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# Gauge field propagator and the number of fermion fields

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The structure of the transverse gluon propagator $D$ of massless quantum chromodynamics is considered in the Landau gauge. The essential differences in the weak-coupling limit $g \to 0$ for $\gamma_0/\beta_0 > 0$ and $\gamma_0/\beta_0 < 0$ are exhibited. Here $\gamma_0$ and $\beta_0$ are coefficients of lowest-order terms of the anomalous dimension and of the $\beta$ function. For SU(3) as the color group and quark triplets, the corresponding flavor conditions are $N_F \leq 9$ and $10 \leq N_F \leq 16$, respectively. It was shown previously that for $\gamma_0/\beta_0 > 0$ there are inconsistencies with the postulates of local quantum field theory and the requirement that the positive-norm contribution $D_+$ to $D$ should approach its free-field value for $g \to 0$. In the present paper, it is investigated in detail how this requirement is violated assuming that the other postulates hold, including invariance under the renormalization group. Using a specific, simple projection into a subspace of positive norm, it is shown that $D_+$ diverges like $(g^2)^{-\gamma_0/\beta_0}$, while the free-field value and higher-order terms of $D$ are entirely due to contributions from negative-norm states. In contradistinction, the required dominance of positive-norm states in the weak-coupling limit prevails for $\gamma_0/\beta_0 < 0$. In particular, the condition is fulfilled that $D_+$ approaches its free-field value for $g^2 \to 0$.

## I. INTRODUCTION

In previous publications we presented a detailed study of quark and gluon propagators for massless quantum chromodynamics in the Landau gauge.[1,2] In particular, we used Lorentz covariance and some minimal spectral conditions in order to obtain the transverse gluon propagator as an analytic function in the cut $k^2$ plane. With the help of these analytic properties and of renormalization-group methods, we derived the explicit asymptotic expression for the propagator in all directions of the complex plane. We find that it satisfies an unsubtracted Lehmann representation[3]

$$D(k^2, g, \kappa^2) = \int_0^\infty dk'^2 \frac{\rho(k'^2, g, \kappa^2)}{k'^2 - k^2}, \tag{1.1}$$

and that

$$\begin{aligned} D(k^2, g, \kappa^2) &\simeq -C_V k^{-2} \left(\ln\left|\frac{k^2}{\kappa^2}\right|\right)^{-\gamma_0/\beta_0}, \\ C_V &= (g^2|\beta_0|)^{-\gamma_0/\beta_0} \exp\left(\int_{g^2}^0 dx\, \tau(x)\right) > 0 \end{aligned} \tag{1.2}$$

for $k^2 \to \infty$ in any direction. Here $\tau(x)$ is an integrable function, $\kappa^2 < 0$ is the renormalization point, and $\beta_0, \gamma_0$ are, respectively, the lowest-order coefficients of the renormalization-group[4] function $\beta(g) = \beta_0 g^4 + \cdots$ and of the anomalous dimension $\gamma(g) = \gamma_0 g^4 + \cdots$ for the gluon field $A^a_\mu$. We consider only the case $\beta_0 < 0$ corresponding to asymptotic freedom.[5]

Later we will also introduce projected propagators where the contributions of negative-norm states are omitted. In order to eliminate the longitudinal part also for the projected case, we use the Fourier transform $G_{\mu\nu\rho\lambda}$ of

$$\langle T A^a_{\mu\nu}(x) A^b_{\rho\lambda}(y)\rangle, \tag{1.3}$$

with

$$A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu, \tag{1.4}$$

and define the transverse structure function $\delta_{ab}D = D_{ab}$ by

$$\begin{aligned} -iG^{ab}_{\mu\nu\rho\lambda}(k) = (&k_\mu k_\rho g_{\nu\lambda} - k_\mu k_\lambda g_{\nu\rho} \\ &+ k_\nu k_\lambda g_{\mu\rho} - k_\nu k_\rho g_{\mu\lambda}) D_{ab}(k^2). \end{aligned} \tag{1.5}$$

For reasons of simplicity, we ignore in this paper the possibility of a spontaneous breakdown of global color symmetry. The generalization is straightforward with the help of the detailed discussion given in Ref. 1.

The discontinuity $\rho$ of $D$ along the positive $k^2$ axis is generally a distribution. In Sec. II we will give a detailed characterization. If appropriately averaged, it approaches the asymptotic expression

$$\rho_{as}(k^2, g, \kappa^2) = -\frac{\gamma_0}{\beta_0} C_V k^{-2} \left(\ln \frac{k^2}{|\kappa^2|}\right)^{-\gamma_0/\beta_0 - 1} \tag{1.6}$$

for $k^2 \to +\infty$, with $C_V > 0$ as given in Eq. (1.2).

Of particular interest for our discussions is the case $\gamma_0/\beta_0 > 0$, where the structure function $D$ becomes superconvergent. We then have the relation

$$\int_{-0}^\infty dk'^2 \rho(k'^2, g, \kappa^2) = 0. \tag{1.7}$$

In addition, we see from Eq. (1.6) that $\rho_{as}$ becomes negative.

In Ref. 1 we defined projected transverse gluon propagators where only states of the type $\tilde{A}^a_{\mu\nu}(k)|0\rangle$ with positive norm contribute to the discontinuity, which is non-negative in the sense of distributions and hence a positive measure. Besides the usual assumptions of nonperturbative gauge theories, we then introduced the requirement that the projected propagator approaches its free-field expression $-1/k^2$ in the weak-coupling limit $g^2 \simeq +0$. We found that this condition could only be fulfilled for $\gamma_0/\beta_0 < 0$, but not for $\gamma_0/\beta_0 > 0$, where the propagator becomes superconvergent and negative-norm states play the dominant role in the limit $g^2 \simeq +0$. For SU(3) as the color group, and with quarks in the fundamental representation, the condition $\gamma_0/\beta_0 < 0$ implies $10 \leq N_F \leq 16$, where $N_F$ is the number of flavors. Hence, our requirement imposes a *lower bound* on this number.

In view of the interesting implications of our requirement for the weak-coupling limit of the projected propagator, we consider in this paper the structure of the propagator in greater detail, in particular in the superconvergent case $\gamma_0/\beta_0 > 0$. We introduce an explicit and simple projection method, which allows the evaluation of many features of the projected propagators. We show explicitly what happens in the weak-coupling limit if the special assumption of Ref. 1 is not imposed.[6]

Besides the main topics mentioned above, we consider in this paper several related problems. In Sec. III A, we use only general projections in order to obtain an inequality for the anomalous dimension of the projected gluon field, and in order to derive a lower bound for the projected gluon propagator in the weak-coupling limit. In Sec. IV, possible massive colored vector-meson poles[7] are discussed, and it is shown that the transverse gluon propagator cannot be a meromorphic function.

## II. TRANSVERSE PROPAGATOR

In this section we consider the complete transverse gluon propagator $D(k^2, g, \kappa^2)$. It is an analytic function of $k^2$, regular in the cut complex plane and real for $k^2 < 0$, with distribution-valued boundary values along the positive real axis. In order to bring out the main points, we will postpone discussions of distribution aspects until later in this section. However, the notation used in the following is perfectly correct in the general case, provided the formulas are appropriately interpreted as relations involving generalized functions.

As we have pointed out in the Introduction, the convergence properties of the representation (1.1) for $D(k^2, g, \kappa^2)$ are qualitatively different depending on whether $\gamma_0/\beta_0 < 0$ or $\gamma_0/\beta_0 > 0$. We consider both cases separately.

(a). $\gamma_0/\beta_0 < 0$ [corresponding to $10 \leq N_F \leq 16$ for the SU(3) gauge group and color-triplet representations of quarks]. For the application of renormalization-group methods it is convenient to separate the representation of the dimensionless quantity

$$R = -k^2 D\,, \qquad R = R(k^2, g, \kappa^2) = R(k^2/\kappa^2, g) \tag{2.1}$$

into two parts with momenta below and above a certain value $K^2 > 0$,

$$R = -k^2 \int dk'^2 \frac{\rho}{k'^2 - k^2} = -k^2 \int_{-0}^{K^2} dk'^2 \frac{\rho}{k'^2 - k^2} - k^2 \int_{K^2}^{\infty} dk'^2 \frac{\rho}{k'^2 - k^2}. \tag{2.2}$$

Rearranging the first term, we obtain

$$R = \int_{-0}^{K^2} dk'^2 \rho - \int_{-0}^{K^2} dk'^2 \frac{k'^2 \rho}{k'^2 - k^2} - k^2 \int_{K^2}^{\infty} dk'^2 \frac{\rho}{k'^2 - k^2}. \tag{2.3}$$

We see from the asymptotic expression (1.6) for the weight function that the sign of $\rho$ is opposite to that of $\gamma_0/\beta_0$. Hence, for sufficiently large $k^2$, we have $\rho > 0$ in the present case, and we assume that the point $K^2$ is chosen such that

$$\rho(k^2, g, \kappa^2) > 0 \quad \text{for } k^2 \geq K^2. \tag{2.4}$$

Under renormalization-group transformations

$$\rho(k^2, g_2, \kappa_2^2) = Z\rho(k^2, g_1, \kappa_1^2), \quad Z > 0\,, \tag{2.5}$$

or under scale transformations

$$\rho(\eta^2 k^2, g, \eta^2\kappa^2) = \eta^{-2}\rho(k^2, g, \kappa^2)\,, \tag{2.6}$$

the sign of the discontinuity remains unchanged. Therefore, the bound $K^2$ for the inequality (2.4) may be chosen in a renormalization-group-invariant fashion

$$K^2 = c_0\kappa^2 \exp\left(\int_{g^2}^{g_0^2} dx\, \beta^{-1}(x)\right) = c_0\kappa^2\left(\frac{g^2}{g_0^2}\right)^{+\beta_1/\beta_0^2} \exp\left[\frac{1}{\beta_0}\left(\frac{1}{g^2} - \frac{1}{g_0^2}\right)\right] \times \exp\left(\int_{g^2}^{g_0^2} dx\, \tau(x)\right). \tag{2.7}$$

$K^2 = \kappa^2 f(g^2)$ behaves as

$$K^2 = O(e^{1/\beta_0 g^2}(g^2)^{+\beta_1/\beta_0^2}) \tag{2.8}$$

for $g \to 0$.

With an invariant choice of $K^2$, we see that the first two terms

$$R_1 = \int_0^{K^2} dk'^2 \rho, \quad R_2 = \int_0^{K^2} dk'^2 \frac{k'^2 \rho}{k'^2 - k^2} \tag{2.9}$$

transform like

$$R_1(g_2) = ZR_1(g_1),$$
$$R_2(k^2, g_2, \kappa_2^2) = ZR_2(k^2, g_1, \kappa_1^2), \tag{2.10}$$
$$Z^{-1} = R(\kappa_2^2/\kappa_1^2, g_1), \quad g^2 = Q^2(\kappa_2^2/\kappa_1^2, g_1)$$

under a renormalization-group transformation. Choosing $k^2, k_0^2 \neq 0$ on a ray through the origin, and setting

$$g_1 = g, \quad g_2 = g_0, \quad \kappa_1^2 = \kappa^2 \frac{k^2}{k_0^2}, \quad \kappa_2^2 = \kappa^2,$$

we obtain the identities

$$R_1(g) = R_1(g_0) \exp\left( \int_{g^2}^{g_0^2} dx\, \gamma\beta^{-1} \right), \tag{2.11}$$

$$R_2(k_0^2, g, \kappa^2) = R_2(k^2, g_0, \kappa^2) \exp\left( \int_{g^2}^{g_0^2} dx\, \gamma\beta^{-1} \right), \tag{2.12}$$

$$\frac{k_0^2}{k^2} = \exp\left( \int_{g^2}^{g_0^2} dx\, \beta^{-1} \right). \tag{2.13}$$

The first equation (2.11) implies

$$R_1(g) = a_0 \left( \frac{g^2}{g_0^2} \right)^{-\gamma_0/\beta_0} \exp\left( \int_{g^2}^{g_0^2} dx\, \sigma \right)$$
$$\sim (g^2)^{-\gamma_0/\beta_0} \quad \text{for } g^2 \to +0, \tag{2.14}$$

where $\sigma$ is integrable at $x=0$, and

$$a_0(g_0^2) = \int_{-0}^{K^2(g_0, \kappa^2)} dk'^2 \rho(k'^2, g_0, \kappa^2)$$

is only a function of $g_0$. The second equation (2.12), in combination with (2.13), allows us to express the limit $g \to +0$ of $R_2(k_0^2, g, \kappa^2)$ by the asymptotic form of $R_2(k^2, g_0, \kappa^2)$ for $k^2 \to -\infty$. From the definition (2.9) it follows that

$$-k^2 R_2(k^2, g_0, \kappa^2) \simeq \int_{-0}^{K^2} dk'^2 k'^2 \rho(k'^2, g_0, \kappa^2)$$
$$\text{for } k^2 \to -\infty.$$

Hence

$$R_2(k^2, g_0, \kappa^2) \sim k^{-2} \sim \exp\left( \int_{g^2}^{g_0^2} dx\, \beta^{-1}(x) \right), \tag{2.15}$$

so that

$$R_2(k_0^2, g, \kappa^2) \sim \exp\left( \int_{g^2}^{g_0^2} dx\, \beta^{-1} \right) \exp\left( \int_{g^2}^{g_0^2} dx\, \gamma\beta^{-1} \right)$$
$$\sim e^{-1/|\beta_0| g^2} (g^2)^{-\gamma_0/\beta_0 + \beta_1/\beta_0^2} \tag{2.16}$$

for $g \to +0$ at fixed $k_0^2$. We see that $R_2$ vanishes exponentially for $g \to +0$ at given $k_0^2$, whereas $R_1$ goes to zero with the power $(g^2)^{-\gamma_0/\beta_0}$, since we have $\gamma_0/\beta_0 < 0$. Of course, this latter term, which may have a nonintegral power in general, will be canceled by contributions from the last integral in Eq. (2.3).

The decomposition (2.3) for $R$ is of considerable interest since we have chosen $K^2(g)$ so that $\rho > 0$ for $k^2 \geq K^2$. Only the third term of (2.3), with a positive weight function, survives for $g \to +0$ and fixed $k^2 \neq 0$:

$$R(k^2, g, \kappa^2) = -k^2 \int_{K^2}^{\infty} dk'^2 \frac{\rho}{k'^2 - k^2}$$
$$+ O((g^2)^{-\gamma_0/\beta_0}) \tag{2.17}$$

for $g \to +0$, with $\rho > 0$ in the interval $k^2 \geq K^2$. Hence, the weak-coupling limit is completely determined by the positive-metric states.

(b). $\gamma_0/\beta_0 > 0$ [corresponding to $N_F \leq 9$ for the SU(3) gauge group and color-triplet representations of quarks]. In this case it follows from Eq. (1.2) that the transverse gluon propagator $D(k^2, g, \kappa^2)$ vanishes asymptotically faster than in the free-field case. We can write an unsubtracted Lehmann representation not only for $D$, but also for the dimensionless function $R = -k^2 D$

$$R = -\int_{-0}^{\infty} dk'^2 \frac{k'^2 \rho}{k'^2 - k^2}, \tag{2.18}$$

so that there are the two alternative forms

$$D = \int_{-0}^{\infty} dk'^2 \frac{\rho}{k'^2 - k^2} = k^{-2} \int_{-0}^{\infty} dk'^2 \frac{k'^2 \rho}{k'^2 - k^2} \tag{2.19}$$

of the spectral representation for the transverse propagator. Combining both representations (2.19), we get the superconvergence relation

$$\int_{-0}^{\infty} dk'^2 \rho = 0. \tag{2.20}$$

Since $k^2 D$ vanishes at infinity in all directions, this sum rule can also be obtained by taking an appropriate limit in Eq. (1.1), for example, in the direction $k^2 \to -\infty$.

As in case (a), we also here choose a point $K^2(g, \kappa^2)$ in a renormalization-group-invariant way. From Eq. (1.3) we see that $\rho < 0$ for $k^2 \geq K^2$ for large enough $K^2$ since we consider the case $\gamma_0/\beta_0 > 0$ and $\beta_0 < 0$. We can again write Eq. (2.18) in the form

$$R = -\int_{-0}^{K^2} dk'^2 \frac{k'^2 \rho}{k'^2 - k^2} - \int_{K^2}^{\infty} dk'^2 \frac{k'^2 \rho}{k'^2 - k^2}$$
$$= -\int_{K^2}^{\infty} dk'^2 \frac{k'^2 \rho}{k'^2 - k^2} + O(e^{1/\beta_0 g^2} (g^2)^{-\gamma_0/\beta_0 + \beta_1/\beta_0^2}), \tag{2.21}$$
$$\rho = \rho(k'^2, g, \kappa^2) < 0 \quad \text{for } k'^2 \geq K^2(g, \kappa^2).$$

Hence $\rho < 0$ in the integrand, and the remainder vanishes exponentially for $g^2 \to +0$, at fixed $k^2 \neq 0$. We see that the weak-coupling limit contains *only* contributions from states

$$\tilde{A}_{\mu\nu}(k) | 0 \rangle \tag{2.22}$$

of *negative* norm. Nevertheless, the free gluon pole term of $D$ is obtained in the limit

$$\lim_{g^2\to+0}\frac{1}{k^2}\int_{K^2}^{\infty}dk'^2\frac{k'^2\rho}{k'^2-k^2}=-\frac{1}{k^2}, \tag{2.23}$$

which involves only negative-norm contributions. In fact, the perturbative expansion of $D(k^2,g,\kappa^2)$ can be recovered from

$$D(k^2,g,\kappa^2)\simeq\frac{1}{k^2}\int_{K^2(g,\kappa^2)}^{\infty}dk'^2\frac{k'^2\rho(k'^2,g,\kappa^2)}{k'^2-k^2} \tag{2.24}$$

for $g^2\to+0$ by inserting the *asymptotic* expansion of $\rho(k'^2,g,\kappa^2)$ for $k'^2\to+\infty$ inside the integral.

The dominance of negative-norm states of the type (2.22), as demonstrated by Eq. (2.21), indicates potential difficulties for the question of unitarity. There is no problem with *perturbative* unitarity which has been shown to hold order by order (apart from infrared infinities of the $S$ matrix). It should be noted, however, that our results go beyond perturbation theory leading to a new type of ghost states enforced by the superconvergence relation (2.20). In view of this, it may be doubtful whether conservation of probability holds in the case $\gamma_0/\beta_0>0$. At least, the question of unitarity of the $S$ matrix should be reconsidered in the light of these results.

We finally discuss some relevant mathematical aspects of the methods developed in this section. The spectral representations (1.1), (2.19), as well as the sum rule (2.20), are valid in the sense of distributions with

$$t=(k'^2-k^2)^{-1},\quad t=k'^2(k'^2-k^2)^{-1},\quad \text{or } t\equiv 1 \tag{2.25}$$

as test functions. Given the boundedness property (1.6) for $k'^2\to\infty$, functions of the type (2.25) are admissible test functions. The notation $-0$ for the lower limit indicates that any value $\eta<0$ may be used as the lower limit since the integrand vanishes identically for negative argument.

Decompositions like (2.2) or (2.21) are permissible if $\rho(k'^2)$ may be considered as an ordinary function around $k'^2=K^2$. It may be assumed that such values $K^2$ can always be found. But even this restriction is not needed for the discussion of this section. For it is always possible to perform a separation by inserting

$$1=t_1\left(\frac{k'^2}{K^2}\right)+t_2\left(\frac{k'^2}{K^2}\right), \tag{2.26}$$

where $t_1$ is a test function with

$$t_1(v)\equiv 1 \quad \text{for } v\le 1,$$
$$t_1(v)\equiv 0 \quad \text{for } v\ge 1+a,\quad a>0. \tag{2.27}$$

Since this yields a renormalization-group-invariant decomposition, the analysis of this section goes through leading to the results (2.17) and (2.21) with the replacement

$$\int_{-0}^{K^2}dk'^2\cdots\to\int_{-0}^{\infty}dk'^2 t_1\left(\frac{k'^2}{K^2}\right)\cdots. \tag{2.28}$$

The precise meaning of the inequality $\rho\gtrless 0$ for $k^2\ge K^2$ is

$$\int dk^2 t(k^2)\rho(k^2)>0 \quad \text{if } \gamma_0/\beta_0<0,$$
$$\int dk^2 t(k^2)\rho(k^2)<0 \quad \text{if } \gamma_0/\beta_0>0 \tag{2.29}$$

for every positive test function $t$ with support above $K^2$. It is an interesting consequence of (2.29) that $\rho$ is a positive (negative) measure for $k^2\ge K^2$ if $\gamma_0/\beta_0<0$ ($>0$).[8] This result excludes derivatives of $\delta$ functions in the asymptotic region $k^2\ge K^2$.

In the distribution sense there are no infrared problems with the type of dispersion integrals considered here. As an example, we take the case where

$$\rho\sim(k'^2)^{-1-\eta},\quad \eta\ge 0. \tag{2.30}$$

Then the Lebesgue integral

$$\int_{-0}^{K^2}dk'^2\frac{\rho}{k'^2-k^2} \tag{2.31}$$

diverges at $k'^2=0$, whereas the expression is regarded as integral over a distribution $\rho$ with the test function $(k'^2-k^2)^{-1}$. It is sometimes useful to convert a dispersion integral into a form which is Lebesgue integrable. We discuss this point in some detail for the infrared behavior of spectral representations.

Consider the expression

$$D=\int_{-0}^{\infty}dk'^2\frac{\rho}{k'^2-k^2} \tag{2.32}$$

and

$$(k^2)^{-n}\int_{-0}^{a}dk'^2\frac{(k'^2)^n\rho}{k'^2-k^2}+\int_{a}^{\infty}dk'^2\frac{\rho}{k'^2-k^2}, \tag{2.33}$$

or a corresponding smooth decomposition by inserting (2.26). Both expressions have the same analyticity and boundedness properties. After multiplication by $(k^2)^n$, their imaginary parts are identical on the real axis. Hence, their difference must be a polynomial in $(k^2)^{-1}$, at most of order $n$. Consequently, the representation

$$D=-\frac{c_1}{k^2}-\cdots-\frac{c_n}{k^{2n}}+\frac{1}{k^{2n}}\int_{-0}^{a}dk'^2\frac{(k'^2)^n\rho}{k'^2-k^2}+\int_{a}^{\infty}dk'^2\frac{\rho}{k'^2-k^2} \tag{2.34}$$

follows with $c_i$ real. In this way the infrared singularity due to (2.30) in the sense of Lebesgue integration may be eliminated, rendering the integral in (2.34) absolutely convergent for sufficiently large $n$.

According to (2.34), the coefficients $c_i$ are uniquely determined by $D$.[9] To illustrate this for $n=1$, we take the example

$$\rho(k'^2)=c(k'^2)^{-1-\eta}+\tilde{\rho}(k'^2)\,, \tag{2.35}$$

with $\tilde{\rho}$ being $L$ (Lebesgue integrable). Then

$$c_1=\int_0^a dk'^2\rho(k'^2)=-\frac{c}{\eta}a^{-\eta}+\int_0^a dk'^2\tilde{\rho}(k'^2)\,, \tag{2.36}$$

where we have used the distribution formula[9]

$$\int_0^a dx\,x^\lambda=\frac{a^{\lambda+1}}{\lambda+1}\,,\quad \lambda\neq-1\,. \tag{2.37}$$

Another, but related way to handle infrared singularities of $\rho$ is via the usual subtracted dispersion representation for $k^{2n}D(k^2)$. For example, if $k^2\rho(k^2)$ is $L$ but not $\rho(k^2)$, we may write in place of Eq. (1.1)

$$D(k^2)=\frac{\kappa^2}{k^2}D(\kappa^2)+\frac{k^2-\kappa^2}{k^2}\int_0^\infty dk'^2\frac{k'^2\rho(k'^2)}{(k'^2-\kappa^2)(k'^2-k^2)}\,, \tag{2.38}$$

with $\kappa^2<0$. If $D$ is normalized so that $-k^2D(k^2)=1$ for $k^2=\kappa^2$, the subtraction term is known.

## III. TRANSVERSE PROJECTED PROPAGATOR

In the first part of this section we will derive some properties which hold for projected propagators in general. For these arguments the specific form of the projection on a subspace of positive-definite metric is not relevant. As in Ref. 1, it is only assumed that the state space contains a linear subspace of positive metric[10] which is Lorentz and translational invariant and does not involve new dimensional parameters in its construction. In the second part of this section a specific construction of such a subspace will be given and studied in detail.

### A. General projection

We first derive an inequality for the anomalous dimension of the transverse projected propagator. In the weak-coupling limit, and under similar hypotheses as in Ref. 1, this inequality can be used to give another derivation of the consistence condition $\gamma_0/\beta_0<0$ for massless quantum chromodynamics, which was first obtained in Ref. 2 and discussed in detail in Ref. 1.

We consider the transverse structure function of a projected propagator as described in Sec. I and introduced in Ref. 1. In the conventional normalization,

$$-k^2D=1 \quad\text{at}\quad k^2=\kappa^2\,,$$

the transverse projected propagator will be denoted by $D^+$. By setting

$$\begin{aligned}&D_+(k^2,g,\kappa^2)=f(g^2)\hat{D}_+(k^2,g,\kappa^2)\,,\\ &f(g^2)=-\kappa^2D_+(\kappa^2,g,\kappa^2)\,,\end{aligned} \tag{3.1}$$

the projected propagator itself becomes normalized to one at the Euclidean point $\kappa^2$, namely

$$-k^2\hat{D}_+=1\quad\text{at}\quad k^2=\kappa^2\,. \tag{3.2}$$

The dimensionless quantity

$$\begin{aligned}&\hat{R}_+(k^2,g,\kappa^2)=-k^2\hat{D}_+(k^2,g,\kappa^2)\,,\\ &\hat{R}_+=\hat{R}_+(u,g),\quad u=k^2/\kappa^2\,,\end{aligned} \tag{3.3}$$

satisfies the renormalization-group equation

$$u\frac{\partial\hat{R}_+}{\partial u}=\hat{\beta}\frac{\partial\hat{R}_+}{\partial g^2}+\hat{\gamma}\hat{R}_+\,. \tag{3.4}$$

The coefficients $\hat{\beta},\hat{\gamma}$ are related to the Callan-Symanzik functions $\beta,\gamma$ of the conventional normalization by

$$\hat{\beta}=\beta,\quad \hat{\gamma}=\gamma+\frac{\beta}{f}\frac{df}{dg^2}\,, \tag{3.5}$$

with $f$ given by (3.1). For the present argument we assume, as in Ref. 1, that the projected propagators approach their free-field values in the weak-coupling limit. Then

$$\hat{\gamma}(g^2)\simeq\gamma(g^2)\simeq\gamma_0g^2\quad\text{for } g^2\to+0\,. \tag{3.6}$$

Using this information, the unsubtracted Lehmann representation

$$\hat{D}_+=\int dk'^2\frac{\hat{\rho}_+}{k'^2-k^2} \tag{3.7}$$

was proved in Ref. 1. By (3.2)–(3.4) the anomalous dimension $\hat{\gamma}$ is alternatively given by

$$\hat{\gamma}=\kappa^2\frac{\partial\hat{R}_+}{\partial k^2}\bigg|_{k^2=\kappa^2}\,. \tag{3.8}$$

Inserting here the Lehmann representation (3.7), we find Wilson's sum rule[11]

$$\begin{aligned}\hat{\gamma}(g^2)&=-\kappa^2\int dk'^2\frac{\hat{\rho}_+}{k'^2-\kappa^2}-\kappa^4\int dk'^2\frac{\hat{\rho}_+}{(k'^2-\kappa^2)^2}\\ &=-\kappa^2\int dk'^2\frac{k'^2\hat{\rho}_+(k'^2)}{(k'^2-\kappa^2)^2}\geq0\,.\end{aligned} \tag{3.9}$$

The equality sign only holds if

$$\hat{\rho}_+(k^2)=c\,\delta(k^2)\,,$$

which, according to (1.6), is not possible for $0<g<g_\infty$. An upper bound for the anomalous dimension follows by using the normalization condition (3.2) in the form

$$-\kappa^2 \int dk'^2 \frac{\hat{\rho}_+}{k'^2-\kappa^2} = 1 .$$

Substitution in the first line of (3.9) implies

$$\hat{\gamma} - 1 = -\kappa^4 \int dk'^2 \frac{\hat{\rho}_+}{(k'^2-\kappa^2)^2} \leqslant 0 , \tag{3.10}$$

which is a special case of a relation we have obtained in Ref. 2. Again the equality sign cannot hold for $0<g<g_\infty$. Thus, the anomalous dimension of the transverse projected propagator is placed between the bounds[12]

$$0<\hat{\gamma}(g^2)<1 \quad \text{if} \quad 0<g<g_\infty. \tag{3.11}$$

Combining this result with the weak-coupling limit (3.6), we find

$$\gamma_0 > 0 \tag{3.12}$$

as a consistency condition under the stated assumption. Since $\beta_0<0$, this is equivalent to the condition $\gamma_0/\beta_0<0$.

It is apparent that the hypothesis on the weak-coupling limit of the projected propagator plays a central part in the derivation of the flavor condition. To some extent it was also used for proving that the projected propagator satisfies an unsubtracted spectral representation. Reversing our point of view, we will now investigate what follows for the weak-coupling limit if the gluon propagator is assumed to exist with a nonvanishing transverse projected structure function $D_+$. If $D_+$ satisfies an unsubtracted Lehmann representation, the inequality

$$D_+ \geqslant \int_{L^2}^{K^2} dk'^2 \frac{\rho_+}{k'^2-k^2}, \quad k^2<0 \tag{3.13}$$

follows. $K^2$ and $L^2$ are chosen as invariants of the renormalization group [see Eq. (2.7)] such that $\rho_+ \neq 0$ for $L^2 \leqslant k'^2 \leqslant K^2$. Equation (3.13) implies the estimate

$$|D_+| \gtrsim |k^2|^{-1} \quad \text{for } k^2 \to -\infty . \tag{3.14}$$

If the behavior of $D_+$ for $k^2\to-\infty$ is such that the spectral representation requires subtractions, the estimate (3.14) holds anyway. The dimensionless function

$$G = -k^2 D_+ \tag{3.15}$$

satisfies an identity similar to (2.12):

$$G(k_0^2, g, \kappa^2) = G(k^2, g_0, \kappa^2) \exp\left( \int_{g^2}^{g_0^2} dx\, \gamma \beta^{-1} \right). \tag{3.16}$$

With (3.14), the estimate

$$|G(k_0^2, g, \kappa^2)| \gtrsim \exp\left( \int_{g^2}^{g_0^2} dx\, \gamma \beta^{-1} \right)$$

follows, or

$$|D_+(k^2, g, \kappa^2)| \gtrsim (g^2)^{-\gamma_0/\beta_0} \quad \text{for } g^2 \to +0 \tag{3.17}$$

at fixed $k^2$. For $\gamma_0/\beta_0<0$, this inequality is compatible with the hypothesis that projected propagators approach their free-field values in the weak-coupling limit. For $\gamma_0/\beta_0>0$, however, a nonvanishing projected gluon propagator must necessarily diverge at least like $(g^2)^{-\gamma_0/\beta_0}$ in the limit $g^2\to+0$.

By explicit construction it will be shown below that a projected propagator can always be defined which is nontrivial and satisfies an unsubtracted Lehmann representation. There may, however, be other definitions of a positive-metric space $H_+$ for which the propagator representations diverge in their unsubtracted form. Of particular interest is the case where the weight function

$$\pi\rho_+(k^2) = \text{Im} D_+(k^2+i0) , \tag{3.18}$$

given by

$$\begin{aligned}\langle \tilde{A}_{\mu\nu}(k) p \tilde{A}_{\rho\lambda}(k) \rangle = {} & 2(2\pi)^4 \theta(k_0)\delta(k+l) \\ & \times (k_\mu k_\rho g_{\nu\lambda} - k_\mu k_\lambda g_{\nu\rho} \\ & - k_\nu k_\rho g_{\mu\nu} + k_\nu k_\lambda g_{\mu\rho}) \text{Im} D_+(k^2)\end{aligned}$$

($p$ projection on $H_+$, $\tilde{A}_{\mu\nu}$ Fourier transform of $\partial_\mu A_\nu - \partial_\nu A_\mu$), increases faster than any power of $k^2$ for $k^2\to\infty$. In such a case even the subtracted forms of the spectral representations diverge and the introduction of time ordering presents difficulties. A similar situation is known from the unitary gauge formulation of the electron propagator in massive quantum electrodynamics. In the following we study what such high momentum behavior would imply for the weak-coupling limit of the weight function. We form the dimensionless expression

$$\begin{aligned} &\sigma_+(k^2, g, \kappa^2) = k^2 \rho_+(k^2, g, \kappa^2) , \\ &\sigma_+ = \sigma_+(u, g), \quad u = k^2/|\kappa^2| , \end{aligned} \tag{3.19}$$

for which an identity similar to (2.12) holds

$$\sigma_+(k_0^2, g, \kappa^2) = \sigma_+(k^2, g_0, \kappa^2) \exp\left( \int_{g^2}^{g_0^2} dx\, \gamma \beta^{-1} \right). \tag{3.20}$$

With

$$\sigma_+(k^2, g_0, \kappa^2) \gtrsim \left( \frac{k^2}{|\kappa^2|} \right)^n , \tag{3.21}$$

the estimate is

$$\sigma_+(k_0^2, g, \kappa^2) \gtrsim \exp\left( -n \int_{g^2}^{g_0^2} dx\, \beta^{-1} \right) \exp\left( \int_{g^2}^{g_0^2} dx\, \gamma \beta^{-1} \right),$$

or

$$\sigma_+(k^2,g,\kappa^2) \gtrsim e^{n/|\beta_0|g^2}(g^2)^{-\gamma_0/\beta_0 - n\beta_1/\beta_0^2} \qquad (3.22)$$

[for $g^2 \to +0$ at given $k^2 > 0$. The conclusion of this analysis is that a projected gluon correlation function (3.18) must be bounded by a polynomial in $k^2$ unless it increases faster than any power of $e^{1/g^2}$ in the weak-coupling limit. All the above relations should be interpreted in the sense of distributions.

### B. Special projection

We now introduce a specific definition of a transverse projected propagator. The absorptive part $\rho$ of the structure function $D$ is proportional to the norm of states of the form

$$C^a_{\mu\nu}(k)\tilde{A}^{*\mu\nu}_a(k)|0\rangle\,, \qquad (3.23)$$

where $A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$, and $\tilde{A}^a_{\mu\nu}(k)$ is the Fourier transform. These states are eigenstates of $P_\mu P^\mu$ and $P_0$ with eigenvalues $k^2 \geq 0$ and $k_0 \geq 0$. Only states of the form (3.23) contribute to $D$. Hence, regarding the transverse propagator, we need only define a projection on positive-norm states within the space $H_T$ spanned by the vectors (3.23). We define the space $H_T^+$ as the subspace of $H_T$ formed by all linear superpositions of these states (3.23) which have positive norm. In accordance with this definition, we decompose the absorptive part into

$$\rho = \rho_+ + \rho_- \qquad (3.24)$$

by setting

$$\rho_+(k^2) = \begin{cases} \rho(k^2) & \text{if } \rho(k^2) > 0\,, \\ 0 & \text{otherwise}\,. \end{cases} \qquad (3.25)$$

(A precise formulation on the basis of distribution theory will be given at the end of this section.) Hence, $\rho_+$ contains all contributions of positive-norm states of the type (3.23). With the definitions (3.24) and (3.25), we introduce projected propagators by

$$D = D_+ + D_-\,, \qquad (3.26)$$

$$D_\pm = \int dk'^2 \frac{\rho_\pm}{k'^2 - k^2}\,. \qquad (3.27)$$

$D_+$ represents the transverse propagator function, projected on the subspace $H_T^+$.

The distribution $\rho_+$ is by definition a positive measure consisting of parts which are Lebesgue integrable except for the neighborhood of singular points and of $\delta$-function contributions with positive coefficients. On the other hand, the negative-norm states contribute to $\rho_-$, which also contains all multipole ghosts corresponding to derivatives of $\delta$ functions. In the work that follows, we discuss the projected propagators separately for the two cases $\gamma_0/\beta_0 \lessgtr 0$.

#### 1. $\gamma_0/\beta_0 < 0$ $(10 \leq N_F \leq 16)$

From Eqs. (1.2) and (2.4) we see that $\rho_-(k^2, g, \kappa^2)$ vanishes for $k^2 \geq K^2(g,\kappa^2)$. We can write Eq. (3.4) for $D_-$ in the form

$$\begin{aligned} D_- &= \int_0^{K^2} dk'^2 \frac{\rho_-}{k'^2 - k^2} \\ &= -\frac{1}{k^2}\int_0^{K^2} dk'^2 \rho_- + \frac{1}{k^2}\int_0^{K^2} \frac{k'^2\rho_-}{k'^2 - k^2}\,. \end{aligned} \qquad (3.28)$$

On the other hand, for $D_+(k^2,g,\kappa^2)$ we have, as an analogous decomposition of Eq. (3.27),

$$\begin{aligned} D_+ = &-\frac{1}{k^2}\int_0^{K^2} dk'^2\rho_+ + \frac{1}{k^2}\int_0^{K^2} dk'^2 \frac{k'^2\rho_+}{k'^2 - k^2} \\ &+ \int_{K^2}^\infty dk'^2 \frac{\rho_+}{k'^2 - k^2}\,. \end{aligned} \qquad (3.29)$$

Since our projection procedure is invariant with respect to the renormalization group, the coefficients of $-1/k^2$ in the first two terms of Eqs. (3.28) and (3.29) can be analyzed in the same way as the corresponding expressions in Eq. (2.3) for the unprojected propagator. Hence, we find

$$\int_0^{K^2(g,\kappa^2)} dk'^2 \rho_\pm(k'^2, g, \kappa^2) = a_0^\pm \left(\frac{g^2}{g_0^2}\right)^{-\gamma_0/\beta_0} \times \exp\left(\int_{g^2}^{g_0^2} dx\, \sigma(x)\right), \qquad (3.30)$$

with integrable $\sigma(x)$ at $x=0$, and

$$\int_0^{K^2(g,\kappa^2)} dk'^2 \frac{k'^2 \rho_\pm(k'^2,g,\kappa^2)}{k'^2 - k^2} \simeq C_\pm(g_0^2) e^{1/\beta_0 g^2}(g^2)^{-\gamma_0/\beta_0 + \beta_1/\beta_0^2} \qquad (3.31)$$

for $g^2 \to +0$, $k^2 \neq 0$ fixed. For a detailed proof, we refer again to Sec. II.

We conclude that for $\gamma_0/\beta_0 < 0$, the weak-coupling limit of the projected propagators is quite reasonable. The ghost part $D_-$ vanishes; the first terms in Eqs. (3.28) and (3.29) with the power $(g^2)^{-\gamma_0/\beta_0}$, and the second terms exponentially. We have a nontrivial example of a theory which satisfies the condition introduced in Ref. 1 that the projected propagator $D_+(k^2,g,\kappa^2)$ approaches its free-field expression $-1/k^2$ for $g^2 \to +0$, $k^2 \neq 0$ fixed. The pole term is generated as the limit of the last term in Eq. (3.29). It may be seen explicitly by inserting the perturbation expansion in the dispersion integral of the last term, as we have discussed in Sec. II for the full propagator.

*2. $\gamma_0/\beta_0>0$ ($N_F\leqslant 9$)*

From the discussion of the unprojected propagator, we know already that in this case the negative-norm states dominate the weak-coupling limit. Since $\rho_+$ vanishes for $k^2\geqslant K^2(g,\kappa^2)$, we have in analogy to Eqs. (3.28) and (3.29)

$$D_+=-\frac{1}{k^2}\int_0^{K^2}dk'^2\rho_+ +\frac{1}{k^2}\int_0^{K^2}dk'^2\frac{k'^2\rho_+}{k'^2-k^2}, \quad (3.32)$$

and

$$D_-=-\frac{1}{k^2}\int_0^{K^2}dk'^2\rho_- +\frac{1}{k^2}\int_0^{K^2}dk'^2\frac{k'^2\rho_-}{k'^2-k^2}+\int_{K^2}^{\infty}dk'^2\frac{\rho_-}{k'^2-k^2}. \quad (3.33)$$

Because of the improved boundedness of $\rho_-$ for $k^2\to\infty$ in the present case, we can also write Eq. (3.33) in the form

$$D_-=-\frac{1}{k^2}\int_0^{\infty}dk'^2\rho_- +\frac{1}{k^2}\int_0^{K^2}dk'^2\frac{k'^2\rho_-}{k'^2-k^2}+\frac{1}{k^2}\int_{K^2}^{\infty}dk'^2\frac{k'^2\rho_-}{k'^2-k^2}. \quad (3.34)$$

In all three equations, the coefficient of $-1/k^2$ in the first term is given by an expression corresponding to (3.30). But since now $\gamma_0/\beta_0>0$, these coefficients *diverge* for $g^2\to+0$ like $(g^2)^{-\gamma_0/\beta_0}$. As indicated in Eq. (3.31), the second terms in all three equations still vanish *exponentially*. In $D=D_++D_-$, the diverging coefficients of $-1/k^2$ cancel because of the superconvergence relation (2.20), as may be seen by adding Eqs. (3.32) and (3.34). For general $\gamma_0/\beta_0>0$, there is no leading or nonleading term in $D_+(k^2,g,\kappa^2)$ which approaches the free-field expression

$$D_+^{\text{free}}(k^2,g,\kappa^2)=-\frac{1}{k^2} \quad (3.35)$$

in the limit $g^2\to+0$. Rather, the last integral of the negative-metric part $D_-$ actually approaches $-1/k^2$, and in fact, as also described in Sec. II, generates the perturbation expansion via the asymptotic expansion of $\rho_-$. Of course, $D_-$ contains the positive-metric free-field pole $-1/k^2$ as a secondary term in the limit $g^2\to+0$. This pole is shielded by the negative-metric contribution of the first term which diverges as described above, so that the overall coefficient of $-1/k^2$ in $D_-$ remains negative for all values of $g^2$, as required by definition.

We finally discuss in greater detail the distribution properties of the weight functions, as far as they are relevant for the purpose of this section. In order to cover all situations to be expected in a gauge field theory which allows negative-norm states, we assume that the discontinuity $\rho(k^2)$ is a distribution of the form

$$\rho(k^2)=\sigma(k^2)+\sum_j D_j(k^2), \quad (3.36)$$

where we suppress other variables like $g^2$, $\kappa^2$, and where

$$\sigma(k^2)=\tau(k^2)+\sum_K C_K\delta(k^2-k_K{}^2), \quad (3.37)$$

$$D_j(k^2)=\sum_{n=1}^{n_j} C_{jn}\delta^{(n)}(k^2-k_j{}^2). \quad (3.38)$$

The function $\tau(k^2)$ is a sufficiently smooth function which we take to be $L$ (absolutely integrable in the sense of Lebesgue) except for isolated singular points. As singularities of $\tau(k^2)$, we think in particular of the possibility of an infrared behavior like

$$\tau(k^2)\sim(k^2)^{-1-\eta}, \quad \eta\geqslant 0. \quad (3.39)$$

Except for such isolated singular points, the part $\sigma(k^2)$ of $\rho(k^2)$ is a real measure. (The general form of a real measure, as the derivative of a function of bounded variation, contains an additional term. This term is an almost everywhere vanishing derivative of a continuous function which need not be constant. Such terms do not appear in the context of dispersion relations.[13]) The positive part $\rho_+$ as described by (3.25) is then defined by the positive part $\sigma_+$ of (3.37) omitting derivatives (3.38) of $\delta$ functions.

Since negative-norm states are possible, we also include the distributions (3.38) which are derivatives of $\delta$ functions. While the $\delta$ functions in (3.37) correspond to normalizable states

$$P^2\Phi_j=k_j{}^2\Phi_j, \quad (3.40)$$

with $C_j\gtrless 0$ for positive- and negative-norm respectively; the derivatives of $\delta$ functions are associated with states $\Phi$ of finite norm, which satisfy

$$(P^2-k_j{}^2)^n\Phi_j\neq 0, \qquad (P^2-k_j{}^2)^{n+1}\Phi_j=0. \quad (3.41)$$

We have pointed out in Sec. I that the weight function $\rho$ is either a positive or a negative distribution for sufficiently large values of $k^2\geqslant K^2$. Hence it must be a measure for these $k^2$, and consequently there are no contributions $D_j(k^2)$ for $k^2\geqslant K^2$, i.e.,

$$0\leqslant k_j{}^2<K^2. \quad (3.42)$$

In particular, the weight function $\rho_+$ of the projected propagator (3.7) is non-negative for all $k^2$

$\geq 0$, and hence a positive measure except for isolated singular points of $\tau_+(k^2)$.

## IV. POLES AND DISCONTINUITIES

In this section we discuss the behavior of the residue of pole terms at points $k^2 = m^2(g)$ as a function of $g^2$, as well as of the discontinuity $k^2\rho(k^2, g, \kappa^2)$ evaluated at a renormalization-group-invariant point $k^2 = K^2(g, \kappa^2)$. Furthermore, we consider the question of propagators $D(k^2, g, \kappa^2)$ which are meromorphic in the complex $k^2$ plane. As promised in Ref. 1, we give a proof that the meromorphic case can be excluded.

We suppose that $D$ has a pole at $k^2 = m^2(g) > 0$, where

$$m^2(g) = m^2(g_0)\exp\left(\int_{g^2}^{g_0^2} dx\,\beta^{-1}(x)\right) = m_0^2 e^{1/\beta_0 g^2}(g^2)^{+\beta_1/\beta_0^2}, \quad (4.1)$$

for $g^2 \to +0$, as we have already seen in Eq. (2.7). Then

$$\lim_{k^2 \to m^2(g)} [m^2(g) - k^2] D(k^2, g, \kappa^2) = r(g) \quad (4.2)$$

is a dimensionless function of $g^2$ which satisfies the renormalization-group equation

$$\beta(g)\frac{\partial r}{\partial g^2} + \gamma(g) r = 0\,. \quad (4.3)$$

Consequently we have

$$r(g) = r(g_0)\exp\left(\int_{g^2}^{g_0^2} dx\,\gamma\beta^{-1}\right) = r(g_0)\left(\frac{g^2}{g_0^2}\right)^{-\gamma_0/\beta_0} \exp\left(\int_{g^2}^{g_0^2} dx\,\sigma(x)\right), \quad (4.4)$$

with an integrable function $\sigma(x)$. Depending upon the sign of $\gamma_0/\beta_0$, the residue diverges or vanishes for $g^2 \to +0$. Therefore, unless $\gamma_0 = 0$, such pole terms cannot approach the free-field gluon pole $-1/k^2$.

It must be noted that for $g^2 \to +0$ also all other singularities of $D(k^2, g, \kappa^2)$, which are at finite points on the positive real $k^2$ axis for $g^2 > 0$, accumulate at $k^2 = 0$ for $g^2 \to +0$. In general we find that the dimensionless function $R(k^2, g, \kappa^2) = -k^2 D(k^2, g, \kappa^2)$ and its discontinuity $-k^2\rho(k^2, g, \kappa^2)$, if evaluated at a point $k^2 = K^2(g, \kappa^2)$ as given by Eq. (2.7), behave as $(g^2)^{-\gamma_0/\beta_0}$:

$$R(K^2(g,\kappa^2), g, \kappa^2) \sim (g^2)^{-\gamma_0/\beta_0}\,,$$
$$-K^2(g,\kappa^2)\rho(K^2(g,\kappa^2), g, \kappa^2) \sim (g^2)^{-\gamma_0/\beta_0}\,, \quad (4.5)$$

for $g^2 \to +0$.

Returning to the problem of pole terms in the structure function $D$, it may be of interest to consider the possibility of massive colored vector mesons as particles associated with asymptotic fields. We can introduce the appropriate operators

$$A_\mu^{\prime a} = Z^{1/2} A_\mu^a\,, \quad (4.6)$$

so that the corresponding structure function $D'$ satisfies

$$\lim_{k^2 \to m^2} (m^2 - k^2) D' = 1\,. \quad (4.7)$$

Here the factor $Z$ is given by Eq. (4.4):

$$Z^{-1} = r(g)\,, \quad (4.8)$$

with $r(g) > 0$. The asymptotic fields $A^{\text{in,out}}$ are then obtained as the appropriate limits of $A'$ for $x_0 = \mp\infty$. If $|k\rangle$ denotes a normalized one-particle state generated by the Fourier transform of $A^{\text{in}}$, we find

$$\langle 0|A_\mu^a(x)|k\rangle = Z^{-1/2}\langle 0|A_\mu^{\prime a}(x)|k\rangle = Z^{-1/2}\langle 0|A_\mu^{\text{in},a}(x)|k\rangle = Z^{-1/2} C_\mu^a e^{ikx}\,. \quad (4.9)$$

From Eqs. (4.4), (4.8), and (4.9), we see then that

$$\langle 0|A_\mu^a|k\rangle \sim (g)^{-\gamma_0/\beta_0} \quad (4.10)$$

for $g \to +0$, a result which is related to our finding for projected propagators discussed in Section III.

If the colored vector mesons are tentatively considered as physical particles, it is of interest to look at the relation

$$S(k_1, \ldots, k_n) = Z^{n/2} \prod_i (k_j^2 - m^2)\tau(k_1, \ldots, k_n) \quad (4.11)$$

between connected $S$-matrix elements and the corresponding connected $\tau$ functions involving time-ordered products of the fields $A_\mu^a(x)$. We know from the discussions in previous sections that our gauge field theory has a reasonable weak-coupling limit for $\gamma_0/\beta_0 < 0$ $(10 \leq N_F \leq 16)$, at least as far as the two-point function (1.3) is concerned. If the same is true for more general $\tau$ functions, the limit $g^2 \to +0$ of $S$-matrix elements is modified by the diverging factors $Z^{1/2} \sim (g)^{\gamma_0/\beta_0}$. Because of the gauge invariance of the $S$ matrix, this statement is not restricted to the Landau gauge.

The previous considerations are, *a priori*, not sufficient to exclude the existence of colored vector mesons. It is quite possible that the appearance of factors $(g)^{\gamma_0/\beta_0}$ in Eq. (4.11) signals a modification of the isolated pole term. For example, a logarithmic modification could be compatible with the renormalization-group equation. A situation like this is familiar from quantum electrodynamics, where there are, however, physical zero-mass quanta which lead to a superposition of poles and of branch points due to many photon thresholds. In a strong interaction theory with only massive particles for $g^2 > 0$, we may not expect such thresholds in the $S$ matrix if colored channels are assumed to be physical.

We now want to show that the propagator $D(k^2, g, \kappa^2)$ cannot be a meromorphic function of $k^2$. We can assume that there are no asymptotic gluon states and hence no positive-norm poles in the propagator. Because, if we allow physical gluons, the unitarity condition will certainly generate branch cuts corresponding to many-gluon states so that $D(k^2, g, \kappa^2)$ would not be meromorphic.

We write then

$$\rho(k^2, g, \kappa^2) = \sum_n c_n(g)\delta(k_n^{\,2}(g, \kappa^2) - k^2), \qquad (4.12)$$

and have $c_n(g) < 0$ in view of the absence of positive-norm states. If $\gamma_0/\beta_0 > 0$, we have the superconvergence relation

$$\int_0^\infty dk'^2 \rho(k'^2, g, \kappa^2) = \sum_n c_n(g) = 0 \qquad (4.13)$$

which implies $c_n(g) = 0$ for all $n$, since all coefficients have the same sign.

If $\gamma_0/\beta_0 < 0$, we have no superconvergence, $k^2 D(k^2, g, \kappa^2)$ increases asymptotically according to Eq. (1.2), and $\rho \geq 0$ for $k^2 \geq K^2(g, \kappa^2) > 0$. Since all $c_n(g) < 0$, we have therefore

$$k_n^{\,2}(g, \kappa^2) \leq K^2(g, \kappa^2), \qquad (4.14)$$

and hence $D(k^2, g, \kappa^2)$ is regular at $k^2 = \infty$ in contradiction to Eq. (1.2) for $\gamma_0/\beta_0 < 0$.

We conclude that $D(k^2, g, \kappa^2)$ cannot be a meromorphic function of $k^2$. There must be branch points on the positive real axis with cuts drawn to infinity.

## V. CONCLUSIONS

We first review the assumptions underlying the results of this paper. They include the usual postulates of quantum field theory as far as they are appropriate for a non-Abelian gauge theory with indefinite-metric states. Only minimal spectral conditions are required: exclusion of negative eigenvalues of $P_\mu P^\mu$ and $P_0$. Solutions of massless quantum chromodynamics are supposed to exist, parametrized by the coupling constant $g$ and the normalization mass $\kappa^2$. The topologically nontrivial sector of the theory is not essential, since our arguments depend only upon general assumptions and the short distance properties. Hence, instantons and associated features need not be considered explicitly. Important are asymptotic freedom and the representation of the general solution in the limit $g^2 \to +0$ by the formal perturbation expansion of the Lagrangian formulation, at least as far as the first few terms of this expansion are concerned.

We now summarize the results for the full transverse propagator function $D(k^2, g, \kappa^2)$. For $\gamma_0/\beta_0 < 0$, we find the conventional situation where the Lehman representation of $D(k^2, g, \kappa^2)$ is dominated by positive-norm states, with all negative-metric contributions vanishing in the weak-coupling limit $g^2 \to +0$, in which we recover the perturbative expansion. But for the case $\gamma_0/\beta_0 > 0$, the situation is just the opposite. The propagator is superconvergent and negative-norm states dominate. For $k^2 > 0$, *all* positive-norm contributions vanish *exponentially*. Nevertheless, the free propagator pole at $k^2 = 0$ is reproduced in the limit $g^2 \to +0$. Due to the superconvergence, it is the limit of an infinite superposition of negative-norm states.

We have also constructed explicit projected propagators $D_\pm(k^2, g, \kappa^2)$. Their properties in the weak-coupling limit reflect the features described above for the full propagator $D = D_+ + D_-$, but there are additional interesting results.

For $\gamma_0/\beta_0 < 0$, $D_-$ is regular at $k^2 = \infty$, and it vanishes for $g^2 \to +0$ and fixed $k^2 \neq 0$. The positive-norm part $D_+$ approaches its free-field expression:

$$D_+(k^2, g, \kappa^2) \simeq -1/k^2 + O((g^2)^{-\gamma_0/\beta_0}) \qquad (5.1)$$

for $g^2 \to +0$, $k^2 \neq 0$. Hence, in this case, the theory satisfies the condition formulated in Ref. 1.

On the other hand, for $\gamma_0/\beta_0 > 0$, the function $D_+$ has a finite cut. For $k^2 \neq 0$ and $g^2 \to +0$, it behaves according to

$$D_+(k^2, g, \kappa^2) \simeq -C_+(g)/k^2 + O\left(e^{1/\beta_0 g^2}(g^2)^{-\gamma_0/\beta_0 + \beta_1/\beta_0^2}\right), \qquad (5.2)$$

with $C_+(g) > 0$ diverging like $(g^2)^{-\gamma_0/\beta_0}$. There is a corresponding term $-C_-(g)k^{-2}$ in $D_-$ so that $C_+ + C_- = 0$ due to the superconvergence relation. But $D_-$ also contains an infinite integral over negative-metric states which approaches the free-field pole $-1/k^2$ for $g^2 \to +0$. Hence this pole term appears as a secondary term in $D_-$, shielded by the diverging negative-norm contribution mentioned above.

Clearly, the case $\gamma_0/\beta_0 > 0$ does *not* satisfy the requirement of Ref. 1 that the projected propagator $D_+$ approaches its free-field value for $g^2 \to +0$. Even apart from the diverging contributions to $D_+$ and $D_-$, there is a marked discontinuity in the weak-coupling limit, which is particularly apparent in comparison with the case $\gamma_0/\beta_0 < 0$ we have discussed above, and where the limit is normal.

As in the previous paper, we have restricted the discussions here to the Landau gauge. In a separate publication,[14] we will present our results, for gluon as well as quark propagators, in other covariant gauges. If the conventional gauge parameter is denoted by $\alpha$, we have $\gamma(g,\alpha)=\gamma_0(\alpha)g^2+\cdots$, with $\gamma_0(\alpha)=\gamma_{00}+\gamma_{01}\alpha$, where $\gamma_{00}$ is the same as the constant denoted by $\gamma_0$ in this paper, and $\gamma_{01}>0$. The constant $\beta_0$ is independent of $\alpha$. It is important to note, and will be shown in Ref. 14, that under certain restrictions the asymptotic properties of the gluon propagator discontinuity involve only $\gamma_{00}$ [and not $\gamma_0(\alpha)$] as far as their functional dependence is concerned. Mainly because of this feature, we find that the abnormal behavior of the theory for $\gamma_{00}/\beta_0>0$ ($N_F\leq 9$) is not restricted to the Landau gauge.

It is also of considerable interest to extend our explorations to noncovariant gauges, in particular to the axial gauge, which would also be helpful for assessing the implications of our findings for the unitarity structure of the theory.


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