem 4 (Georgi-Weinberg). Suppose that $t_{2\nu}=0$ and that all vacuum expectation values are such that $t_{1a}t_{2a}=0$, then the neutral-current interaction of the neutrino is precisely the same as if the gauge group were $G_1 \times U(1)$.

The proof is obvious by referring to Eqs. (5.9) and (5.10).

Clearly the conditions $K_i = K_i(|t_{2a}| \to \infty,\ t_{1a}=0$, for some a) can be satisfied because of an endless variety of restrictions on v_a, $t_{1\nu}$, $t_{2\nu}$, t_{1a}, and t_{2a}. For instance, one could imagine cancellation between the various terms in Eq. (5.10) because some vacuum expectation values are equal. The real utility of Eqs. (5.9) through (5.15) is that the reader can readily check whether any given model will have the same neutral-current interaction as the standard $SU(2) \times U(1)$ theory. If we remove the restriction that G_2 contains only one generator, the analogs of Eqs. (5.9), (5.10), and (5.15) could still be derived but the resulting expressions would be rather unwieldy and we would not present them here.

VI. DEVIATIONS FROM THE STANDARD SU(2) × U(1) THEORY

One rather attractive feature of the SU(5) theory is the relative economy of Higgs fields. A $\underline{24}$ with huge vacuum expectation value is needed to break SU(5) into $SU(3)_c \times SU(2) \times U(1)$. The electroweak $SU(2) \times U(1)$ is then broken to $U(1)_{em}$ by a $\underline{5}$ of Higgs fields. The $\underline{5}$ couples the fermion $\underline{\bar{5}}$ to the fermion $\underline{\overline{10}}$ and also the fermion $\underline{\overline{10}}$ to itself. The first coupling leads to masses for the electron and the down quark, while the second makes the up quark massive.

When we generalize the theory of GL(5, c) or U(5) this coupling of the fermion $\underline{\overline{10}}$ to itself via the $\underline{5}$ of Higgs fields is no longer allowed, since it involves the $\epsilon^{\mu\nu\gamma\rho\sigma}$ symbol which is only invariant under SU(5). Instead, we are forced to introduce in addition to the $\underline{5}$ and $\underline{24}$ of Higgs fields a totally antisymmetric tensor field $\phi^{\mu\nu\gamma\rho}$ with four indices. This Higgs field $\phi^{\mu\nu\gamma\rho}$, which couples the fermion $\underline{\overline{10}}$ to itself, transforms as an irreducible five-dimensional representation of GL(5, c) or U(5). Within the SU(5) subgroup $\phi^{\mu\nu\gamma\rho}$ is equivalent to $\underline{\bar{5}}$, ϕ_σ, but under GL(5, c) or U(5) it behaves quite differently. In particular, it has quinticity $\tilde{Y}/2=2$.

Referring to the theorems in Sec. V, we see that the inclusion of $\phi^{\mu\nu\gamma\rho}$ implies that the neutrino neutral-current interaction is no longer precisely the same as in the standard SU(2)× U(1) theory. In fact, using Eq. (5.9) and (5.10) we can readily read off the deviations from the standard SU(2)× U(1) theory. We obtain the deviation as

$$\Delta\mathcal{L} = \eta \frac{4G_F}{\sqrt{2}} \bar{\nu}_L \gamma_\mu \nu_L \left[T_3 - \sin^2\theta_W Q - \frac{\tilde{Y}}{2}\right]^\mu + \epsilon \frac{4G_F}{\sqrt{2}} \bar{\nu}_L \gamma_\mu \nu_L [T_3 - \sin^2\theta_W Q]^\mu , \qquad (6.1)$$

where

$$\eta = \frac{1}{5} \frac{v'^2/\tilde{v}^2}{1+[v^2 v'^2/\tilde{v}^2(v^2+v'^2)]} , \qquad (6.2)$$

$$\epsilon = -\frac{v'^4/\tilde{v}^2(v^2+v'^2)}{1+[v^2 v'^2/\tilde{v}^2(v^2+v'^2)]} . \qquad (6.3)$$

In order for the deviations to be small we have to demand that the vacuum expectation value v' of $\frac{1}{4}\phi^{\mu\nu\lambda\rho}$ be small compared with that of ϕ^μ. Neglecting the second-order terms in v'^2, we note that the chiral structure change in neutrino scattering is $\eta[T_3 - \sin^2\theta_W Q - \tilde{Y}/2]$ which can be negligible for small v', since η is further suppressed by a factor of $\frac{1}{5}$. Full expressions for the $\epsilon_{L,R}(u,d)$ parameters are

$$\epsilon_L(u) = \kappa[(\tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_W)(1+\tfrac{4}{5}\xi) - \tfrac{1}{5}\xi] , \qquad (6.4)$$

$$\epsilon_L(d) = \kappa[(-\tfrac{1}{2} + \tfrac{1}{3}\sin^2\theta_W)(1+\tfrac{4}{5}\xi) - \tfrac{1}{5}\xi] , \qquad (6.5)$$

$$\epsilon_R(u) = \kappa[-\tfrac{2}{3}\sin^2\theta_W(1+\tfrac{4}{5}\xi) + \tfrac{1}{5}\xi] , \qquad (6.6)$$

$$\epsilon_R(d) = \kappa[\tfrac{1}{3}\sin^2\theta_W(1+\tfrac{4}{5}\xi) - \tfrac{1}{10}\xi] , \qquad (6.7)$$

where

$$\kappa = \frac{1+b^2}{1+b^2+a^2b^2} , \qquad (6.8)$$

$$\xi = a^2 b^2 , \qquad (6.9)$$

$$a^2 = v^2/\tilde{v}^2 , \qquad (6.10)$$

$$b^2 = v'^2/v^2 . \qquad (6.11)$$

In Fig. 2, we show this chiral structure dependence on v'/v and $v/\tilde{v}$ for $\sin^2\theta_W = 0.2$ and 0.25. The solid straight lines correspond to the Weinberg-Salam limit. Experimental error bars are also shown on the vertical axis. From Fig. 2, we notice that both v'/v and $v/\tilde{v}$ should be small to agree with the experiments

$$v'/v \lesssim 0.5 ,$$

$$v/\tilde{v} \lesssim 0.5 ,$$

with $\sin^2\theta_W = 0.20 \sim 0.25$.

The $\nu,\bar{\nu}$-electron scattering is characterized by two parameters g_V^e and g_A^e which are

$$g_V^e = \kappa[(-\tfrac{1}{2} + 2\sin^2\theta_W)(1+\tfrac{4}{5}\xi) + \tfrac{3}{10}\xi] , \qquad (6.12)$$

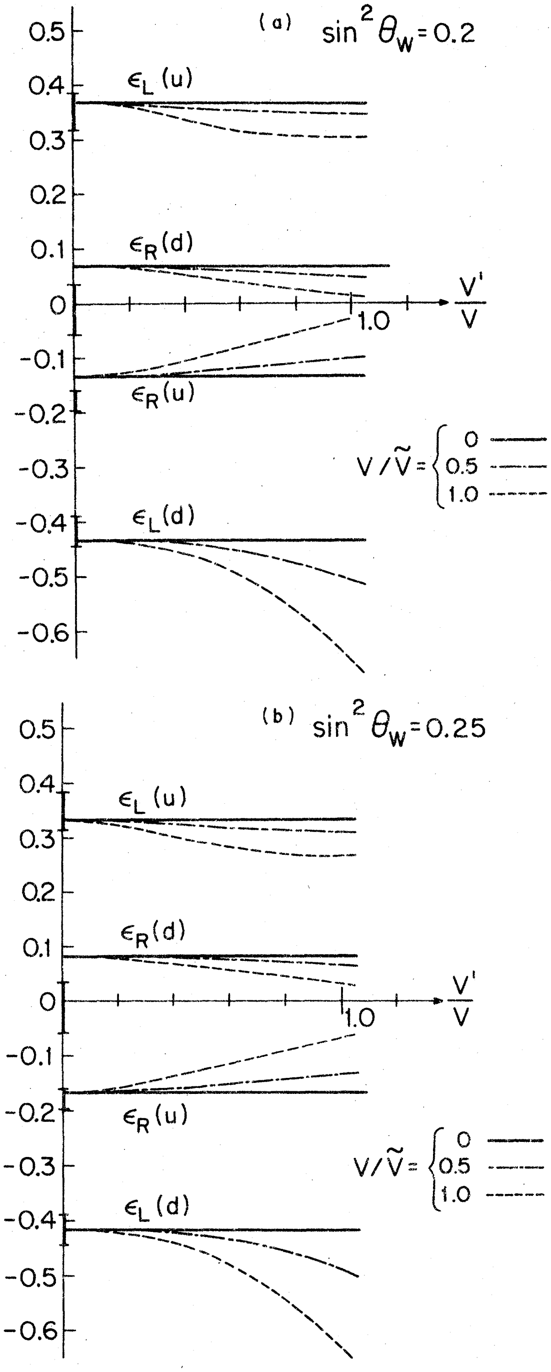

FIG. 2. (a) $\epsilon_{L,R}(u,d)$ for several values of $v/\tilde{v}$ and v'/v, for $\sin^2\theta_W = 0.2$. Both $v/\tilde{v} = 0$ and $v'/v = 0$ are the Weinberg-Salam limit. Experimental errors determined by Ref. 9 are shown on the vertical axis. (b) Same as (a) but $\sin^2\theta_W = 0.25$.

$$g_A^e = \kappa[-\tfrac{1}{2}(1+\tfrac{4}{5}\xi) - \tfrac{1}{10}\xi] . \qquad (6.13)$$

In Fig. 3, we show the $g_V^e - g_A^e$ plane with lines given by the above expressions. Also shown are

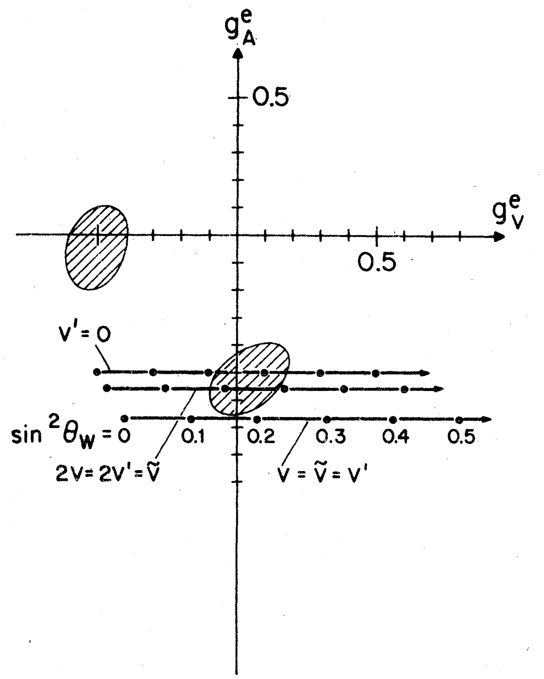

FIG. 3. g_A^e vs g_V^e plot compared to the experimentally allowed region (90% C.L.) by $\nu_\mu e$, $\bar{\nu}_\mu e$, and $\bar{\nu}_e e$ scatterings.

the two regions which are allowed (90% C.L.) by $\nu, \bar{\nu}$-electron scattering.[12] The theory chooses the g_A^e dominant region, which will be reproduced in the present model with $\sin^2\theta_W = 0.2-0.25$, $v'/v \lesssim 0.5$, $v/\tilde{v} \lesssim 0.5$.

The electron-hadron interaction parameters $C_{1,2}^i$ are given as

$$C_1^u = \kappa(-\tfrac{1}{2} + \tfrac{4}{3}\sin^2\theta_W)(1 + \tfrac{4}{5}\xi), \tag{6.14}$$

$$C_1^d = \kappa[(\tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_W)(1 + \tfrac{4}{5}\xi) + \tfrac{3}{10}\xi], \tag{6.15}$$

$$C_2^u = \kappa[(-\tfrac{1}{2} + 2\sin^2\theta_W)(1 + a^2/5) - \tfrac{3}{10}a^2], \tag{6.16}$$

$$C_2^d = \kappa[(\tfrac{1}{2} - 2\sin^2\theta_W)(1 + \tfrac{4}{5}\xi) - \tfrac{3}{10}\xi]. \tag{6.17}$$

Comparisons with the SLAC[13] polarized-electron–deuterium scattering are given in Figs. 4(a), 4(b), 5(a), and 5(b).

In Fig. 4(a), C_1^u vs C_1^d are plotted for $v'/v = 0.5$. The solid line corresponds to the Weinberg-Salam limit. The variation over $v/\tilde{v}$ is so negligible that only dots are shown in the figure. In Fig. 4(b), we show C_1^u vs C_1^d plotted for a larger value of $v'/v = 1$. In this case deviations from the Weinberg-Salam model are expected for an appreciable value of $v/\tilde{v}$. For the range of vacuum expectation values allowed by neutrino experiments, $v'/v \lesssim 0.5$ and $v/\tilde{v} \lesssim 0.5$, the present model is practically indistinguishable from the standard theory in the parameters C_1^u and C_1^d.

In Fig. 5(a), we show C_2^u vs C_2^d plots for v'/v

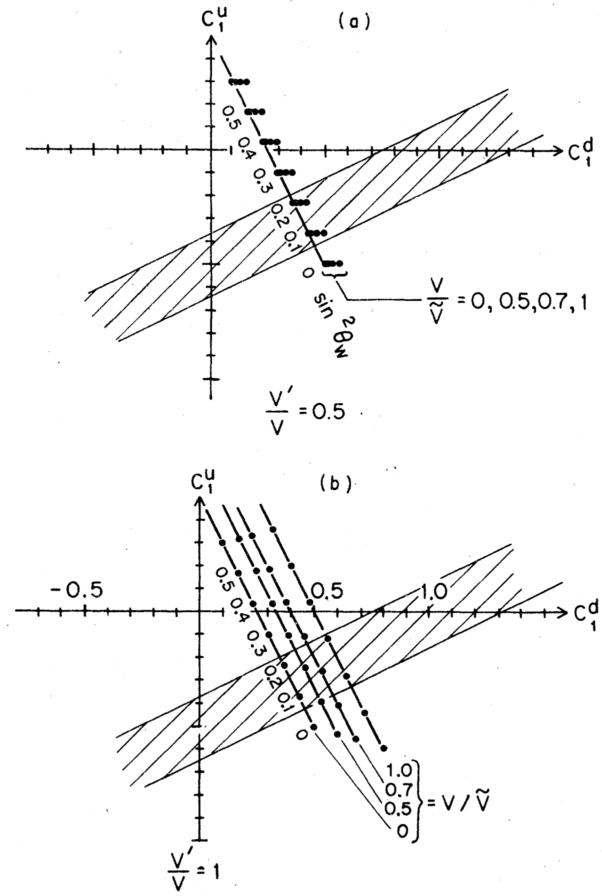

FIG. 4. (a) C_1^u and C_1^d curves for $v'/v = 0.5$ for four different values of $v/\tilde{v} = 0$, 0.5, 0.7, 1. The Weinberg-Salam limit is $v/\tilde{v} = 0$. The shaded region is 1σ limit. (b) Same as above but with $v'/v = 1$.

$= 0.5$. Comparing this with Fig. 1, we note that the change of parameters, $\sin^2\theta_W$ and $v/\tilde{v}$, going from $v'/v = 0$ to $v'/v = 0.5$ is very mild in the region allowed by the experiment. In Fig. 5(b), we plot C_2^u vs C_2^d for $v'/v = 1$. From Figs. 1 and 5, we note that the increase of v'/v allows a slightly better fit to $C_2^u - C_2^d$ for fixed $\sin^2\theta_W = 0.23$, but then neutrino scattering is not very well fit for too large value of v'/v.

From the above considerations, we note that the currently available neutral-current data can be fitted in the present model as successfully as the standard model with the parameters

$$\sin^2\theta_W \cong 0.23, \tag{6.18}$$

$$v'/v \lesssim 0.5, \tag{6.19}$$

$$v/\tilde{v} \lesssim 0.5, \tag{6.20}$$

where $v'/v = |\langle\phi^{\mu\nu\rho\sigma}\rangle / \langle\phi^\mu\rangle|$ and $v/\tilde{v} = |\langle\phi^\mu\rangle / \langle\phi^{\alpha\beta\gamma\delta\epsilon}\rangle|$.

This theory will also predict quite different be-

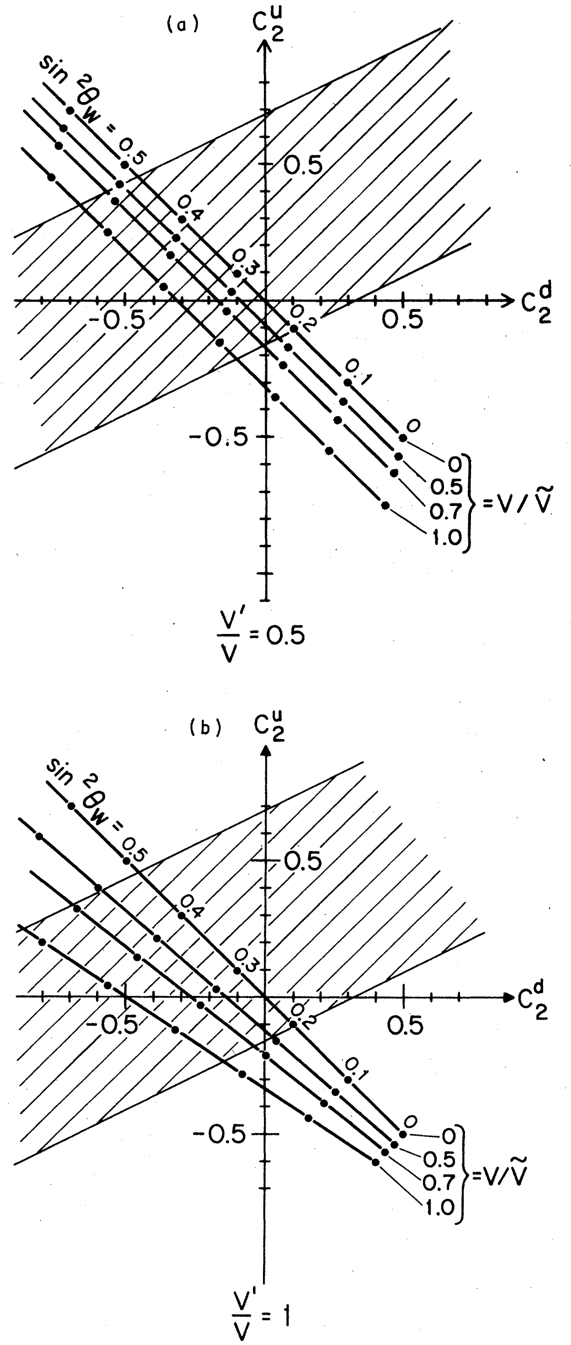

FIG. 5. (a) and (b) C_2^u vs C_2^d with the same conventions as in Fig. 4.

havior for the production of neutral gauge bosons and for the interference effect between the electromagnetic and weak interactions in the process[17] $e^+e^- \to \mu^+\mu^-$. In particular, resonance will occur at $M_Z \geq M_{Z_1} \gtrsim 93.9$ GeV, $M_Z \leq M_{Z_2} \lesssim 590$ GeV given by (6.21) rather than at $M_Z = 94.4$ GeV for $\sin^2\theta_W = 0.23$. The lower (upper) limit of M_{Z_1} (M_{Z_2}) corresponds to $v'/v = 0$ and the upper (lower) limit of M_{Z_1} (M_{Z_2}) corresponds to $v'/v = 0.5$. We note that $M_{Z_1} = M_{Z_2} = M_Z$ for $v'/v = 0.5$. Explicit expressions of M_{Z_1} and M_{Z_2} are, for $v'/v \neq \frac{1}{2}$,

$$M_{Z_i}{}^2 = M_Z{}^2\left(1 + y_i r^2 \; \frac{1-4b^2}{1+b^2}\right), \tag{6.21}$$

where

$$y_i = \frac{1 + \dfrac{25}{a^2} - \dfrac{1}{r^2} + b^2\left(16 - \dfrac{1}{r^2}\right) \mp \sqrt{A}}{2(1-4b^2)}, \tag{6.22}$$

$$A = \left(\rho - \frac{1}{r^2}\right)^2 + \frac{4}{r^2} + 2\frac{b^2}{a^2}\left(16 - \frac{1}{r^2}\right) + 2b^2\left(16 - \frac{33}{r^2} + \frac{1}{r^4}\right) + b^4\left(16 + \frac{1}{r^2}\right)^2 \tag{6.23}$$

with the definitions given in Eqs. (2.5), (2.6), (6.10), and (6.11). A remarkable fact of this theory allowed by Eqs. (6.18)–(6.20) is that the resonance of Z_1 is practically indistinguishable from the standard Z, $0.99 \leq M_{Z_1}/M_Z \leq 1$, while the possibility of Z_2 resonance is quite open, $1 \leq M_{Z_2}/M_Z \leq 6.3$. A verification of two narrowly separated neutral gauge bosons around 90–95 GeV imply the saturation of the relation (6.19). Experimentally, it may be difficult to separate a few-GeV mass difference of neutral Z bosons by a mass hunt since their decay width is expected to be a few GeV. Other observations, such as the integrated cross section $\int ds\,\sigma_{\text{tot}}$, should be considered to distinguish the single-Z-boson or two-Z-boson hypotheses around 90 GeV. If M_{Z_2} turned out to be much larger than M_{Z_1}, i.e., $M_{Z_2} \approx 6M_{Z_1}$, we would have a mystery as to why nature breaks the gauge symmetry so unnaturally, $v'/v \approx 0$.

The parity-violating electron-muon interaction is for $Q^2 \approx 0$,

$$\mathcal{L}_{\text{int}} = -\frac{G_F}{\sqrt{2}} \; \frac{(\frac{1}{2} - 2\sin^2\theta_W) + \xi(\frac{1}{10} - \frac{8}{5}\sin^2\theta_W)}{1 + \xi/(1+b^2)} \times (\bar{e}\gamma_\alpha e\bar{\mu}\gamma^\alpha\gamma_5\mu + \bar{e}\gamma_\alpha\gamma_5 e\bar{\mu}\gamma^\alpha\mu), \tag{6.24}$$

where ξ and b^2 are defined in (6.9)–(6.11). Remarkably, for $\xi = 0$ the expression reduces to the Weinberg-Salam case, which can be verified by use of (3.3) and (3.4). In the standard model with the allowed value of $\sin^2\theta_W$, the asymmetries, such as average helicity and front-back asymmetry, are expected to be small due to the fortuitous cancellation by the term $(\frac{1}{2} - 2\sin^2\theta_W) \approx 0.02$ for $\sin^2\theta_W = 0.23$. In the present case, with the allowed limits on v'/v and $v/\tilde{v}$, $\xi(\frac{1}{10} - \frac{8}{5}\sin^2\theta_W) \approx -0.017$ which almost cancels the standard term, we expect an order suppression of the asymmetries compared to the standard ones. For the range of $\frac{1}{16} < \sin^2\theta_W < \frac{1}{4}$, where the current determination[12] of the mixing angle lies, the suppression of the asymmetries in $e^+e^- \to \mu^+\mu^-$ is a general feature of the present model. For completeness, the exact expression for the parity-violating electron-muon interaction is given for nonzero Q^2,

$$\mathcal{L}_{\rm int} = -\frac{G_F}{\sqrt{2}}\frac{N}{D}(\bar{e}\gamma^\alpha\gamma_5 e\bar{\mu}\gamma_\alpha\mu + \bar{e}\gamma^\alpha e\bar{\mu}\gamma_\alpha\gamma_5\mu), \quad (6.25)$$

where

$$N = [\tfrac{1}{2} - 2\sin^2\theta_W + \xi(\tfrac{1}{10} - \tfrac{8}{5}\sin^2\theta_W)] + \frac{Q^2}{M_Z^2}\frac{a^2(1+b^2)}{25r^2}(-\tfrac{1}{2} + 2\sin^2\theta_W + \tfrac{3}{2}r^2), \quad (6.26)$$

$$D = \left(1 + \frac{\xi}{1+b^2}\right) - \frac{Q^2}{M_Z^2}\left\{1 + \frac{a^2}{25}\left[1 + 16b^2 + \frac{1+b^2}{r^2}\left(1 - \frac{Q^2}{M_Z^2}\right)\right]\right\}. \quad (6.27)$$

Note the Q^2-dependent term in the numerator N. For $|Q| \simeq \frac{1}{2}M_Z$ which is the highest possible energy range at PETRA and PEP, this Q^2-dependent term is suppressed by $\sim(|Q|/M_Z)^2 a^2/25 \lesssim = .0025$. Therefore, the aforementioned conclusion derived from Eq. (6.24) is still valid for $|Q| \lesssim \frac{1}{2}M_Z$.

Though we do not expect large asymmetries in either case, a failure to detect the asymmetries at the level of the standard prediction ($\sin^2\theta_W = 0.25$ is almost certainly ruled out) may shed light on the scheme of the general linear gauge symmetries.

VII. ANOMALIES

We must mention that the U(5) theory presented here is afflicted with anomalies due to the extra U(1), thus rendering the theory nonrenormalizable. One not entirely satisfactory way of avoiding this difficulty is to simply introduce appropriate numbers of sufficiently massive "mirror" fermions[18] transforming according to $\underline{\bar{5}}$ and $\underline{10}$ of U(5). Mirror fermions are in fact suggested by certain theorectical ideas[19] on unification. However, this way out may not appeal to some people. In the case of SU(N) theory, we may have many more superheavy fermions in SU(5) × U(1) to cancel the anomaly if we judiciously started from appropriate representations in SU(N).[4] It should be noted here, as is explained in the Appendix, that GL(n, c) gauge theories are probably nonrenormalizable.

Because of the anomalies generated by the U(1) of quinticity, we are not, strictly speaking, able to calculate the value of $\tilde{g}$ using the renormalization group. However, because of Theorem 1 in Sec. V the neutral-current interaction at zero momentum transfer is actually independent of $\tilde{g}$. Of course, at nonzero momentum transfer, the value of $\tilde{g}$ would be relevant.

If we are allowed to use in effect the lowest-order renormalization-group equation to describe the change in $\tilde{g}$, we could calculate the value of $\tilde{g}$. We assume that the U(5) theory is effectively cut off at the unification scale of say 10^{16} GeV. The behavior of $g_{\rm strong}$, g, and g' as one goes from the unification scale to ordinary mass scales has been discussed by Georgi *et al.*[7] and is unchanged here. At the unification scale Λ the relative normalization of Y^2 and of $\tilde{Y}^2$ imply that $g' = (\frac{3}{2})^{1/2}\tilde{g}$. At ordinary mass scale μ, we have[20]

$$\frac{4\pi}{\tilde{g}^2} - \frac{3}{2}\frac{4\pi}{g'^2} = \frac{1}{3\pi}\sum\left[\left(\frac{\tilde{Y}}{2}\right)^2 - \frac{3}{2}\left(\frac{Y}{2}\right)^2\right]\ln\frac{\Lambda}{\mu}, \quad (7.1)$$

where the summation is over all fermions, each helicity state counted separately. With $\Lambda/\mu \sim 10^{16}$, we find, for three families of quarks and leptons, $\tilde{g}^2/g^2 \sim \frac{1}{2}$.

VIII. CONCLUSION

In conclusion, we have examined the neutral-current phenomenology of an SU(5) × U(1) theory in some detail. The assignment of U(1) quantum number was motivated by embedding SU(5) in bigger groups, SU(N) and GL(5, c). Whether this proposal that an additional local U(1) is relevant would have to be decided by experiment. Though the detailed predictions depend on the Higgs representations chosen to break the theory, we have shown that for a suitable choice it is possible to have three-parameter fit to neutral-current data. The three parameters correspond to two ratios of vacuum expectation values and $\sin^2\theta_W$. In certain limits of these parameters, the neutral-current phenomenology becomes indistinguishable from that of the standard $SU(2)_L \times U(1)$ theory. The small deviations predicted[21] will be tested, at least heopefully, by future neutral-current experiments, in particular $e^+e^- \to \mu^+\mu^-$, in addition to precision measurements of the usual neutral-current processes. A direct confirmation of two neutral Z bosons, one lower than 95 GeV and the other between 95 ~ 600 GeV (presumably 100 ~ 200 GeV), and only one charged $W^\pm$ around 85 GeV will reveal that nature might choose SU(5) × U(1) gauge symmetry, and will suggest considerations of bigger gauge groups such as SU(N) or GL(5, c). In the latter case, one can generalize other successful gauge theories to GL(n, c) theories. For example, if one would consider a generalization of $SU(3)_c$ to GL(3, c), after the symmetry breaking he will have an $SU(3)_c \times U(1)$ theory where the extra U(1) gauge boson couples to the number of color indices.[22] If all the quarks are color triplets, this extra U(1) gauge boson couples to the baryon number. As another example, a $GL(2, c)\times GL(1, c)$ theory can be considered as a generalization of the $SU(2)\times U(1)$ electroweak theory.

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APPENDIX: GL(n,c) THEORY

Let us extend a simple unitary gauge theory $SU(n)$ to a general linear gauge theory $GL(n, c)$. As mentioned in the text, the usual restriction to simple unitary groups comes about because the usual kinetic energy terms are invariant only under unitary transformations. For example, the kinetic-energy term for an n-component field ϕ, $\partial_\mu \phi^\dagger \partial^\mu \phi$, is invariant under transformation $\phi \to z\phi$ only if z is an $n\times n$ transformation by requiring the theory to be invariant under the transformation

$$\phi \to z\phi \,, \tag{A1}$$

where z is an arbitrary $n\times n$ complex matrix whose inverse z^{-1} exists. The set of such matrices clearly forms a group, known as the general linear group over the complex numbers $GL(n, c)$. In order for the theory to be invariant under $GL(n, c)$, the kinetic-energy term clearly has to be modified. This can be accomplished by introducing a "metric" field P, which is an $n\times n$ Hermitian matrix field transforming under $GL(n, c)$ as

$$P \to z^{\dagger -1} P z^{-1} . \tag{A2}$$

Then the $GL(n, c)$ invariant

$$\partial_\mu \phi^\dagger P \partial^\mu \phi \tag{A3}$$

is a candidate kinetic-energy term. The requirement that the kinetic-energy term be positive-definite implies that P has no negative eigenvalues. Thus P may be written in the form $P = p^\dagger p$ (the transformation in Eq. (A2) preserves this form).

This global $GL(n, c)$ symmetry could readily be extended to local symmetry.[23] We introduce gauge fields A_μ which transform under $GL(n, c)$ as

$$A_\mu \to z A_\mu z^{-1} + \frac{1}{ig} (\partial_\mu z) z^{-1} . \tag{A4}$$

Then the covariant derivative $D_\mu \phi$, defined as

$$D_\mu \phi = \partial_\mu \phi - i g A_\mu \phi \tag{A5}$$

transforms just as ϕ, namely

$$D_\mu \phi \to z D_\mu \phi \, . \tag{A6}$$

For $GL(n, c)$, A_μ denotes an $n\times n$ complex matrix and corresponds to $2n^2$ real gauge fields. It is useful to decompose A_μ as $A_\mu = B_\mu + iC_\mu$, where B_μ, C_μ are Hermitian matrices. We see that B_μ corresponds to the gauge fields associated with the unitary subgroup $U(n)$ of $GL(n, c)$.

A covariant derivative can also be introduced for the "metric" field P as

$$D_\mu P = \partial_\mu P + i g P A_\mu - i g A_\mu^\dagger P \, . \tag{A7}$$

Then $D_\mu P$ transforms as P and a kinetic-energy term for P can be introduced, viz.,

$$\mathrm{Tr}(D_\mu P) P^{-1} (D^\mu P) P^{-1} \, . \tag{A8}$$

Under the transformation in Eq. (1.2) the field P can be transformed to the unit matrix 1. The term in Eq. (A8) then generates a mass term for the anti-Hermitian part of A_μ, namely the gauge fields C_μ introduced above. We are then left with a gauge theory based on the group $U(n) = SU(n)\times U(1)$ with gauge fields B_μ.

In fact, the group $GL(n, c)$ is not simple and factorizes into $SL(n, c)\times GL(1, c)$, where $SL(n, c)$ denotes the multiplicative group of $n\times n$ complex matrices with determinant 1. The group $GL(1, c)$ factorizes into $U(1)\times GL(1,R)$. Thus, we should gauge the three factors $SL(n, c)$, $U(1)$, and $GL(1,R)$ separately and each of these factor groups has its own gauge coupling constant. Fortunately, the phenomenological analysis presented in the text does not depend on this fact.

Gauged general linear symmetries have been discussed previously in the literature, and most recently, by Cahill.[5] However, Cahill does not discuss the incorporation of fermions and possible application to the real world.

To construct a local $GL(n, c)$ theory, the covariant field strength is defined as usual as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu] \, . \tag{A9}$$

One easily verifies that $[D_\mu, D_\nu]\phi = -i g F_{\mu\nu}\phi$ and thus

$$F_{\mu\nu} \to z F_{\mu\nu} z^{-1} . \tag{A10}$$

The transformation law (A4) indicates that A_μ is in general not Hermitian. We decompose it into Hermitian components

$$A_\mu = B_\mu + iC_\mu \, . \tag{A11}$$

To construct a kinetic-energy term for ϕ we introduce a Hermitian matrix field P transforming as

$$P \to z^{\dagger -1} P z^{-1} . \tag{A12}$$

We can also take P to be of the form $P=p^\dagger p$. Then

$$D_\mu\phi^\dagger PD^\mu\phi \tag{A13}$$

is $\mathrm{GL}(n,c)$ invariant. Similarly, an invariant kinetic-energy term for the gauge fields may be written

$$\mathrm{Tr}F^\dagger_{\mu\nu}PF^{\mu\nu}P^{-1} \tag{A14}$$

which we note is Hermitian.

A covariant derivative can also be constructed from metric field P. One checks readily that

$$D_\mu P=\partial_\mu P+igPA_\mu-igA^\dagger_\mu P \tag{A15}$$

is Hermitian and transforms as P. Thus

$$\mathrm{Tr}D_\mu PP^{-1}D^\mu PP^{-1} \tag{A16}$$

is a suitable kinetic-energy term for P.

Finally, we can introduce a fermion field transforming as $\psi\to z\psi$ and the covariant derivative $D_\mu\psi=\partial_\mu\psi-igA_\mu\psi$. One must take care to write the Hermitian form for the fermion kinetic energy

$$\tfrac{1}{2}[\bar\psi Pi\gamma^\mu D_\mu\psi-(D_\mu\bar\psi)i\gamma^\mu P\psi] \tag{A17}$$

which could also be written upon integration by parts in the form

$$\bar\psi Pi\gamma^\mu D_\mu\psi+\tfrac{1}{2}\bar\psi i\gamma^\mu(D_\mu P)\psi\,. \tag{A18}$$

Note the second term in Eq. (A18).

A $\mathrm{GL}(n,c)$-invariant Lagrangian can now be constructed by various terms (A13), (A14), (A16), and (A17). Potential terms $V(\phi^\dagger P\phi)$ and fermion mass term $m\bar\psi P\psi$ can also be included. Note that the coefficient of the term (A16) in the Lagrangian cannot be normalized to unity; we denote it by κ. The Lagrangian is thus characterized by κ, g, m, and any parameters appearing in the scalar potential V.

We could, of course, construct a more complicated theory with many sets of scalar fields ϕ_i and fermions ψ_i transforming differently under $\mathrm{GL}(n,c)$. We could also have ψ_L and ψ_R transforming differently.

We now show that the "metric" field P can actually be removed entirely from the theory. As a result the (Hermitian) fields C_μ (the "anti-Hermitian" part of A_μ) become massive.

One way of seeing this is to note that the transformation law $P\to z^{\dagger-1}Pz^{-1}$ implies the gauge equivalence of any field configuration for P with the configuration $P=1=n\times n$ unit matrix. In other words, P can always be "gauged away." More precisely, note the identity

$$F_{\mu\nu}(A')=pF_{\mu\nu}(A)p^{-1}\,, \tag{A19}$$

where

$$A'_\mu=pA_\mu p^{-1}+\frac{1}{ig}\partial_\mu pp^{-1}\,. \tag{A20}$$

Thus, writing P in the form $P=p^\dagger p$, the gauge-field kinetic energy becomes

$$\mathrm{Tr}F^\dagger(A)PF(A)P^{-1}=\mathrm{Tr}F^\dagger(A')F(A')\,. \tag{A21}$$

Also,

$$pD_\mu(A)\psi=D_\mu(A')\psi'\,, \tag{A22}$$

where $\psi'=p\psi$ and thus

$$\bar\psi Pi\gamma^\mu D_\mu(A)\psi=\bar\psi' i\gamma^\mu D_\mu(A')\psi'\,. \tag{A23}$$

Similarly, the scalar-field kinetic-energy term can be rewritten as

$$[D_\mu(A')\phi']^\dagger[D_\mu(A')\phi']\,. \tag{A24}$$

Finally, the covariant derivative

$$D_\mu P=p^\dagger ig(A'_\mu-A'^\dagger_\mu)p=-2gp^\dagger C'_\mu p \tag{A25}$$

and the "kinetic-energy" term for P, (A16), becomes a mass term for C'_μ

$$4g^2\kappa\mathrm{Tr}C'_\mu C'^\mu\,. \tag{A26}$$

Thus, the n^2 components of P emerge as the longitudinal components of the n^2 C_μ fields.

For notational simplicity we drop the primes from now on. It should be stressed that it would be somewhat misleading to refer to the phenomenon just described as a spontaneous symmetry breakdown. We have simply rewritten the theory by gauging P away completely; we are able to do so because P is always a pure gauge.

An interesting point is that the massive gauge fields C_μ do not couple to the fermions as one can see by inspecting the fermion kinetic energy (A23).

We now turn our attention to the gauge fields. Note that the field strength can be decomposed into Hermitian and anti-Hermitian parts as

$$F^h_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu+ig[B_\mu,B_\nu]+ig[C_\mu,C_\nu]\,, \tag{A27}$$

$$F^a_{\mu\nu}=\partial_\mu C_\nu-\partial_\nu C_\mu+g[B_\mu,C_\nu]+g[C_\mu,B_\nu]\,. \tag{A28}$$

Thus, the massive gauge fields C_μ interact with the massless fields B_μ.

Gauged $\mathrm{GL}(1,c)$ theory provides an amusing special case in which C_μ decouples from both fermions and the massless gauge field B_μ. Thus, the quantum electrodynamics of fermions actually has a hidden $\mathrm{GL}(1,c)$ gauge symmetry. We do not know whether this hidden symmetry in an apparent U(1) gauge symmetric theory is of any relevance. It is amusing to note that historically[24] Weyl first sought (in 1918) to construct a theory of electromagnetism by gauging $\mathrm{GL}(1,R)$. It was only after the invention of quantum mechanics that Fock and London inserted the appropriate i and gauged U(1). Incidentally, the complete decoupling of C_μ does not hold if there are charged scalars in the Lagrangian. The coupling of A_μ to a scalar field reads

$$igA_\mu^\dagger\phi^\dagger\partial^\mu\phi - igA_\mu\partial^\mu\phi^\dagger\phi + g^2A_\mu^\dagger A^\mu\phi^\dagger\phi$$
$$= igB_\mu(\phi^\dagger\partial^\mu\phi - \partial^\mu\phi^\dagger\phi) + gC_\mu(\phi^\dagger\partial^\mu\phi + \partial^\mu\phi^\dagger\phi)$$
$$+ g^2(B_\mu{}^2 + C_\mu{}^2)\phi^\dagger\phi\,. \tag{A29}$$

The decoupling of C_μ from fermions strongly suggests that $\mathrm{GL}(n, c)$ theories are not renormalizable. In particular, the equality of various couplings between gauge bosons, such as the BBB and BCC couplings, would be destroyed by a cutoff-dependent amount in one-loop order.

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[1]S. L. Glashow, Nucl. Phys. 22, 579 (1961); A. Salam and J. C. Ward, Phys. Lett. 13, 168 (1964); S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); 27, 1688 (1971); A. Salam, in ***Elementary Particle Theory: Relativistic Groups and Analyticity*** (*Nobel Symposium No. 8*), edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1969), p. 367.

[2]H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974).

[3]S. Weinberg, Phys. Rev. D 13, 974 (1976); L. Susskind, *ibid.* 20, 2619 (1979).

[4]H. Georgi, Nucl. Phys. B156, 126 (1979).

[5]K. Cahill, Phys. Rev. D 18, 2930 (1978) and earlier references cited therein.

[6]Yet a different direction involves the very interesting concept of supersymmetry. Y. A. Golfand and E. P. Likhtman, Zh. Eksp. Teor. Fiz. Pis'ma Red. 13, 452 (1971) [JETP Lett. 13, 323 (1971)]; J. Wess and B. Zumino, Phys. Lett. B49, 54 (1974).

[7]H. Georgi, H. Quinn, and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).

[8]Our notation is such that there should not be any confusion between the gauge fields B_μ in the Appendix and the gauge field B_μ which appears in the standard SU(2) × U(1) model.

[9]The recent literature includes N. G. Deshpande and D. Iskandar, Phys. Rev. Lett. 42, 20 (1979); E. Ma, Phys. Lett. 65B, 468 (1976); H. Fritzsch and P. Minkowski, Nucl. Phys. B103, 61 (1976); S. Barr and P. Langacker, Phys. Rev. Lett. 42, 1654 (1979); A. Davidson, M. Koca, and K. D. Wali, *ibid.* 43, 92 (1979); K. Kang and J. E. Kim, Phys. Rev. D 14, 1903 (1976); 18, 3467 (1978); D. Darby and G. Grammer, Nucl. Phys. B349, 490 (1978); H. Georgi and A. Pais, Phys. Rev. D 10, 539 (1974); D. Darby, *ibid.* 11, 2552 (1975); J. Schechter and Y. Ueda, *ibid.* 2, 736 (1970).

[10]Φ^4 is taken to be the electrically neutral component of ϕ^α.

[11]Note that the relevant physics is not altered if we had chosen $\phi^{\alpha\beta\gamma\delta\epsilon}$ instead of $\phi_{\alpha\beta\gamma\delta\epsilon}$.

[12]The notation is the same as that of a recent paper by P. Langacker, J. E. Kim, M. Levine, H. H. Williams, and D. P. Sidhu, Univ. of Pennsylvania Report No. COO-3071-243, 1979 (unpublished); J. E. Kim, P. Langacker, M. Levine, D. P. Sidhu, and H. H. Williams (unpublished).

[13]C. Y. Prescott *et al.*, Phys. Lett. 77B, 347 (1978); 84B, 524 (1979).

[14]H. Georgi and S. Weinberg, Phys. Rev. D 17, 275 (1978).

[15]Some version of this theorem was also known to H. Fritzsch and P. Minkowski, Nucl. Phys. B103, 61 (1976); J. C. Pati, S. Rajpoot, and A. Salam, Phys. Rev. D 17, 131 (1978).

[16]The theorem is stated in Ref. 14 without restricting G_1 and G_2 to have one neutral generator each. The proof follows simply because $(\mu^{-2})_{ij}$ is block diagonal.

[17]J. Godine and A. Hankey, Phys. Rev. D 6, 3301 (1972); V. Cung, A. Mann, and E. Paschos, Phys. Lett. 41B, 355 (1972); A. Love, Lett. Nuovo Cimento 5, 113 (1972); R. Bundy, Phys. Lett. 45B, 340 (1973); A. McDonald, Nucl. Phys. B75, 343 (1974); B. Kayser, S. P. Rosen, and E. Fischbach, Phys. Rev. D 11, 2547 (1975).

[18]This terminology was introduced by A. Salam.

[19]F. Wilczek and A. Zee (unpublished).

[20]For example, A. Zee, Phys. Rev. D 18, 2600 (1978).

[21]These deviations would have to be disentangled from other effects, such as the strong interaction effects discussed by J. Collins, F. Wilczek, and A. Zee, Phys. Rev. D 18, 242 (1978).

[22]P. Langacker, G. Segrè, and A. Weldon, Phys. Lett. 73B, 87 (1978).

[23]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[24]For a review on the history of gauge theories, see C. N. Yang, Proc. N. Y. Acad. Sci. 294, 86 (1977).