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Current-algebra analysis of CP violations in $K \to 3\pi$ decay in the six-quark Weinberg-Salam model

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The standard current-algebra technique is used to relate the parameter $\eta_{+-0} \equiv \langle \pi^+\pi^-\pi^0 | H_w | K_S\rangle / \langle \pi^+\pi^-\pi^0 | H_w | K_L\rangle$ in $K \to 3\pi$ decay to the CP-violating parameters of $K \to 2\pi$ decays in the context of the six-quark Weinberg-Salam theory (Kobayashi-Maskawa model).

The gauge theory of weak and electromagnetic interactions has not only aesthetic appeal but also quite substantial experimental support. Even though the experimental data has not ruled out all the other models, it is rather impressive that the simplest Weinberg-Salam model[1] with three doublets of quarks and leptons is able to describe most, if not all, of the weak-interaction data accumulated over the last few years. As pointed out by Kobayashi and Maskawa,[2] one of the nice features of the Weinberg-Salam model with three left-handed doublets of quarks is the presence of the CP violation in the gauge coupling to the quarks. So far, most of the predictions on CP-violating processes from this model are hampered by strong-interaction dynamics.[3] Recently, there has been some experimental interest in measuring CP violations in $K \to 3\pi$ decays.[4] In this paper, the standard current-algebra technique is used to relate the CP violations in $K \to 3\pi$ decays to the better known CP-violating $K \to 2\pi$ decays in the context of the Weinberg-Salam model. From the successes of the current-algebra analysis of the usual CP-conserving $K \to 3\pi$ decays,[5] one can expect that the result from this type of analysis for the CP-violating part of the $K \to 3\pi$ decays will not depend very much on the details of strong-interaction dynamics.

In the $K \to 3\pi$ amplitudes, with the kinematics given by

$$K^i(p) \to \pi_1^a(k_1) + \pi_2^b(k_2) + \pi_3^c(k_3),$$

we can define the Lorentz-invariant energy variables s_i as

$$s_i = (p - k_i)^2, \tag{1}$$

with $s_1 + s_2 + s_3 = 3s_0 = 3(m_\pi^2 + \frac{1}{3}m_k^2)$. The indices i label the isospin of the kaon, and a, b, c label the isospin of the pions. In the usual approximation that the dependence of the Dalitz plot on the energies s_i is at most linear, the CP-conserving part of the $K \to 3\pi$ amplitudes can be parametrized as[6]

$$\langle \pi^+\pi^+\pi^- | H_w^+ | K^+\rangle = i\left[(2\alpha_1 - \alpha_3) + (\beta_1 - \tfrac{1}{2}\beta_3 + \sqrt{3}\gamma_3)\left(\frac{s_3 - s_0}{m_\pi^2}\right)\right], \tag{2a}$$

$$\langle \pi^0\pi^0\pi^+ | H_w^+ | K^+\rangle = i\left[(-\alpha_1 + \tfrac{1}{2}\alpha_3) + (\beta_1 - \tfrac{1}{2}\beta_3 - \sqrt{3}\gamma_3)\left(\frac{s_3 - s_0}{m_\pi^2}\right)\right], \tag{2b}$$

$$\langle \pi^+\pi^-\pi^0 | H_w^+ | K_1\rangle = \frac{-2i\gamma_3}{\sqrt{3}m_\pi^2}(s_1 - s_2), \tag{2c}$$

$$\langle \pi^+\pi^-\pi^0 | H_w^+ | K_2\rangle = i\left[(\alpha_1 + \alpha_3) - (\beta_1 + \beta_3)\left(\frac{s_3 - s_0}{m_\pi^2}\right)\right], \tag{2d}$$

$$\langle \pi^0\pi^0\pi^0 | H_w^+ | K_2\rangle = -3i(\alpha_1 + \alpha_3), \tag{2e}$$

where α_1, β_1 (α_3, β_3) come from the transition into the $I=1$ state of three pions caused by the $\Delta I = \frac{1}{2}$ ($\Delta I = \frac{3}{2}$) part of the CP-conserving weak Hamiltonian H_w^+, and γ_3 comes from the $I=2$ state of three pions. The CP eigenstates of the neutral kaons are defined by

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle), \quad |K^2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle, \tag{3}$$

with

$$|\bar{K}^0\rangle = CP|K^0\rangle.$$

Note that the CP-even state K_1 can decay into the $\pi^+\pi^-\pi^0$ state through the $\Delta I = \frac{3}{2}$ part of the Hamiltonian without CP violation. Similarly, the CP-violating part of the $K \to 3\pi$ amplitudes can be parametrized as

$$\langle\pi^+\pi^+\pi^-|H_w^-|K^+\rangle = i^2\left[(2\alpha_1' - \alpha_3') + (\beta_1' - \tfrac{1}{2}\beta_3' + \sqrt{3}\gamma_3')\left(\frac{s_3 - s_0}{m_\pi^2}\right)\right], \quad (4a)$$

$$\langle\pi^0\pi^0\pi^+|H_w^-|K^+\rangle = i^2\left[(-\alpha_1' + \tfrac{1}{2}\alpha_3') + (\beta_1' - \tfrac{1}{2}\beta_3' - \sqrt{3}\gamma_3')\left(\frac{s_3 - s_0}{m_\pi^2}\right)\right], \quad (4b)$$

$$\langle\pi^+\pi^-\pi^0|H_w^-|K_1\rangle = i^2\left[(\alpha_1' + \alpha_3') - (\beta_1' + \beta_3')\left(\frac{s_3 - s_0}{m_\pi^2}\right)\right], \quad (4c)$$

$$\langle\pi^+\pi^-\pi^0|H_w^-|K_2\rangle = i^2\left(\frac{-2\gamma_3'}{\sqrt{3}\,m_\pi^2}\right)(s_1 - s_2), \quad (4d)$$

$$\langle\pi^0\pi^0\pi^0|H_w^-|K_1\rangle = 3(\alpha_1' + \alpha_3'), \quad (4e)$$

where H_w^- is the CP-violating part of the weak Hamiltonian H_w, and $\alpha_i', \beta_i', \gamma_i'$ are defined in a similar way as $\alpha_i, \beta_i, \gamma_i$. The parameters α_i, α_i', etc. are real if final-state interactions are neglected but in general have complex phases completely determined by CPT and unitarity from the final-state strong-interaction S matrix. The final 3π states can be classified by the isospin and the permutation symmetry of three pions. To a good approximation[7] the phase is determined by the isospin and the permutation symmetry of the 3π states; thus there is one phase for all the α's, the amplitudes of the completely symmetrical $I=1$ 3π state, one for all the β's, the amplitudes of the $I=1$ 3π state with mixed symmetry, and one for all the γ's, the amplitudes of the $I=2$ state.

The experimental quantity of interest is

$$\eta_{+-0}(s_1, s_2, s_3) = \frac{\langle\pi^+\pi^-\pi^0|H_w|K_S\rangle}{\langle\pi^+\pi^-\pi^0|H_w|K_L\rangle}, \quad (5)$$

where

$$|K_S\rangle = (|K_1\rangle + \rho|K_2\rangle)/\sqrt{1+|\rho|^2}, \quad (6a)$$

$$|K_L\rangle = (|K_2\rangle + \rho|K_1\rangle)/\sqrt{1+|\rho|^2}. \quad (6b)$$

Because of the CP-conserving amplitude (2c), the quantity η_{+-0} is not in general a CP-violating quantity. We therefore limit ourselves to the case $s_1 = s_2$ to eliminate the amplitude (2c) so that η_{+-0} depends only on s_3. From Eqs. (3)–(6), we then find

$$\eta_{+-0}(y) = \rho + i\frac{(\alpha_1' + \alpha_3') - (\beta_1' + \beta_3')y}{(\alpha_1 + \alpha_3) - (\beta_1 + \beta_3)y}, \quad y \equiv (s_3 - s_0)/m_\pi^2. \quad (7)$$

The isospin decomposition of the $K\to 2\pi$ amplitudes are

$$\langle\pi^+\pi^0|H_w^+|K^+\rangle = \tfrac{1}{2}(\tfrac{3}{2})^{1/2}f_3, \quad (8a)$$

$$\langle\pi^+\pi^-|H_w^+|K_0\rangle = -(\tfrac{1}{6})^{1/2}f_1 + \frac{1}{\sqrt{12}}f_3, \quad (8b)$$

$$\langle\pi^0\pi^0|H_w^+|K_0\rangle = (\tfrac{1}{6})^{1/2}f_1 + \frac{1}{\sqrt{3}}f_3, \quad (8c)$$

for the CP-conserving part and

$$\langle\pi^+\pi^0|H_w^-|K^+\rangle = \frac{i}{2}\left(\frac{3}{2}\right)^{1/2}g_3, \quad (9a)$$

$$\langle\pi^+\pi^-|H_w^-|K_0\rangle = i\left[-\left(\frac{1}{6}\right)^{1/2}g_1 + \frac{1}{\sqrt{12}}g_3\right], \quad (9b)$$

$$\langle\pi^+\pi^-|H_w^-|K_0\rangle = i\left[\left(\frac{1}{6}\right)^{1/2}g_1 + \frac{1}{\sqrt{3}}g_3\right], \quad (9c)$$

for the CP-violating part. The parameters f_1, g_1 denote the $\Delta I = \frac{1}{2}$ transitions, and f_3, g_3 denote the $\Delta I = \frac{3}{2}$ transitions. They are related to the standard notation for the $K\to 2\pi$ amplitudes[8] by

$$f_1 = -2\,\mathrm{Re}a_0e^{i\delta_0}, \quad f_3 = 2\,\mathrm{Re}a_2e^{i\delta_2}, \quad g_1 = -2\,\mathrm{Im}a_0e^{i\delta_0}, \quad g_3 = 2\,\mathrm{Im}a_2e^{i\delta_2}, \quad (10)$$

where

$$\langle I=n|H_w|K^0\rangle = a_ne^{i\delta_n}. \quad (11)$$

The experimentally measured quantities are

$$\eta_{+-} = \epsilon + \epsilon'/(1 + \omega/\sqrt{2}), \quad (12a)$$

$$\eta_{00} = \epsilon - 2\epsilon'/(1 - \sqrt{2}\,\omega), \quad (12b)$$

where[9]

$$\epsilon = \rho + ig_1/f_1, \quad (13a)$$

$$\sqrt{2}\,\epsilon' = -i(f_3/f_1)(g_3/f_3 - g_1/f_1), \quad (13b)$$

$$\omega = -f_3/f_1. \quad (13c)$$

The weak Hamiltonian responsible for $K\to 3\pi$ decays in the six-quark Kobayashi-Maskawa model can be written $H_w = H_w^+ + H_w^-$ with

$$H_w^+ = \frac{G_F}{\sqrt{2}}A\bar{d}\gamma^\lambda(1-\gamma_5)[(u\bar{u} - c\bar{c}) + K(c\bar{c} - t\bar{t})]\times\gamma_\lambda(1-\gamma_5)s, \quad (14a)$$

$$H_w^- = \frac{G_F}{\sqrt{2}}iAK'\bar{d}\gamma^\lambda(1-\gamma_5)(c\bar{c} - t\bar{t})\gamma_\lambda(1-\gamma_5)s, \quad (14b)$$

where[10] A, K, and K' involve the three mixing angles and the CP-violating phase δ. With the phase convention chosen by Kobayashi and Maskawa,[2] the CP-violating piece H_w^- satisfies the $\Delta I = \frac{1}{2}$ rule so that

$$\alpha_3' = \beta_3' = \gamma_3' = g_3 = 0, \quad (15)$$

and only the CP-conserving piece H_w^+ violates the $\Delta I = \frac{1}{2}$ rule.

Since both H_w^+ and H_w^- consist purely of left-handed currents, we have for both the usual relation

$$[Q_5^\gamma, H_{PC}] = -[Q^\gamma, H_{PV}], \tag{16}$$

where H_{PV} is the parity-violating piece and H_{PC} the parity-conserving piece of the Hamiltonian and

$$Q_5^\gamma = \int d^3x A_0^\gamma(x),$$

$$Q^\gamma = \int d^3x V_0^\gamma(x).$$

The soft-pion limits of the $K\to 3\pi$ amplitude can then be written using the usual current-algebra technique as

$$\lim_{k_3\to 0} \langle \pi_1^\alpha \pi_2^\beta \pi_3^\gamma | H_{PC} | K^i\rangle = -\frac{i}{f_\pi}\langle \pi_1^\alpha \pi_2^\beta | [Q_5^\gamma, H_{PC}] | K^i\rangle = \frac{i}{f_\pi}\langle \pi_1^\alpha \pi_2^\beta | [Q^\gamma, H_{PV}] | K^i\rangle . \tag{17}$$

When the isospin operator Q^γ acts on the states $|K_i\rangle$ and $|2\pi\rangle$, the $K\to 3\pi$ amplitudes are related to the $K\to 2\pi$ amplitudes in the soft-pion limits. For example,

$$\begin{aligned}\langle \pi_1^+ \pi_2^- \pi_3^0 | H_{PC} | K^0\rangle &\xrightarrow[k_1\to 0]{} \frac{-i}{f_\pi}[\langle \pi^0\pi^0 | H_{PV} | K^0\rangle + \langle \pi^+\pi^- | H_{PV} | K^0\rangle],\\ &\xrightarrow[k_2\to 0]{} \frac{i}{f_\pi}[\langle \pi^0\pi^0 | H_{PV} | K^0\rangle + \langle \pi^+\pi^- | H_{PV} | K^0\rangle - \frac{1}{\sqrt{2}}\langle \pi^+\pi^0 | H_{PV} | K^+\rangle],\\ &\xrightarrow[k_3\to 0]{} \frac{i}{f_\pi}\tfrac{1}{2}\langle \pi^+\pi^- | H_{PV} | K^0\rangle .\end{aligned} \tag{18}$$

The linear dependence on the energies of the $K\to 3\pi$ amplitudes given in Eqs. (2) and (4) can then be used to extrapolate from the unphysical soft-pion limits to the physical region. In terms of the parameters previously defined, we then obtain

$$\frac{\alpha_1}{f_1} = \frac{\alpha_1'}{g_1} = -\frac{1}{f_\pi}\frac{1}{6\sqrt{3}}, \tag{19a}$$

$$\frac{\alpha_3}{f_3} = \frac{\alpha_3'}{g_3} = -\frac{1}{f_\pi}\frac{1}{3\sqrt{6}}, \tag{19b}$$

$$\frac{\beta_1}{f_1} = \frac{\beta_1'}{g_1} = \frac{1}{f_\pi}\frac{1}{2\sqrt{3}}\left(\frac{m_\pi^2}{m_K^2 - m_\pi^2}\right), \tag{19c}$$

$$\frac{\beta_3}{f_3} = \frac{\beta_3'}{g_3} = -\frac{1}{f_\pi}\frac{5}{4\sqrt{6}}\left(\frac{m_\pi^2}{m_K^2 - m_\pi^2}\right), \tag{19d}$$

$$\frac{\gamma_3}{f_3} = \frac{\gamma_3'}{g_3} = \frac{1}{f_\pi}\left(\frac{9}{8}\frac{1}{\sqrt{2}}\right)\left(\frac{m_\pi^2}{m_K^2 - m_\pi^2}\right). \tag{19e}$$

We first apply these results to the center of the Dalitz plot where $y=0$. From Eqs. (7) and (15),

$$\eta_{+-0}(0) = \rho + i\alpha_1'/(\alpha_1 + \alpha_3). \tag{20}$$

We now use one of the soft-pion results, the first part of Eq. (19a),

$$\alpha_1'/\alpha_1 = g_1/f_1 . \tag{21}$$

In general, soft-pion results have problems with phases; thus we do not expect from unitarity that α_1 has the same phase as f_1 as implied by Eq. (19a). However, Eq. (21) does not have this problem since both sides are real by unitarity.

Using this result together with Eq. (13a) we obtain

$$\eta_{+-0}(0) - \epsilon = i\frac{g_1}{f_1}\frac{\alpha_3}{\alpha_1 + \alpha_3}. \tag{22}$$

The result that this difference is purely imaginary has been noted before.[9] Using Eqs. (12) and (13) to eliminate g_1/f_1 and remembering $g_3 = 0$, we find

$$\eta_{+-0} - \eta_{+-} = -(\eta_{+-} - \eta_{00})\left(\frac{1}{3} + \frac{\sqrt{2}}{3}\frac{\alpha_3/\alpha_1}{f_3/f_1}\right), \tag{23}$$

where we have neglected terms of the order α_3/α_1 and f_3/f_1 compared to unity. Since α_3/α_1 and f_3/f_1 are known approximately from experiment, Eq. (23) may be used directly to estimate $\eta_{+-0} - \eta_{+-}$. By combining Eqs. (19a) and (19b) we can obtain a second soft-pion result

$$\alpha_3/\alpha_1 = \sqrt{2} f_3/f_1 . \tag{24}$$

Substituting Eq. (24) into Eq. (23), we obtain

$$\eta_{+-0} - \eta_{+-} = \eta_{00} - \eta_{+-}, \tag{25}$$

or $\eta_{+-0} = \eta_{00}$. Equation (24), unlike Eq. (21), is inconsistent with the unitarity constraints since the left-hand side is real whereas the right-hand side has the phase factor $\exp[i(\delta_2 - \delta_0)]$. Thus, Eq. (25) also is incorrect as to phase. If we use only the magnitude relation in Eq. (24), we obtain

$$|\eta_{+-0} - \epsilon| = \tfrac{2}{3}|\eta_{+-} - \eta_{00}| . \tag{26}$$

Final-state interaction effects would also be expected to affect the magnitude relation of Eq. (24); both theory[11] and experiment indicate it should be good within a factor of 2 so that we expect Eq. (26) to be valid within this factor.

In the same way, we can estimate η_{+-0} at other values of y. Using Eqs. (19) in the form $\beta_1'/\beta_1 = \alpha_1'/\alpha_1$, we find in place of Eq. (23)

$$\eta_{+-0}-\eta_{+-}=-(\eta_{+-}-\eta_{00})\left(\frac{1}{3}+\frac{\sqrt{2}}{3}\frac{\alpha_3/\alpha_1}{f_3/f_1}-\frac{\sqrt{2}}{3}\frac{\beta_3/\alpha_1}{f_3/f_1}y\right), \tag{27}$$

where we have also neglected β_1/α_1 and β_3/α_1 compared to unity. Again this result is in terms of *CP*-conserving observables. From Eqs. (19c) and (19d)

$$\frac{\beta_3/\alpha_1}{f_3/f_1}=\tfrac{15}{4}\sqrt{2}\frac{m_\pi^2}{m_K^2}, \tag{28}$$

so that Eq. (27) becomes

$$\eta_{+-0}-\eta_{+-}=-(\eta_{+-}-\eta_{00})\left(1-\frac{5}{2}\frac{s_3-s_0}{m_K^2-m_\pi^2}\right). \tag{29}$$

Equation (29), like Eq. (25), is inconsistent with the unitarity constraints. It should, however, give the approximate magnitude of the variation of η_{+-0} over the Dalitz plot.

Recently, it has been suggested[12] that the penguin diagrams coming from gluon exchange might help to explain the $\Delta I=\frac{1}{2}$ rule in the nonleptonic decay. If an effective Hamiltonian is defined as involving purely left-handed pieces plus the penguin operator

$$H'=C[\bar{s}\gamma_\mu(1-\gamma_5)\lambda^\alpha d](\bar{u}\gamma^\mu\lambda^\alpha u+\bar{d}\gamma^\mu\lambda^\alpha d+\cdots)+\text{H.c.},$$

where C is some constant and λ^α is the color SU(3) matrix, our results still hold. This is because H' also satisfies the commutation relation of Eq. (16) because the factor $(\bar{u}\gamma^\mu\lambda^\alpha u+\bar{d}\gamma^\mu\lambda^\alpha d)$ is an isospin singlet.

Our major result is given as either Eq. (23) or (26). This shows that $\eta_{+-0}-\eta_{+-}$ (or $\eta_{+-0}-\epsilon$), which vanishes in the superweak model, is expected to be approximately equal in magnitude to $(\eta_{+-}-\eta_{00})$ in the Kobayashi-Maskawa (KM) model. Experimentally we know that $|\eta_{+-}-\eta_{00}|$ is less than 0.06 times $|\eta_{+-}|$, and theoretically we expect a small nonzero value of the order of 0.01 to 0.05, the exact value depending on the KM parameters and the treatment of penguin terms.[13] Thus, an experimental determination of $\eta_{+-0}-\eta_{+-}$ to the same accuracy as $\eta_{+-}-\eta_{00}$ would be of interest. Proposed experiments[14] aim at measuring η_{00}/η_{+-} to 1%, but the most optimistic proposal[4] to measure η_{+-0} aims at a 25% accuracy. From the point of view of the KM model, this η_{+-0} proposal is therefore not competitive.

The reason for the small value of $\eta_{+-}-\eta_{00}$ is well known: As long as the $\Delta I=\frac{1}{2}$ rule holds at least as well for *CP*-odd amplitudes as it does for *CP*-even, ϵ' is bound to be small. From a general phenomenological point of view, this approximate $\Delta I=\frac{1}{2}$ rule does not require that $|\eta_{+-0}-\epsilon|$ be small compared to ϵ. However, for the relatively simple KM Hamiltonian, this follows from the soft-pion arguments; for more complicated Hamiltonians, this need not be true. In particular, for the $SU(2)_L\times SU(2)_R$ model of *CP* violation,[15] $\eta_{+-}-\eta_{00}$ vanishes but $|\eta_{+-0}-\epsilon|$ may be of the order of magnitude of ϵ itself.

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