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Noncontribution of $n \neq 4$ axial-vector anomaly

Kenzo Ishikawa and Kimball A. Milton*
Department of Physics, University of California, Los Angeles, California 90024
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It is argued that any term in the axial-vector anomaly proportional to $n-4$, in the n-dimensional regularization scheme, does not contribute to higher-order processes, in particular those with overlapping divergences, provided the usual cancellation of the $n=4$ anomaly by compensating fermions is effected.

I. INTRODUCTION

Recently, Frampton has questioned[1] the renormalizability of gauge theories of weak interactions, or more generally, what is now called quantum flavor dynamics. His point is that there is an additional contribution to the familiar axial-vector anomaly[2] which depends on fermion masses and so cannot be canceled by standard methods.[3,4] He argues that if one uses n-dimensional regularization, a further anomalous term arises proportional to $n-4$, hitherto disregarded,[5] which can contribute to processes with overlapping divergences, for example, violating naive Ward identities and therefore destroying renormalizability.

An immediate objection to Frampton's work is that no satisfactory definition of γ_5 exists in other than four dimensions.[6] However, we will show here that, even if Frampton is correct, the extra term does not yield nonzero contributions as $n\to 4$ in higher-order graphs. Moreover, we will provide a general argument based on unitarity that only the usual anomaly can contribute to any process, and therefore that the standard cancellations are quite sufficient.

II. DIAGRAMMATIC ANALYSIS

The triangle process is represented by Fig. 1. At the vertices we have $\Gamma_\mu=(\gamma_\mu, \gamma_\mu\gamma_5)$; the processes in question involve either one or three γ_5's. For ease of discussion, let us consider a case in which all the internal fermion masses are equal to, say, m_1. Then, if $I_{\mu\nu\lambda}$ represents the triangle amplitude, and $I_{\mu\nu}$ represents the same amplitude with the replacements

$$\begin{aligned}&\gamma_5\gamma_\lambda \to 2m_1\gamma_5,\\ &\gamma_\lambda\to 0,\end{aligned} \tag{1}$$

the usual triangle anomaly is

$$q^\lambda I_{\mu\nu\lambda}-I_{\mu\nu}=\frac{1}{4\pi^2}\epsilon_{\mu\nu\alpha\beta}p_1^\alpha p_2^\beta\,. \tag{2}$$

By working in n dimensions, Frampton[1] finds an additional term on the right-hand side of (2),

$$\frac{1}{4\pi^2}\epsilon_{\mu\nu\alpha\beta}p_1^\alpha p_2^\beta(n-4)f(p_1,p_2,m_1)\,. \tag{3}$$

We will not require the detailed form of f in the following, but merely note that as $p_1\to\infty$, it satisfies

$$f(p_1,p_2,m_1)-f(p_1,p_2,m_2)\sim B\,\frac{m_1^2-m_2^2}{p_1^2}\,, \tag{4}$$

where B is a function involving p_1 only logarithmically. Because of the mass dependence of (4), it cannot be canceled by the standard mechanisms.[3] The question is, can this new term cause problems in higher-order graphs, for example those with overlapping divergences?

We can answer in the negative by examining some examples. For simplicity we will use the Feynman gauge, and carry out the estimation in four dimensions—any divergence corresponds to a pole at $n=4$ which would cancel the zero at $n=4$ in the new anomaly. In Fig. 2(a) we have a simple example of an overlapping diagram. To cancel the usual anomaly we introduce fermions of masses m_1 and m_2 with opposite γ_5 couplings. Then the anomalous part of the divergence of the amplitude has the form

$$\begin{aligned}q^\lambda I_\lambda=\int\frac{(dp_1)}{(2\pi)^4}\{&q^\lambda[I_{\mu\nu\lambda}(p_1,q-p_1,m_1)-I_{\mu\nu\lambda}(p_1,q-p_1,m_2)]\\ &-2[m_1I_{\mu\nu}(p_1,q-p_1,m_1)-m_2I_{\mu\nu}(p_1,q-p_1,m_2)]\}\frac{1}{p_1^2}\frac{1}{(q-p_1)^2}\gamma^\mu\frac{1}{\gamma(l-p_1)+m}\gamma^\nu\\ \sim(n-4)\int&\frac{(dp_1)}{(2\pi)^4}B\epsilon_{\mu\nu\alpha\beta}p_1^\alpha q^\beta\,\frac{m_1^2-m_2^2}{p_1^2}\,\frac{1}{(p_1^2)^3}\gamma^\mu\gamma p_1\gamma^\nu\to 0,\quad\text{as } n\to 4\end{aligned} \tag{5}$$

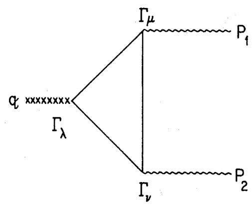

FIG. 1. The "triangle" amplitude. Crosses denote the vertex where multiplication by q^λ takes place.

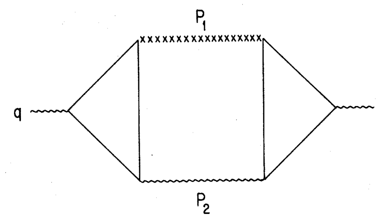

FIG. 3. Example in which the pp part of a boson propagator (denoted by crosses) couples anomalous triangle diagrams.

since the integral is convergent. Similarly, we can consider the processes in Figs. 2(b) and 2(c). The anomalous divergence of the first behaves as

$$q^\lambda F_{\lambda ..}(q) \sim (n-4) \int \frac{(dp_1)}{(2\pi)^4} B\epsilon_{\mu\nu\alpha\beta} p_1^\alpha q^\beta \times \frac{m_1^2 - m_2^2}{p_1^2} \frac{1}{p_1^2} \frac{1}{(p_1-q)^2} = 0, \tag{6}$$

while that of the second also vanishes identically,

$$q^\lambda \Pi_\lambda^\gamma \sim (n-4) \int \frac{(dp_1)}{(2\pi)^4} B\epsilon_{\mu\nu\alpha\beta} p_1^\alpha q^\beta \times \frac{m_1^2 - m_2^2}{p_1^2} \frac{1}{p_1^2} \frac{1}{(p_1-q)^2} I^{\mu\nu\gamma}, \tag{7}$$

$$I^{\mu\nu\gamma} = (p_1 - p_2)^\gamma g^{\mu\nu} + (2p_2 + p_1)^\mu g^{\nu\gamma} - (2p_1 + p_2)^\nu g^{\mu\gamma}. \tag{7a}$$

Although the integral in (7) is formally logarithmically divergent, it is actually zero by virtue of the antisymmetry of the ϵ symbol.

Problems can arise when, for example, the external fermion line in Fig. 2(a) is part of a further divergent subdiagram. However, insofar as that subdiagram is nonanomalous, it may be rendered finite by the standard renormalization procedure.[7] What about the high-p_1 behavior of such a subdiagram? We consider examples of closed fermion loops. The anomalous divergence part of Fig. 2(d) involves a superficially logarithmically divergent integral, since, by Weinberg's theorem,[8] the second fermion loop behaves no worse than $\sim p_1$. However, only the parity-violating part of that loop contributes here, the leading behavior of which is softened by the occurrence of fermions with opposite γ_5 couplings, leaving the amplitude in the form of Eq. (7) with

$$I^{\mu\nu\gamma} \sim \frac{m_1^2 - m_2^2}{p_1}, \quad p_1 \to \infty \tag{7b}$$

yielding a finite integral. Diagrams such as 2(e), with two or more external bosons attached to the second fermion loop, have vanishing anomalous divergence: The fermion loop grows no worse than logarithmically as $p_1 \to \infty$, so the boson loop integral is convergent.

Our ability to work in the Feynman gauge is predicated on the preservation of the gauge invariance of the theory. That gauge invariance will be broken by any noncanceled anomaly. So, it behooves us to consider the diagram in Fig. 3, in which the $p_1^\alpha p_1^\beta$ part of one of the boson propagators acts on two fermion triangles; because of the anomaly, we may not be able to eliminate this unwanted term by use of Ward identities. We consider the anomaly part of both fermion loops, for, otherwise, the Ward identities may be used. Again we cancel the anomaly to the extent possible by additional fermions, leaving us with the amplitude

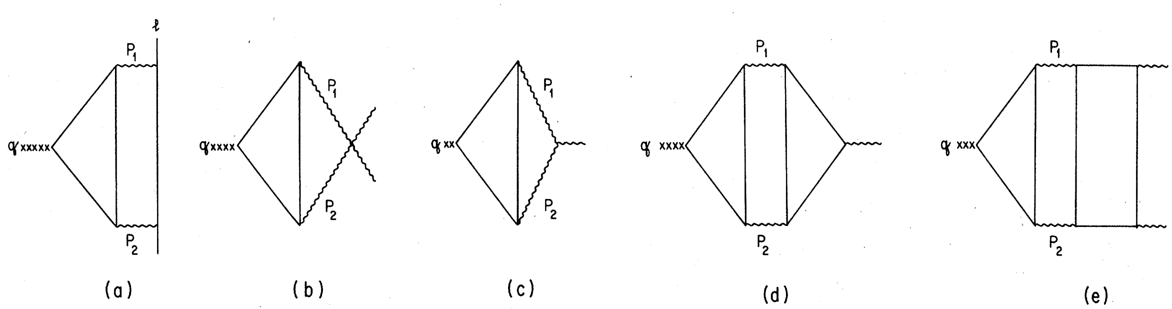

FIG. 2. Examples of processes involving the anomalous triangle diagram as part of a process with overlap.

$$\Pi'_{\mu\nu} \sim (n-4)^2 \int \frac{(dp_1)}{(2\pi)^4} B^2 \epsilon_{\mu\lambda\alpha\beta} p_1^\alpha q^\beta \epsilon_\nu{}^\lambda{}_{\rho\sigma} p_1^\rho q^\sigma \times \frac{(m_1{}^2 - m_2{}^2)^2}{(p_1{}^2)^2} \frac{1}{p_1{}^2} \frac{1}{(p_1-q)^2} \to 0, \quad n \to 4 \tag{8}$$

for again the integral converges.

Thus we see no evidence for any of the problems foretold by Frampton.[1] Clearly, it should not be hard, merely tedious, to construct a general proof along these lines, that there are no consequences of any anomaly term proportional to $n-4$. Instead, we shall now offer a general, independent argument, based on unitarity that the standard anomaly-canceling schemes are sufficient.

III. UNITARITY ARGUMENT

The argument is based on the fact that one can analyze any process in terms of all possible causal exchanges of real particles; the resulting absorptive parts determine the full amplitude up to contact terms. First, we must remind ourselves of how the triangle anomaly emerges from this point of view.[9,10] Suppose, for example, we consider a triangle graph coupling a virtual W to a real W and γ.[11] A spectral form (dispersion relation) may be obtained by considering a process in which an incoming timelike W produces a real exchanged lepton pair. The corresponding amplitude, in the Weinberg-Salam theory, is proportional to[12]

$$\tilde{I}_{\mu\nu\lambda} = i \operatorname{Tr} \int d\omega_p d\omega_{p'} (2\pi)^4 \delta(p+p'-q) \times (\gamma_\lambda \gamma p \gamma_\mu \gamma_5 + \gamma_\lambda \gamma_5 \gamma p \gamma_\mu) \times \frac{1}{m+\gamma(p'-k)} \gamma_\nu (m - \gamma p'), \tag{9}$$

as far as the γ_5 part is concerned. Carrying out the trace and phase-space integration, and removing the causal restriction that the exchanged particles be real (space-time extrapolation), leads to the amplitude[10]

$$I_{\mu\nu\lambda} = \int_{m^2}^\infty \frac{dM^2}{q^2+M^2-i\epsilon} \times \left(\frac{q^2+m_W{}^2}{m_W{}^2 - M^2} B_1(M^2) \epsilon_{\mu\lambda\nu\alpha} k^\alpha + B_2(M^2) q_\lambda \epsilon_{\mu\nu\alpha\beta} q^\alpha k^\beta \right). \tag{10}$$

The factor in front of $B_1(M^2)$ is required by electromagnetic gauge invariance, as shown in Ref. 10; also given there are explicit forms for the weight functions B_1 and B_2. The anomaly arises because if we were to compute $q^\lambda \tilde{I}_{\mu\nu\lambda}$ we would obtain a causal amplitude which differs from (9) by the replacement $\gamma_\lambda \to -m$. Space-time extrapolation of that amplitude yields

$$I_{\mu\nu} = -m \int_{m^2}^\infty \frac{dM^2}{q^2+M^2-i\epsilon} I(M^2) \epsilon_{\mu\nu\alpha\beta} q^\alpha k^\beta, \tag{11}$$

with

$$-mI(M^2) = -B_1(M^2) - M^2 B_2(M^2). \tag{12}$$

This, however, is clearly not the same as $q^\lambda I_{\mu\nu\lambda}$, the difference, the triangle anomaly, arising because $-q^2 \neq M^2$:

$$q^\lambda I_{\mu\nu\lambda} - I_{\mu\nu} = c\epsilon_{\mu\nu\alpha\beta} q^\alpha k^\beta, \tag{13}$$

$$c = \int_{m^2}^\infty dM^2 \left(\frac{B_1(M^2)}{M^2 - m_W{}^2} + B_2(M^2) \right) = \frac{1}{4\pi^2}. \tag{14}$$

Now we are prepared to discuss the cancellation of the anomaly. As usual, we have in mind cancellation of the anomaly by compensating fermions, as first suggested by Gross and Jackiw.[3] We will show that this cancellation occurs even when the triangle process is part of a larger, complicated graph with overlapping divergences. It is sufficient to consider the diagram in Fig. 4. Con-

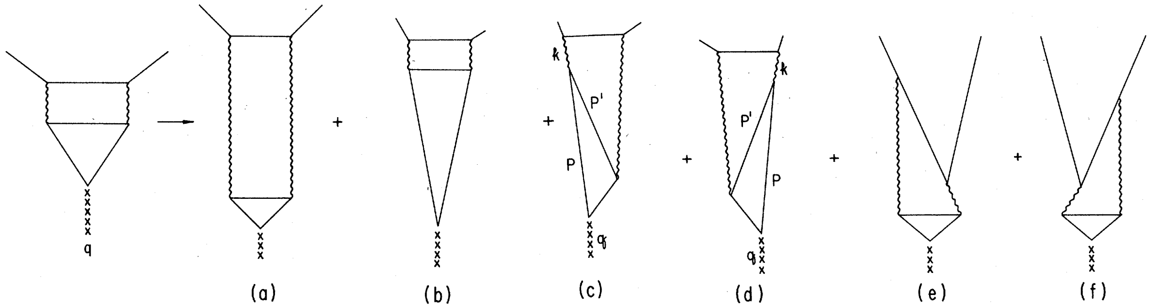

FIG. 4. Causal decomposition of a general process in which the fermion triangle couples to an arbitrary fermion line. Long lines represent real particles, that is, indicate where the general amplitude is "cut".

traction with the external momentum is to be taken with respect to the lower boson leg, and the external fermions may be virtual, so this diagram may be part of a larger process. In particular, q_λ may arise from the $q_\lambda q_\mu$ part of the boson propagator. We use unitarity[13] to analyze this process in terms of all possible causal exchanges of real particles,[14] as also shown in Fig. 4. The triangle anomaly discussed above occurs in Fig. 4(a), because the three-boson vertex is the full amplitude given, for example, in its axial-vector part by (10). Figure 4(b), on the other hand, has no anomaly since the fermions to which the divergence couples are real. Likewise, processes Fig. 4(c) and 4(d) give rise to no anomaly because, for example ($p^2=0$),

$$q_\lambda \operatorname{Tr} \frac{1}{m+\gamma(q-p)} \gamma^\lambda \gamma_5 \gamma p \gamma_\mu (m-\gamma p')\gamma_\nu = -m \operatorname{Tr} \frac{1}{m+\gamma(q-p)} \gamma_5 \gamma p \gamma_\mu (m-\gamma p')\gamma_\nu + \operatorname{Tr} \gamma_5 \gamma p \gamma_\mu (m-\gamma p')\gamma_\nu \,, \quad (15)$$

where any anomaly would be due to the last trace, which vanishes when integrated over phase space:

$$\int d\omega_p d\omega_{p'} (2\pi)^4 \delta(p+p'-k)\, \epsilon_{\mu\nu\alpha\beta} p^\alpha p'^\beta \propto \epsilon_{\mu\nu\alpha\beta}(k^\alpha k^\beta, g^{\alpha\beta}) \,. \quad (16)$$

It remains for us to analyze the processes of Figs. 4(e) and 4(f), which we can again do by a causal analysis. As before, the triangle anomaly comes only from the process where the full, noncausally analyzed, fermion loop occurs, coupled to two real bosons. This is then just the usual triangle anomaly, completely cancellable by the standard method.

In summary, provided $\operatorname{Tr}[T_a\{T_b, T_c\}]=0$, leading to cancellation of the usual anomaly,[3,4] Lee and Zinn-Justin's proof[7] of renormalizability is not disturbed. For, according to the above, we can (in an R gauge) (i) analyze the process in question in terms of all possible causal exchanges (cut the diagram in all possible ways), (ii) use naive Ward identities to compute these causal processes, and (iii) determine the full amplitude by space-time extrapolation (write the amplitude in terms of a dispersion relation). Since this scheme proceeds in a step-by-step manner, in which amplitudes are expressed in terms of subamplitudes, which can in turn be causally analyzed, one will never encounter any remnant of the triangle anomaly.

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*Present address: Department of Physics, The Ohio State University, Columbus, Ohio 43210.

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[11] Also treated in Ref. 10 is the $ZZ\gamma$ anomaly. A detailed discussion of a symmetrical example, ZZZ, will appear elsewhere.

[12] Here the phase-space momentum element is $d\omega_p = (d\vec{p})/(2\pi)^3 2p^0$.

[13] See, for example, K. A. Milton, Phys. Rev. D 8, 3434 (1973).

[14] Since, in discussing renormalizability of Green's functions, it is necessary to work in an R gauge, we must in general include in our sum over states ghost contributions. See G. 't Hooft, Nucl. Phys. B33, 173 (1971); B35, 167 (1971). Inclusion of these ghosts in no way affects our argument.