PHYSICAL REVIEW D VOLUME 21, NUMBER 6 15 MARCH 1980

Asymptotic behavior of composite-particle form factors and the renormalization group

A. Duncan and A. H. Mueller
Department of Physics, Columbia University, New York, New York 10027
(Received 27 August 1979)

Composite-particle form factors are studied in the limit of large momentum transfer Q. It is shown that in models with spinor constituents and either scalar or gauge vector gluons, the meson electromagnetic form factor factorizes at large Q^2 and is given by independent light-cone expansions on the initial and final meson legs. The coefficient functions are shown to satisfy a Callan-Symanzik equation. When specialized to quantum chromodynamics, this equation leads to the asymptotic formula of Brodsky and Lepage for the pion electromagnetic form factor. The nucleon form factors $G_M(Q^2)$, $G_E(Q^2)$ are also considered. It is shown that momentum flows which contribute to subdominant logarithms in $G_M(Q^2)$ vitiate a conventional renormalization-group interpretation for this form factor. For large Q^2, the electric form factor $G_E(Q^2)$ fails to factorize, so that a renormalization-group treatment seems even more unlikely in this case.

I. INTRODUCTION

The discovery[1] of asymptotic freedom in non-Abelian gauge theories has led to a great number and variety of interesting predictions for inclusive hadronic processes at high energy. Quantities such as the total hadronic e^+-e^- annihilation cross section, and moments of structure functions in deep-inelastic lepton scattering,[2] inclusive e^+-e^- annihilation,[3] and even lepton-pair production in hadron-hadron collisions[4] have all been shown to be controlled by the renormalization group in the standard Gell-Mann–Zweig quark model with strong interactions mediated by colored non-Abelian gauge gluons [quantum chromodynamics (QCD)]. In the case of exclusive processes, on the other hand, there have been so far no convincing applications of the renormalization group. Indeed, it has been generally supposed that in order to obtain precise predictions in exclusive processes, it would be necessary to augment the renormalization group with detailed information about the mass singularities of the theory—a conclusion which, in the case of a non-Abelian gauge theory, could reasonably lead to pessimism.

The form factor of a composite particle at large momentum transfer provides perhaps the simplest example of an exclusive process which has long resisted renormalization-group analysis. Callan and Gross[5] analyzed the pion electromagnetic form factor in a scalar gluon model at a conformal fixed point of the theory; however, the leading terms $\sim(Q^2)^{-1-\gamma_\psi-\gamma_O/2}$ (where γ_ψ is the fermion anomalous dimension and γ_O that of the leading odd-chirality operator $\bar\psi\gamma_5\psi$) which they found were shown by Menotti[6] to have vanishing coefficient at the conformal point. A more thorough analysis along these lines was performed by Goldberger, Guth, and Soper (GGS)[7] who found, in addition to terms admitting a purely short-distance interpretation [asymptotically $\sim(Q^2)^{-1-(\gamma_O+\gamma_{O'})/2}$, where O, O' refer to operators appearing across the initial and final meson legs], a potential behavior $(Q^2)^{-1-\gamma_O/2-\gamma_\psi}$. These authors also pointed out that the dominant operators at large Q^2 were the twist-2 even-chirality operators $\bar\psi\gamma_5\gamma_\mu\overleftrightarrow{\partial}_{\mu_1}\cdots\overleftrightarrow{\partial}_{\mu_n}\psi$. Unfortunately, the approach used by GGS necessarily had recourse to technical assumptions on the behavior of the pion Bethe-Salpeter wave function which could only be verified in a highly model-dependent fashion (e.g., in a ladder model for the pion).[8] Furthermore, it is virtually impossible to decide within the GGS approach whether the "wave-function poles" giving rise to the $(Q^2)^{-1-\gamma_O/2-\gamma_\psi}$ terms actually appear with zero residue, as we shall later argue they must. Such terms could not appear as such in the asymptotic expansion of a physical form factor in a gauge theory (γ_ψ is a gauge-dependent quantity). In fact, we shall show below that in either scalar or gauge gluon models, with spinor constituent quarks, the pion electromagnetic form factor is indeed controlled by the renormalization group at large momentum transfer and given by terms of the pure short-distance variety $(Q^2)^{-1-(\gamma_O+\gamma_{O'})/2}$.

The renormalization-group approach to exclusive processes divides naturally into two logically distinct steps. First, one must demonstrate that when the relevant invariant gets large, the momentum flows responsible for the dominant asymptotic behavior (up to powers of the large invariant) are of a structure which allows the process to be factorized into a soft part, in which the momenta are typically of the order of the internal masses and renormalization scale of the theory, and a hard subprocess through which the large momentum flows exclusively. This factorization, if possible, will in general lead to an interpretation of the soft part in terms of hadronic matrix elements of renormalized local operators, whereas the hard part is reduced, either directly (for short-distance processes) or by taking moments (for light-cone processes) to a coefficient function of the large invariant solely.

The second step is the derivation of equations of Callan-Symanzik type for the coefficient functions. We wish to stress the logical independence of this step from the factorization proof. There may exist momentum flows which spoil the renormalization-group interpretation of a factorized large-momentum amplitude. An excellent example of this situation occurs in the asymptotically free $(\phi^3)_6$ theory, originally studied in this context by Appelquist and Poggio.[9] An exact evaluation of the asymptotic behavior of the graph shown in Fig. 1 for large $Q^2 \equiv 8p\cdot p'$ gives

$$F(Q^2) \sim \frac{\lambda^4}{2^7\pi^3}\frac{1}{Q^4}\ln^2\left(\frac{Q^2}{m^2}\right) .$$

The appearance of a double logarithm in a one-loop graph (as in the case of elementary fermion form factors in gauge theories) is generally a signal of the failure of the renormalization group. One easily sees that the double logarithm arises from momentum flows which are sensitive to the internal mass of the top line only (referring to Fig. 1) when the external lines are fixed off the mass shell. Nevertheless, the arguments presented in Sec. II can be applied also in this theory to establish factorization of the form factor for large Q^2. The renormalization group, however, does *not* control the asymptotics of this process—we shall see in Sec. III that momentum flows of the sort mentioned above vitiate the derivation of the Callan-Symanzik equation and are thus fatal to the renormalization-group interpretation.

Recently, there have been a number of investigations[10] which suggest that even in gauge theories meson form factors may be controlled by the renormalization group. In particular, Brodsky and Lepage[11] have calculated to all orders the leading logarithms for large Q^2 in the pion electromagnetic form factor in QCD. Their result strongly suggests the existence of a light-cone expansion in the large-Q^2 regime, as well as a renormalization-group interpretation for the coefficient functions. Similar claims have also been made for the nucleon magnetic form factor $G_M(Q^2)$. We present in this paper a derivation of the factorization and renormalization-group behavior of meson vector form factors in both Yukawa and gauge theories.[12] Our results establish that the structures hypothesized by Brodsky and Lepage indeed persist in all the subleading logarithms. However, our techniques also lead us to conclude that in the case of baryon form factors, the renormalization-group interpretation breaks down in subleading logarithms.

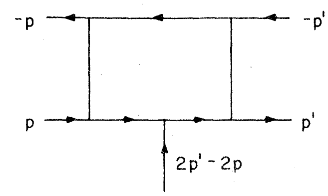

FIG. 1. A one-loop contribution to the scalar form factor in $(\phi^3)_6$ theory.

A brief summary of the paper follows. In Sec. II we establish the factorization of the electromagnetic form factor of a composite pseudoscalar meson with fermion constituents interacting either via scalar or Abelian gauge vector gluons. The complete set of subtractions needed for large Q^2 are described, and a Zimmermann identity used to demonstrate the factorization of the asymptotic part. The result is that the large-Q^2 behavior of the meson electromagnetic form factor is given by independent light-cone expansions on the initial and final meson legs.[13] In Sec. III, we derive the Callan-Symanzik equations which control the large-Q^2 behavior of the meson form factor. We also discuss briefly a number of other cases in which factorization or the renormalization group fail. In Sec. IV we specialize to QCD and derive the asymptotic formula for the pion electromagnetic form factor found by Brodsky and Legage. In Sec. V we discuss the nucleon form factors $G_E(Q^2)$, $G_M(Q^2)$. Factorization fails immediately for $G_E(Q^2)$, though not for $G_M(Q^2)$. Unfortunately, there appear to be momentum flows, leading to subdominant logarithms, which prevent a derivation of a Callan-Symanzik equation for $G_M(Q^2)$. Consequcntly, we think it unlikely that baryon form factors have a conventional renormalization-group interpretation.

II. FACTORIZATION OF THE MESON FORM FACTOR

We begin by deriving a factorization theorem for the meson form factor (with a *vector* current) in the limit of large momentum transfers. This result will hold both in theories with Yukawa-coupled scalar gluons and in gauge gluon theories (it will be crucial, however, as we shall see in Sec. III, that the constituents be spinor objects in order to obtain factorization of the mass-inserted amplitude). The result is simply stated: the large-momentum-transfer regime is given by independent light-cone expansions on each of the meson legs.

Consider the following five-point function (illustrated in Fig. 2), with external fermion legs amputated:

$$\mathcal{T}(p,k,p',k') \equiv \tfrac{1}{16}(p+p')_\mu(\gamma_5\gamma_+)_{\alpha\beta}(\gamma_5\gamma_-)_{\alpha'\beta'}\int d^4x\,d^4y\,d^4x'\,d^4y'\,e^{-i(p+k)\cdot x}e^{i(k-p)\cdot y}e^{-i(k'-p')\cdot x'}e^{i(p'+k')\cdot y'}$$
$$\times\langle 0|Tj^\mu(0)\psi_{\beta_1}(y)\psi_{\beta'_1}(y')\bar{\psi}_{\alpha_1}(x)\bar{\psi}_{\alpha'_1}(x')|0\rangle_c$$
$$\times S^{-1}{}_{\alpha_1\alpha}(p+k)S^{-1}{}_{\beta\beta_1}(k-p)S^{-1}{}_{\beta'\beta'_1}(p'+k')S^{-1}{}_{\alpha'_1\alpha'}(k'-p')\,. \qquad (2.1)$$

The Dirac projections at the ends of the meson legs are introduced to extract precisely the tensor component of $\mathcal{T}$ which will factorize for large $Q^2 \equiv -4(p-p')^2$. As we eventually go to the bound-state pole, these projections do not entail any loss of generality. Neither, clearly, does the inclusion of the factor $(p+p')_\mu$, as current conservation forces the meson form factor to be proportional to $(p-p')^\mu$.

We shall be studying the Green's function (2.1) in the Breit frame for $Q^2 \gg$ renormalization scale and all masses of the theory. Namely,

$$\begin{aligned} &p_-, k_-, p'_+, k'_+ \sim Q\,,\\ &p_+, k_+, p'_-, k'_- \sim m^2/Q\,. \end{aligned} \qquad (2.2)$$

Transverse components of the external relative momenta are taken $O(m)$, and $\vec{p} = \vec{p}' = 0$. We shall use the notation $\tilde{q}_\mu \equiv q_\mu$ for $q_\mu \sim Q$, $\tilde{q}_\mu \equiv 0$ for $q_\mu \ll Q$. It is convenient to begin by considering a scalar-gluon theory. Then we shall discuss how to generalize the subtraction procedure to handle vector-gauge-gluon theories.

The asymptotic behavior of the meson form factor will be obtained by first determining the asymptotic behavior (up to powers of Q^2) of $\mathcal{T}(p,k,p',k')$, and then going to the meson bound-state pole. In analogy to standard proofs of the operator-product expansion,[14] the factorization theorem displays the leading asymptotic behavior of $\mathcal{T}$ as a convolution of subtracted four-point functions on each meson leg (which contain the meson poles) with a five-point Green's function evaluated at a special momentum point. The latter coefficient function will turn out in our case to be merely $\mathcal{T}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')$.

The leading asymptotic behavior of $\mathcal{T}(p,k,p',k')$ can be determined by finding a set of subtractions (including, of course, the usual ones required for renormalization) which extract the leading momentum flows for large Q^2. In other words, we need a generalization of Zimmermann's "Δ-forest" prescription[14] appropriate to this process. As usual, a forest is simply a set of nonoverlapping "subtraction parts" Γ. These subtraction parts consist of the usual renormalization parts γ [(i.e., one-particle-irreducible (1PI) superficially divergent subgraphs, with the exception of photon-photon scattering in an Abelian gauge theory], together with the following three classes of subdiagrams (see Fig. 3 for a typical decomposition of $\mathcal{T}$) corresponding to the momentum flows responsible for the large-Q^2 behavior:

(a) All connected five-point subdiagrams τ of $\mathcal{T}$. We will take τ to be completely amputated.

(b) Connected four-point subdiagrams σ on the incoming meson leg: Such subdiagrams are assumed amputated on the left, but not on the right.

(c) Connected four-point subdiagrams σ' on the outgoing meson leg, amputated on the right only.

Next, we define subtraction operators T_Γ for the various types of subtraction part. For the conventional renormalization parts γ we take $T_\gamma = t_\gamma$, the usual subtraction operator extracting the leading Taylor coefficient(s) at zero external momentum. The subtraction operator T_τ replaces the τ integrand by $(\gamma_5\gamma_-)\otimes(\gamma_5\gamma_+)\mathcal{T}_\tau(\tilde{p}_1,\tilde{k}_1,\tilde{p}'_1,\tilde{k}'_1)$, where the Dirac structures $\gamma_5\gamma_-$ $(\gamma_5\gamma_+)$ are those appearing across the incoming (outgoing) meson legs, and p_1, k_1, etc., are the external momenta of the τ subgraph. The subtraction operator T_σ replaces the left-hand external momenta p_1, k_1 of a σ-type subdiagram by $\tilde{p}_1, \tilde{k}_1$ (i.e., drops + and transverse components of p_1, k_1). Similarly, $T_{\sigma'}$ evaluates a σ' subgraph with right-hand external momenta p'_1, k'_1 at $\tilde{p}'_1, \tilde{k}'_1$.

Constructing Δ forests $\mathfrak{U}_\Delta(\mathcal{T})$ in the usual fashion

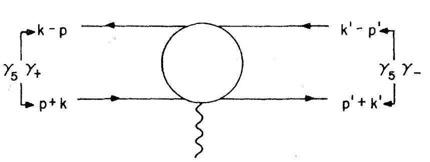

FIG. 2. Five-point function leading to meson form factor.

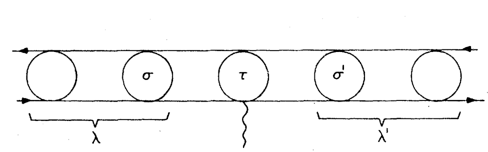

FIG. 3. Forest structure appropriate for subtractions at large Q^2 (scalar gluon theory).

from the subtraction parts $\Gamma \equiv \{\gamma, \tau, \sigma, \sigma'\}$, the expression for $\mathcal{T}(p,k,p',k')$ *oversubtracted* at large Q^2 is just

$$\mathcal{T}_{\rm reg} \equiv \sum_{U\in \mathfrak{U}_\Delta(\mathcal{T})} \prod_{\Gamma\in U} (-T_\Gamma)\mathcal{T}^u , \tag{2.3}$$

where $\mathcal{T}^u$ is the unrenormalized Feynman integrand. The asymptotic behavior can be immediately extracted from (2.3) by a Zimmermann identity. Before doing this, however, it may be helpful to the reader to show by examining an explicit class of graphs how the subtraction procedure (2.3) actually succeeds in removing the dominant momentum flows for large Q^2. Consider therefore the class of graphs shown in Fig. 4(a), in which the rectangular blobs denote two-particle-irreducible subgraphs. All internal subtractions are assumed to have been done—we shall only display explicitly the additional subtractions needed to remove the dominant large-Q^2 piece. These are indicated in Fig. 4(b); zeros denote application of the subtraction operators discussed above.

Consider first the region of large momentum flow through the entire graph shown in Fig. 4(a); specifically, $k_{1\mu} \sim Q$ ($\mu = +, -$, or $i = 1, 2$). The two-particle irreducibility of the rectangular blobs forces all internal lines off-shell by an amount of order Q^2. Hence there is a cancellation of the leading asymptotic power, separately between graphs (i) and (ii), (iii) and (iv), (v) and (vi).

Next, consider a region of momentum flow for which the quark line immediately to the right of the electromagnetic vertex insertion is off-shell of $O(Q^2)$, but the line to the left is close to on-shell, namely, $(k_1 - p)^2 \sim (k_1 + p)^2 \sim m^2$, but $(2p' - p + k_1)^2 \sim Q^2$. Straightforward tensor decomposition followed by power counting shows that graphs (v) and (vi) are separately suppressed and give a contribution to $\mathcal{T}$ of order $1/Q^4$ in this regime. It is also easy to see that graphs (i) and (iii) cancel to leading power in this regime; the large momentum flowing to the right allows us to ignore the small momentum components, and also extracts the appropriate Dirac structure on the left-hand side of the τ subgraph. Since graphs (ii) [(iv)] are simply (i) [(iii)] at special momentum points, they also cancel to leading order in this regime.

We shall point out here that there is also a *potentially* dominant momentum flow, certainly present in scalar field theories (and also in spinor field theories for the scalar form factor), characterized by

$$\begin{aligned}(k - k_1)^2 &\sim (p + k_1)^2 \sim (2p' - p + k_1)^2 \\ &\sim (p' - p + k_1 - k')^2 \sim mQ ,\\ (k_1 - p)^2 &\sim m^2 .\end{aligned}$$

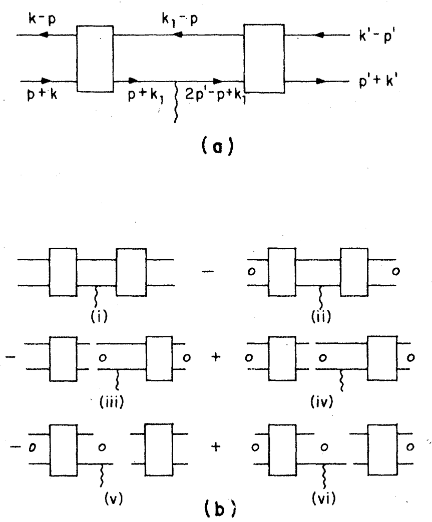

FIG. 4. (a) A class of diagrams contributing to the five-point function of Fig. 2. (b) Large-Q^2 subtractions for the diagrams of (a).

However, as has been noted previously,[7] this regime actually leads to a contribution suppressed by powers of Q for the meson vector form factor in spinor field theories. In fact, all the terms listed in Fig. 4(b) are separately small in this regime. We shall see in Sec. III that the absence of flows of this type is crucial to the derivation of Callan-Symanzik equations.

We now resume the derivation of the asymptotic factorization formula. The renormalized five-point function $\mathcal{T}$ is constructed from the conventional forests $\mathfrak{U}(\mathcal{T})$ (consisting of renormalization parts of $\mathcal{T}$) of the unrenormalized integrand $\mathcal{T}^u$ by writing

$$\mathcal{T} = \sum_{U\in\mathfrak{U}(\mathcal{T})} \prod_{\gamma\in U} (-t_\gamma)\mathcal{T}^u . \tag{2.4}$$

The leading part of $\mathcal{T}$ for large Q^2, up to powers of Q^2, is evidently

$$\mathcal{T}_{\rm asym} = \mathcal{T} - \mathcal{T}_{\rm reg} \tag{2.5}$$

and may be written explicitly (see Fig. 3)

$$\mathcal{T}_{\rm asym} = \sum_{\tau} \sum_{\substack{U_1\in\mathfrak{M}_\lambda \\ U_1'\in\mathfrak{M}_{\lambda'}}} \sum_{U\in\mathfrak{U}(\tau)} \prod_{\substack{\Gamma_1\in U_1 \\ \Gamma_1'\in U_1'}} (-T_{\Gamma_1})(-T_{\Gamma_1'}) T_\tau \prod_{\gamma\in U} (-t_\gamma)\mathcal{T}^u . \tag{2.6}$$

In Eq. (2.6), λ and λ' are the subgraphs of $\mathcal{T}$ (amputated left and right, respectively) external to τ. $\mathfrak{M}_\lambda$ is the set of all forests of λ built from subtraction parts of γ or σ type. $\mathfrak{U}(\tau)$ refers to the set of normal forests of τ, built from conventional renormalization parts γ only.

The final result is obtained by summing (2.6) over all graphs. It is straightforward to verify that this sum generates precisely $\mathcal{T}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')$, sandwiched between two four-point functions, each with a single oversubtraction on the inner momenta. Define the forests of an oversubtracted four-point function $K_{\text{reg}}(p,k,k_1)$ by means of the decompositions indicated in Fig. 5. Thus, if K^u is the unrenormalized integrand, K the conventional renormalized integrand (i.e., containing all the subtractions of type γ), K_{reg} is given by

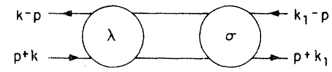

FIG. 5. Forest structure appropriate for defining an oversubtracted four-point function.

$$K_{\text{reg}}=K-\sum_{\sigma}\sum_{U_1\in\mathfrak{M}_\lambda}\sum_{U_2\in\mathfrak{U}(\sigma)}\prod_{\Gamma_1\in U_1}(-T_{\Gamma_1})T_\sigma\prod_{\gamma_2\in U_2}(-t_{\gamma_2})K^u. \tag{2.7}$$

In the notation of Zimmermann,[14] K_{reg} is simply

$$K_{\text{reg}\,\beta\alpha\alpha_1\beta_1}(p,k,k_1)=\int d^4x\,d^4y\,d^4x_1\,e^{-i(p+k)\cdot x}e^{i(k-p)\cdot y}e^{-i(k_1-p)\cdot x_1}\langle 0|T\psi_{\beta'}(y)\bar{\psi}_{\alpha'}(x)N_3(\psi_{\alpha_1}(0)\bar{\psi}_{\beta_1}(x_1))|0\rangle$$
$$\times S^{-1}{}_{\beta\beta'}(k-p)S^{-1}{}_{\alpha'\alpha}(p+k). \tag{2.8}$$

Defining in mirror fashion a left over-subtracted $K_{\text{reg}}\,(p',k',k_1')$, our final result for the leading asymptotic behavior of $\mathcal{T}(p,k,p'k')$ is

$$\sum_{\text{graphs}}\mathcal{T}_{\text{asym}}(p,k,p',k')=\tfrac{1}{16}(\gamma_5\gamma_+)_{\alpha\beta}(\gamma_5\gamma_-)_{\alpha'\beta'}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_1'}{(2\pi)^4}K_{\text{reg}\,\beta\alpha\alpha_1\beta_1}(p,k,k_1)(\gamma_5\gamma_-)_{\beta_1\alpha_1}$$
$$\times\mathcal{T}(\tilde{p},\tilde{k}_1,\tilde{p}',\tilde{k}_1')(\gamma_5\gamma_+)_{\beta'_1\alpha'_1}K_{\text{reg}\,\alpha'_1\beta'_1\beta'\alpha'}(p',k',k_1'). \tag{2.9}$$

Now as we go to the meson pole, $(2p)^2-m_B{}^2\to 0$,

$$K_{\text{reg}\,\beta\alpha\alpha_1\beta_1}(p,k,k_1)\to\frac{i}{(2p)^2-m_B{}^2+i\epsilon}\chi_{\text{reg}\,\alpha_1\beta_1}(p+k_1,k_1-p)\bar{\chi}_{\text{amp}\,\beta\alpha}(p+k,k-p), \tag{2.10}$$

where χ_{reg} is an oversubtracted Bethe-Salpeter wave function

$$\chi_{\text{reg}\,\alpha_1\beta_1}(p+k_1,k_1-p)\equiv\int d^4x\,\langle 0|N_3(\psi_{\alpha_1}(-\tfrac{1}{2}x)\bar{\psi}_{\beta_1}(\tfrac{1}{2}x))|2p\rangle\,e^{-ik_1\cdot x} \tag{2.11}$$

and $\bar{\chi}_{\text{amp}}$ is the conjugate minimally subtracted wave function, with external legs amputated:

$$\bar{\chi}_{\text{amp}\,\beta\alpha}(p+k,k-p)\equiv\int d^4x\langle 2p|T[\psi_{\beta'}(-\tfrac{1}{2}x)\bar{\psi}_{\alpha'}(\tfrac{1}{2}x)]0\rangle e^{-ik\cdot x}S^{-1}{}_{\beta\beta'}(k-p)S^{-1}{}_{\alpha'\alpha}(p+k). \tag{2.12}$$

Our meson states here are covariantly normalized, $\langle p|p'\rangle=(2\pi)^3 2E_p\delta^3(\vec{p}-\vec{p}')$, and internal bound-state degrees of freedom, such as isospin, have been suppressed.

The meson vector form factor $F(Q^2)$ is defined by the matrix element

$$\langle 2p'|j^\mu(0)|2p\rangle\equiv 2(p+p')^\mu F(Q^2) \tag{2.13}$$

From (2.1), the behavior of $\mathcal{T}$ at the meson poles is given by

$$\mathcal{T}(p,k,p',k')\underset{\substack{(2p)^2-m_B{}^2\to 0\\(2p')-m_B{}^2\to 0}}{\sim}\frac{Q^2}{32}F(Q^2)\frac{1}{(2p)^2-m_B{}^2+i\epsilon}\,\frac{1}{(2p')^2-m_B{}^2+i\epsilon}$$
$$\times\text{tr}[\gamma_5\gamma_+\bar{\chi}_{\text{amp}}(p+k,k-p)]\text{tr}[\gamma_5\gamma_-\chi_{\text{amp}}(p'+k',k'-p')]. \tag{2.14}$$

On the other hand, from (2.9) and (2.10), the behavior of $\mathcal{T}_{\text{asym}}$ at the meson poles is just

$$\mathcal{T}_{\text{asym}}(p,k,p',k')\underset{\substack{(2p)^2-m_B{}^2\to 0\\(2p')^2-m_B{}^2\to 0}}{\sim}-\frac{1}{(2p)^2-m_B{}^2+i\epsilon}\,\frac{1}{(2p')^2-m_B{}^2+i\epsilon}$$
$$\times\tfrac{1}{16}\text{tr}[\gamma_5\gamma_+\bar{\chi}_{\text{amp}}(p+k,k-p)]\text{tr}[\gamma_5\gamma_-\chi_{\text{amp}}(p'+k',k'-p')]$$
$$\times\int\frac{d^4k_1d^4k'_1}{(2\pi)^8}\,\text{tr}[\gamma_5\gamma_-\chi_{\text{reg}}(p+k_1,k_1-p)]\mathcal{T}(\tilde{p},\tilde{k}_1,\tilde{p}',\tilde{k}_1')\text{tr}[\gamma_5\gamma_+\chi_{\text{reg}}(p'+k_1',k_1'-p')]. \tag{2.15}$$

Comparing (2.14) and (2.15) we obtain the desired factorization theorem for the asymptotic form factor

$$F(Q^2) \underset{Q^2\to\infty}{\sim} -\frac{2}{Q^2}\int \frac{d^4k\,d^4k'}{(2\pi)^8}\,\mathrm{tr}[\gamma_5\gamma_-\chi_{\mathrm{reg}}(p+k,k-p)]\,\mathcal{T}(\tilde{p},\tilde{k}',\tilde{p}',\tilde{k}')\,\mathrm{tr}[\gamma_5\gamma_+\chi_{\mathrm{reg}}(p'+k',k'-p')]$$

$$+\text{terms smaller by powers of }\frac{1}{Q}. \tag{2.16}$$

This result may be written in a form which displays explicitly the connection with a light-cone expansion by introducing variables

$$x\equiv\frac{k_-}{p_-},\quad x'\equiv\frac{k'_+}{p'_+}.$$

Expand:

$$\mathcal{T}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')=\sum_{nn'}x^nT_{nn'}(Q^2)(x')^{n'}. \tag{2.17}$$

(We expect $\mathcal{T}$ to be analytic at $x=x'=0$ with the nearest thresholds at $x=\pm1$, $x'=\pm1$.) Then, noting that

$$\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}[\gamma_5\gamma_-\chi_{\mathrm{reg}}(p+k,k-p)]x^n = -\frac{1}{p_-{}^n}\langle 0|N_3(\bar{\psi}\gamma_5\gamma_-(\tfrac{1}{2}i\overleftrightarrow{\partial}_-)^n\psi)|2p\rangle\,, \tag{2.18}$$

we see that for large Q^2 the vector form factor is governed by independent light-cone sums over twist-two operators[15] on each meson leg. In Sec. III, we shall derive renormalization-group equations for the coefficient functions $T_{nn'}(Q^2)$.

The factorization derived above for scalar gluon theories can be generalized in the following way to gauge theories. For the present we shall consider a massive Abelian gluon theory, in the Feynman gauge, as we shall then be able to subtract at zero external squared momenta, without encountering infrared divergences. Once the renormalization-group equations for the coefficient functions have been derived, it will be easy to generalize the result to the non-Abelian case. (An analogous derivation for the non-Abelian case, with an appropriate subtraction scheme, should be possible, though technically non trivial.) Essentially the only change in the argument given above for the scalar gluon case is that it is now necessary to include subtraction parts σ, σ', and τ involving arbitrary numbers of gluons collinear with the initial meson momentum p on the left and with the final meson momentum p' on the right (see Fig. 6). These collinear gluons are either emitted from the initial meson state with polarization + (i.e., they couple at the point of emission to a $\gamma-$) or they are absorbed by the final meson with polarization −. For such gluons, and such gluons only, can the extra hard denominator resulting from the gluon insertion be canceled by additional numerator factors.

The proper prescription for subtracting σ, σ', and τ subgraphs involving collinear gluons is obtained by using the Ward identity to compute the effect of a soft polarized gluon entering a subgraph consisting solely of hard lines. As a specific example, consider the σ-type subgraph shown in Fig. 7, in which a single collinear gluon with momentum $q\parallel p\parallel k$ couples with − polarization to a diagram with all internal lines off-shell by Q^2 [take $(k_1+p)^2\sim Q^2$, $(k_1-p)^2\sim m^2$]. As before, we amputate fermion legs on the left, but not on the right. Then, if

$$S_-(p+k-q,k-p,q,p+k_1,k_1-p)$$

denotes the sum of all insertions of a − polarization gluon into the hard subprocess, the Ward identity for the insertion of q at the gluon vertex yields, up to powers of $1/Q$,

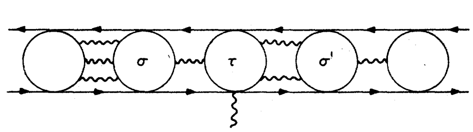

FIG. 6. Forest structure appropriate for subtractions at large Q^2 (gauge vector gluon theory).

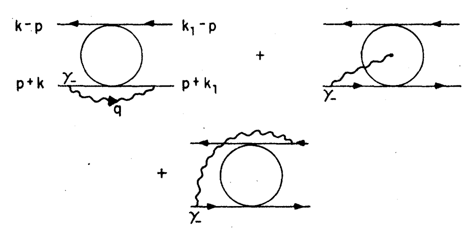

FIG. 7. Collinear gluon insertions in a σ-type subdiagram.

$$S_-(p+k-q, k-p, q, p+k_1, k_1-p) \sim \frac{1}{q_-}[S(\tilde{p}+\tilde{k}-\tilde{q}, \tilde{k}-\tilde{q}-\tilde{p}, p+k_1, k_1-p) - S(\tilde{p}+\tilde{k}, \tilde{k}-\tilde{p}, p+k_1, k_1-p)], \tag{2.19}$$

where

$$S(\tilde{p}+\tilde{k}, \tilde{k}-\tilde{p}, p+k_1, k_1-p)$$

is just the usual four-point subgraph with no gluon entering from the left evaluated at the usual subtraction point. We now define the subtraction operator T_σ applied to the subgraph of Fig. 7 as the right-hand side of (2.19). An exactly analogous definition holds for T_τ in the case where a single collinear gluon enters τ. It is evident that the subtractions for the multigluon case can be related to those needed when no gluons enter the subtraction part by iterative use of the Ward identity.

The only effect of these additional subtractions on the light-cone expansion for the asymptotic form factor is the replacement of the twist-2 tower of operators $N_3(\bar{\psi}\gamma_5\gamma_-(\frac{1}{2}i\overleftrightarrow{\partial}_-)^n\psi)$ appearing in the scalar case [see Eq. (2.18)] by the corresponding gauge-invariant twist-2 tower

$$N_3(\bar{\psi}\gamma_5\gamma_-(\tfrac{1}{2}i\overleftrightarrow{\partial}_- + gA_-)^n\psi).$$

Consider, for example, the contribution to $\mathcal{T}_{\text{asym}}$ in (2.6) arising from decompositions in which a quark, antiquark, and a single gluon line connect σ and τ (but, for simplicity, only a quark and antiquark line connecting σ' and τ). The sum of such contributions leads to a term in $F(Q^2)$ of the form

$$F_{10}(Q^2) = -\frac{2}{Q^2}\int d^4k\, d^4q\, d^4k' \operatorname{tr}[\gamma_5\gamma_-\chi_{\text{reg}}(p+k-q,\ k-p,\ q)]\frac{1}{q_-}[\mathcal{T}(\tilde{p},\tilde{k}-\tilde{q},\tilde{p}',\tilde{k}') - \mathcal{T}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')]$$
$$\times \operatorname{tr}[\gamma_5\gamma_+\chi_{\text{reg}}(p'+k', k'-p')], \tag{2.20}$$

where

$$\chi_{\text{reg}\,\alpha\beta\mu}(p+k-q, k-p, q) \equiv \int d^4x\, d^4y\, e^{-i(k-p)\cdot x}e^{iq\cdot y}\langle 0|N_3(\psi_\alpha(0)\bar{\psi}_\beta(y)A_\mu(y))|2p\rangle. \tag{2.21}$$

If we expand $\mathcal{T}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')$ as in Eq. (2.17), then noting that

$$\int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{(k_- - q_-)^n - k_-^n}{q_-}\operatorname{tr}[\gamma_5\gamma_-\chi_{\text{reg}}(p+k-q, k-p, q)]$$
$$= \left\langle 0\middle| N_3\left(\bar{\psi}(0)\gamma_5\gamma_-\sum_{j=0}^{n-1}(\tfrac{1}{2}i\overleftrightarrow{\partial}_-)^j A_-(0)(\tfrac{1}{2}i\overleftrightarrow{\partial}_-)^{n-j-1}\psi(0)\right)\middle|2p\right\rangle, \tag{2.22}$$

we see that this set of contributions generates precisely the parts linear in the gauge field of the twist-2 operators in the light-cone expansion on the initial meson. Higher-gluon emissions are handled similarly in this Abelian theory.

III. CALLAN-SYMANZIK EQUATION FOR THE MESON FORM FACTOR

In this section we shall show that the asymptotic coefficient functions $\mathcal{T}_{nn'}(Q^2)$ [see Eq. (2.17)] are controlled by the renormalization group for large Q^2. We shall consider a massive Abelian gauge theory (with, for simplicity, equal gluon and fermion masses); the Callan-Symanzik[16] equation (3.7) derived, however, holds equally for scalar gluon theories. The generalization to non-Abelian gauge theories will be made in Sec. IV when we diagonalize the Callan-Symanzik equations and derive the asymptotic form of the pion electromagnetic form factor.

Consider the effect of an application of the Callan-Symanzik operator $\mathfrak{D} \equiv m^2\partial/\partial m^2 + \beta(g)\partial/\partial g - 4\gamma_\psi(g)$ to the five-point Green's function $\mathcal{T}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')$:

$$\hat{\mathcal{T}}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}') \equiv \mathfrak{D}\mathcal{T}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}'). \tag{3.1}$$

Graphically, $\hat{\mathcal{T}}$ corresponds to the set of all possible soft (i.e., minimally subtracted) mass insertions into the graphs contributing to $\mathcal{T}$. In order to obtain a Callan-Symanzik equation, we must extract the asymptotic behavior of $\hat{\mathcal{T}}$ by determining the set of subtraction parts corresponding to the dominant momentum flows of $\hat{\mathcal{T}}$ for large Q^2. Once again, the forest notation concisely expresses the necessary subtractions:

$$\hat{\mathcal{T}}_{\text{reg}}(\tilde{p},\tilde{k},\tilde{p},\tilde{k}') = \sum_{U\in\mathfrak{U}_\Delta(\hat{\mathcal{T}})}\prod_{\Gamma\in U}(-T_\Gamma)\hat{\mathcal{T}}^u(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}'). \tag{3.2}$$

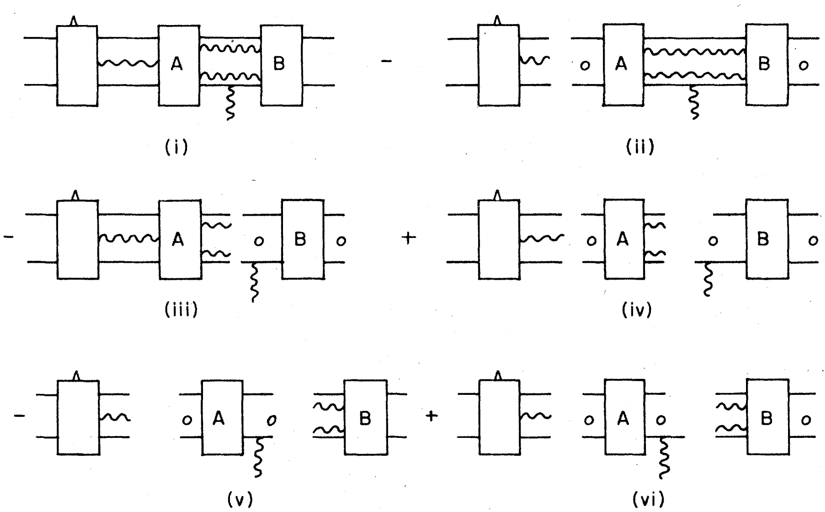

FIG. 8. Large-Q^2 subtraction for a class of mass-inserted diagrams.

Here the subtraction parts Γ are of the τ, σ, σ', and γ types discussed above in the course of the factorization proof, and the T_Γ are as defined there. The $\hat{\Delta}$ forests are built in the usual manner from these Γ subgraphs, with the proviso that no σ, σ', or τ subgraph with a mass insertion is to be considered a subtraction part (γ subgraphs with mass insertions are given the minimum number of subtractions required to make them convergent). We claim that $\hat{\mathcal{T}}_{\text{reg}}$ is suppressed in the large-Q^2 regime by inverse powers of Q compared to $\hat{\mathcal{T}}$.

The subtractions prescribed by (3.2) for a particular class of mass-inserted diagrams are shown in Fig. 8. The reader may easily verify that when large momentum flows through both kernels A and B, (i) and (ii), (iii) and (iv), and (v) and (vi) in Fig. 8 cancel separately. Note that we envisage using the Ward identity to pull the collinear gluons exhibited out of the large-momentum subprocesses, and that any large-momentum flow through the mass-inserted kernel on the left is power suppressed. The discussion for the other momentum flows follows in precise analogy to that given in the preceding section for the subtractions of Fig. 4(b).

A potential disaster for the renormalization-group interpretation of this process may be detected in the class of mass-inserted diagrams shown in Fig. 9(a). Our subtraction scheme (3.2) implies for these diagrams the subtractions shown in Fig. 9(b). The success of these subtractions in removing the dominant momentum flows for large Q^2 depends crucially on the absence of "double-flow" regimes of the sort mentioned in the preceding section. Consider, for example, $(p+k_1)^2$, $(2p'-p+k_1)^2$, $(k-k_1)^2$, and $(p'-p+k_1-k')^2$ all of order mQ. Were such a flow to give rise to a dominant asymptotic contribution, it would clearly not be properly subtracted by the terms displayed in Fig. 9(b), and would result in an inhomogeneous term in the Callan-Symanzik equation. [We shall see below that the subtractions contained in (3.2) are precisely such as to allow us to write the asymptotic part of $\hat{\mathcal{T}}$ in terms of $\mathcal{T}$.] Given the absence of such flows in models with spinor constituents, we must still verify that the subtractions of Fig. 9(b) are successful in the single-flow regimes. Consider a large momentum flow to the left: $(2p'-p+k_1)^2 \sim m^2$, $(k_1-p)^2 \sim m^2$, $(p+k_1)^2 \sim (k-k_1)^2 \sim Q^2$. Evidently (i) and (iii) cancel (to leading power in Q) in this regime. We claim

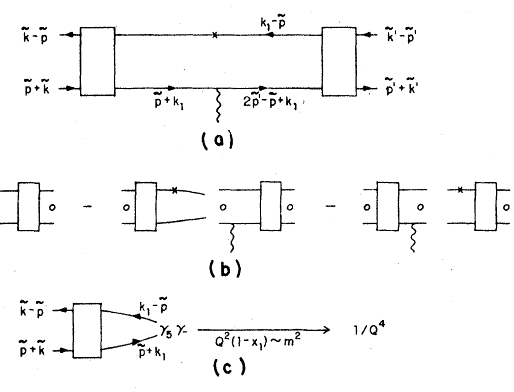

FIG. 9. (a) A class of diagrams contributing to $\hat{\mathcal{T}}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')$. (b) Large-$Q_2$ subtractions for the diagrams of (a). (c) Asymptotic behavior of central diagram of (b) when large momentum flows to the left.

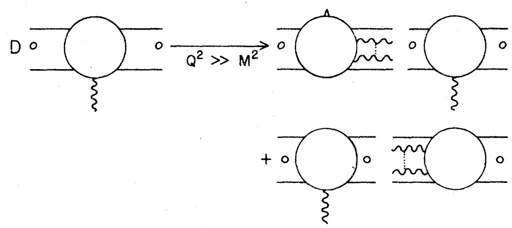

FIG. 10. Callan-Symanzik Equations for $\hat{T}(p,k,p',k')$.

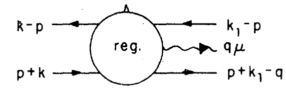

FIG. 11. Oversubtracted five-point function in vector gluon theory.

that (ii) is separately suppressed by a power of Q^2 in this regime. First note that the tree contribution to the structure on the right of (ii) is just

$$(\gamma_5\gamma_-)_{\beta\alpha}(\gamma_5\gamma_+)_{\beta'\alpha'}\frac{1}{Q^2(1-x_1)}\quad\left(x_1\equiv\frac{k_{1-}}{p_-}\right).$$

Furthermore, the regime of interest is characterized by $Q^2\gg m^2$, $Q^2(1-x_1)\sim m^2$. One may check that crossed-channel thresholds in higher-order graphs do not enhance the tree behavior in this regime by more than logarithms. So, up to powers, we are led to a net contribution in this regime of the form shown in Fig. 9(c), which is readily shown to be of order $1/Q^4$.

The derivation of the Zimmermann identity and the extraction of the asymptotic behavior $\hat{T}_{\rm asym}=\hat{T}-\hat{T}_{\rm reg}$ proceeds in complete analogy to the factorization discussion. The result is shown graphically in Fig. 10. Let us first consider only the graphs where the mass insertion is made on the left-hand side of the electromagnetic vertex (a sensible distinction provided double-flow configurations are absent). The contribution of graphs in which only a quark and antiquark, but no gluons, connect the mass-inserted subgraphs to the τ-type hard subprocess give an asymptotic contribution [in analogy to (2.9)]

$$\hat{T}_{\rm asym\ (no\ gluons)}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')=-\tfrac{1}{4}(\gamma_5\gamma_+)_{\alpha\beta}\int\hat{K}_{{\rm reg}\beta\alpha\alpha_1\beta_1}(\tilde{p},\tilde{k},k_1)(\gamma_5\gamma_-)_{\beta_1\alpha_1}\mathcal{T}(\tilde{p},\tilde{k}_1,\tilde{p}',\tilde{k}')\frac{d^4k_1}{(2\pi)^4},\tag{3.3}$$

where

$$\hat{K}_{\rm reg}(p,k,k_1)\equiv\left(m^2\frac{\partial}{\partial m^2}+\beta\frac{\partial}{\partial g}\right)K_{\rm reg}(p,k,k_1)\tag{3.4}$$

is simply the oversubtracted four-point function defined in Eq. (2.8) with a soft mass insertion at all possible points. The subtractions for the graphs in which a single gluon connects the mass-inserted (left-hand) side of the diagram to the τ subdiagram are related to those with no gluons by the Ward identity in the usual manner. Asymptotically this class gives

$$\begin{aligned}\hat{T}_{\rm asym\ (1\ gluon)}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')=&-\tfrac{1}{4}g(\gamma_5\gamma_+)_{\alpha\beta}\int\hat{K}_{{\rm reg}\beta\alpha\alpha_1\beta_1}(\tilde{p},\tilde{k},k_1,q)(\gamma_5\gamma_-)_{\beta_1\alpha_1}\\&\times\frac{1}{q_-}[\mathcal{T}(\tilde{p},\tilde{k}_1-\tilde{q},\tilde{p}',\tilde{k}')-\mathcal{T}(\tilde{p},\tilde{k}_1,\tilde{p}',\tilde{k}')]\frac{d^4k_1d^4q}{(2\pi)^8},\end{aligned}\tag{3.5}$$

where $\hat{K}_{{\rm reg}\beta\alpha\alpha_1\beta_1\mu}(p,k,k_1,q)$ is the following mass-inserted five-point function (see Fig. 11):

$$\begin{aligned}\hat{K}_{{\rm reg}\beta\alpha\alpha_1\beta_1\mu}(p,k,k_1,q)\equiv&\int d^4x\,d^4y\,d^4x_1d^4z\,e^{-i(p+k)\cdot x}e^{i(k-p)\cdot y}e^{-i(k_1-p)\cdot x_1}e^{iq\cdot z}\\&\times\langle 0|T\psi_{\beta'}(y)\bar{\psi}_{\alpha'}(x)N_3(\psi_{\alpha_1}(0)\bar{\psi}_{\beta_1}(x_1)A_\mu(z))|0\rangle S^{-1}{}_{\beta\beta'}(k-p)S^{-1}{}_{\alpha'\alpha}(p+k).\end{aligned}\tag{3.6}$$

In analogy to the factorization discussion, the multigluon exchanges give contributions related to $\mathcal{T}$ $\mathcal{T}(\tilde{p},\tilde{k},\tilde{p}',\tilde{k}')$ by iterative application of the Ward identity. Equations (3.3) and (3.5) display the classic homogeneous Callan-Symanzik structure; application of the operator $\mathfrak{D}$ to the asymptotic function $\mathcal{T}$ yields contributions involving $\mathcal{T}$ linearly. The integrodifferential Eqs. (3.3) and (3.5) (and their multigluon relatives) can be reduced to differential form by performing the expansion (2.17). One then finds

$$\left(m^2\frac{\partial}{\partial m^2}+\beta\frac{\partial}{\partial g}-4\gamma_\psi\right)\mathcal{T}_{nn'}(Q^2)=\sum_{n_1}\gamma_{nn_1}\mathcal{T}_{n_1n'}(Q^2)+\sum_{n_1'}\gamma_{n'n_1'}\mathcal{T}_{nn_1'}(Q^2).\tag{3.7}$$

We have defined an infinite-dimensional anomalous-dimensional matrix γ_{nn_1} by

$$\sum_{n=0}^{n_1}\gamma_{nn_1}x^n=\tfrac{1}{4}i(\gamma_5\gamma_+)_{\alpha\beta}\int d^4x\,d^4y\,e^{-i(\tilde{p}+\tilde{k})\cdot x}e^{i(\tilde{k}-\tilde{p})\cdot y}$$
$$\times\int d^4z\left\langle 0\left|T\{a(g)mN_3[\bar{\psi}\psi(z)]+b(g)m^2N_2[A_\mu(z)A^\mu(z)]\}\right.\right.$$
$$\left.\left.\times\psi_{\beta'}(y)\bar{\psi}_{\alpha'}(x)N_3\left[\bar{\psi}(0)\gamma_5\gamma_-\left(-\frac{i}{2p_-}\tilde{D}_-\right)^n\psi(0)\right]\right|0\right\rangle S^{-1}{}_{\beta\beta'}(\tilde{k}-\tilde{p})S^{-1}{}_{\alpha'\alpha}(\tilde{p}+\tilde{k})\,. \tag{3.8}$$

Here $a(g)$ and $b(g)$ are defined in perturbation theory by the requirement that insertion of the soft mass operator

$$a(g)mN_3(\bar{\psi}\psi)+b(g)m^2N_2(A_\mu A^\mu)$$

in an arbitrary renormalized Green's function be equivalent to the application of the Callan-Symanzik operator for that Green's function. Thus, to lowest order, $a(g)=-\frac{1}{2}+O(g^2)$ and $b(g)=1+O(g^2)$.

It is clear that the γ_{nn_1} are pure numbers; the right-hand side of (3.8) is dimensionless, and the only external momentum invariants $(p^2, p\cdot k, k^2)$ vanish. Furthermore, the maximum power of k_- generated by an insertion of the operator

$$N_3\left(\bar{\psi}\gamma_5\gamma_-\left(-\frac{i}{2p_-}\tilde{D}_-\right)^{n_1}\psi\right)$$

is evidently $k_-^{n_1}$ so the sum over n in (3.8) terminates at $n=n_1$. Our final result, then, is an infinite set of coupled equations of Callan-Symanzik type involving infinite-dimensional upper triangular anomalous-dimension matrices. This mixing of operators with the same twist but different dimension is of course a consequence of the fact that our light-cone expansion is being performed in a channel carrying nonzero total momentum.

We wish to conclude this section by commenting briefly on the situation in a number of other models than that considered explicitly above. In ϕ^3 theory in six spacetime dimensions (an asymptotically free theory), the proof of factorization carries through, but we are unable to derive Callan-Symanzik equations for the form factor because of the presence of double momentum flows. In the case of the scalar form factor in spinor field theories the situation is even worse. The Born contribution to the five-point function here is proportional to a quark mass. The appearance of such a mass in processes of this type is a signal for the failure of factorization; for purposes of power counting, mass terms and transverse numerator factors behave similarly. As a consequence, momentum routings in which the large momentum flows through (say) the Born contribution to τ cannot be factorized—the hard and soft parts of the graph are still entangled via the transverse-loop-momenta integrations.

IV. ASYMPTOTICS OF THE PION FORM FACTOR IN QCD

Given the results of the preceding section, it is straightforward to generalize to the case of quarks with color and flavor interacting via a non-Abelian gauge gluon theory. We shall take the local color gauge symmetry group to be $SU(N_c)$, and will consider only two flavors of quarks (u and d). Our object will be to show that the results for the pion electromagnetic form factor at large Q^2 recently derived by Brodsky and Lepage[11] in leading-logarithmic approximation do in fact follow rigorously from the renormalization group.

We begin by describing our notation. Color, flavor (in this case, isospin), and Dirac indices of the quarks will be denoted by letters $(a,b,a',b',\ldots)$, $(A,B,A',B',\ldots)$, $(\alpha,\beta,\alpha',\beta',\ldots)$, respectively. The isospin label of the initial and final pion will be I and I'. Suppressing isospin and Dirac indices, the QCD Lagrangian is

$$-\tfrac{1}{4}(\partial_\mu A_\nu^i-\partial_\nu A_\mu^i+gC_{ijk}A_\mu^jA_\nu^k)^2$$
$$+\bar{\psi}_a[i\not{\partial}\delta_{ab}+g(T_i)_{ab}\not{A}_i]\psi_b-\bar{\psi}m\psi\,. \tag{4.1}$$

We shall use the notation

$$\begin{aligned}&\mathrm{tr}(T_iT_j)=T(F)\delta_{ij}\,,\\&T_iT_i=C_2(F)1\,,\\&\mathrm{tr}(\tau_I\tau_{I'})=2\delta_{II'}\,,\end{aligned} \tag{4.2}$$

where τ_I are Pauli isospin matrices.

In analogy to (2.1), define

$$\begin{aligned}\mathcal{T}_{II'}(p,k,p',k')\equiv{}&\tfrac{1}{16}(p+p')_\mu(\gamma_5\gamma_+)_{\alpha\beta}(\gamma_5\gamma_-)_{\alpha'\beta'}\\&\times(\tau_I)_{AB}(\tau_{I'})_{A'B'}\delta_{ab}\delta_{a'b'}\\&\times\mathcal{T}^\mu_{\alpha aA,\alpha'a'A'\beta bB,\beta'b'B'}(p,k,p',k')\,,\end{aligned} \tag{4.3}$$

where we have explicitly displayed the Dirac, color, and isopin indices of the external quark lines of the amputated five-point function $\mathcal{T}^\mu$. For

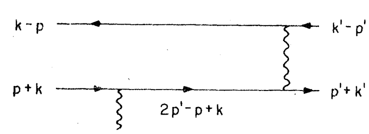

FIG. 12. A Born contribution to $\mathcal{T}(p,k,p',k')$.

example, the tree diagram shown in Fig. 12 leads to the following contribution to $\mathcal{T}_{II'}$:

$$\mathcal{T}_{II'}^{\text{tree}} = C_{II'} N_c C_2(F) \frac{g^2}{Q^2} \frac{1}{(1-x)(1-x')} \tag{4.4}$$

with $C_{II'} \equiv \operatorname{tr}(\tau_I \tau_{I'} \tau_Q)$, $\tau_Q = \frac{1}{6} + \frac{1}{2}\tau_3 =$ quark charge matrix.

The next step is to compute (to one-loop order) the anomalous-dimension matrix γ_{nn_1} [see Eqs. (3.7), and (3.8)] in this theory. The relevant diagrams are displayed in Fig. 13. It is convenient to absorb the fermion anomalous dimension

$$\gamma_\psi = \frac{g^2}{32\pi^2} C_2(F)$$

into the matrix γ_{nn_1}, so we define

$$\Delta_{nn_1} \equiv 2\gamma_\psi \delta_{nn_1} + \gamma_{nn_1}. \tag{4.5}$$

The computation yields

$$\Delta_{nn} = -\frac{g^2}{8\pi^2}\left[\frac{1}{(n+1)(n+2)} - \frac{1}{2} - 2\sum_{j=1}^{n} \frac{1}{j+1}\right] C_2(F). \tag{4.6}$$

Note that these diagonal elements are precisely the nonsinglet anomalous dimensions[2] familiar from deep-inelastic scattering. The off-diagonal matrix elements are

$$\Delta_{nn_1} = -\frac{g^2}{8\pi^2}\left[\frac{1}{(n_1+1)(n_1+2)} + \frac{2(n+1)}{(n_1+1)(n_1-n)}\right] C_2(F) \text{ for } n_1 - n = 2, 4, \ldots$$
$$\Delta_{nn_1} = 0 \text{ otherwise}. \tag{4.7}$$

The terms with odd n or n' in (2.17) correspond to operators with even G parity, so their vacuum-to-single-pion matrix elements will vanish in the limit of exact isospin conservation, which we henceforth assume. The diagonalization of Δ_{nn_1} (Ref. 17) in the even sector is readily accomplished—the eigenvectors generate the coefficients of the Gegenbauer polynomials, as follows. Let

$$\sum_{n_1=0}^{n} \Delta_{nn_1} e_{n_1}^{(N)} = \Delta_{NN} e_n^{(N)} \equiv \lambda_N e_n^{(N)} \text{ for } N = 0, 2, 4, \ldots \tag{4.8}$$

with the normalization

$$e_N^{(N)} = 1. \tag{4.9}$$

Then one finds

$$\sum_{m=0}^{N/2} e_{2m}^{(N)} x^{2m} = \frac{N!}{2N+1!!} C_N^{3/2}(x). \tag{4.10}$$

Thus, if we take

$$\mathcal{T}_{nn'}(Q^2) = \sum_{NN'} e_n^{(N)} \mathcal{T}^{(NN')}(Q^2) e_{n'}^{(N')} \tag{4.11}$$

so that the $\mathcal{T}^{(NN')}(Q^2)$ evolve simply under the renormalization group $[t \equiv \ln(Q^2/Q_0^2)]$

$$\mathcal{T}^{(NN')}(Q^2; g) = \frac{Q_0^2}{Q^2} \mathcal{T}^{(NN')}(Q_0^2; g_{\text{eff}}(t,g)) \times \exp\left[-\int_0^t dx (\lambda_N + \lambda_{N'}) g_{\text{eff}}(x,g)\right], \tag{4.12}$$

we may recover $\mathcal{T}^{(NN')}$ trivially from $\mathcal{T}(x, Q^2, x)$ by using the orthogonality relations of the Gegenbauer polynomials:

$$\mathcal{T}^{(NN')}(Q^2) = \frac{(2N+3)!!(2N'+3)!!}{4(N+2)!(N'+2)!} \times \int_{-1}^{1} dx (1-x^2) \int_{-1}^{1} dx' (1-x'^2) C_N^{3/2}(x) \times \mathcal{T}(x, Q^2, x') C_{N'}^{3/2}(x'). \tag{4.13}$$

In an asymptotically free theory the leading behavior for large Q^2 (up to inverse whole powers of $\ln Q^2$) will be given in terms of the Born diagrams contributing to $\mathcal{T}$. In the case of the charged pion electromagnetic form factor, these three diagrams are readily evaluated [choosing $\tau_I = (\tau_1 + i\tau_2)/\sqrt{2}$, $\tau_{I'} = (\tau_1 - i\tau_2)/\sqrt{2}$] and yield

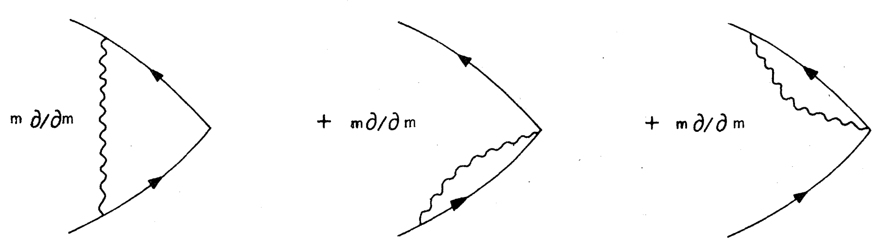

FIG. 13. One-loop graphs contributing to anomalous-dimension matrix γ_{nn_1}.

$$\mathcal{T}_{\mathrm{Born}+-}(x, Q^2, x') = \tfrac{4}{3} N_c C_2(F) \frac{g^2}{Q^2} \times \left(\frac{2}{(1-x)(1-x')} + \frac{1}{(1+x)(1+x')} \right), \tag{4.14}$$

The even sector part of this is

$$\mathcal{T}^{\mathrm{even}}_{\mathrm{Born}+-}(x, Q^2, x') = 4 N_c C_2(F) \frac{g^2}{Q^2} \frac{1}{(1-x^2)(1-x'^2)}. \tag{4.15}$$

Referring to (4.12), (4.13), and (4.15), we obtain finally the asymptotic behavior $\mathcal{T}^{(NN')}(Q^2)$, up to inverse powers of logarithms:

$$\mathcal{T}^{(NN')}_{+-}(Q^2) \underset{Q^2\to\infty}{\sim} \frac{4N_c C_2(F) g_{\mathrm{eff}}{}^2(t,g)}{Q^2} \times \frac{(2N+3)!!}{N+2!} t^{-C_N} \frac{(2N'+3)!!}{N'+2!} t^{-C_{N'}} \tag{4.16}$$

where $t \equiv \ln(Q^2/Q_0{}^2)$, and the constants C_N are explicitly

$$C_N = \frac{\lambda_N}{b_0 g^2},$$

$$\lambda_N = -\frac{g^2}{8\pi^2}\left[\frac{1}{(N+1)(N+2)} - \tfrac{1}{2} - 2\sum_{j=0}^{N}\frac{1}{j+1}\right] C_2(F),$$

$$b_0 g^2 = \frac{g^2}{16\pi^2}\left[\tfrac{11}{3} C_2(G) - \tfrac{4}{3} T(F)\right]. \tag{4.17}$$

Note that $C_0 = 0$, the anomalous dimension vanishes for the minimum dimension operators $\bar{\psi}\gamma_5\gamma_\pm\psi$, which are components of the partially conserved weak axial-vector current. Since C_N are monotone increasing with N, operators of higher dimension give contributions for large Q^2 suppressed by fractional powers of logarithms.

The matrix elements arising in the light-cone expansion governing this process are of the form

$$\langle 0 | N_3(\bar{\psi}\gamma_5\gamma_-\tau_-(\tfrac{1}{2} i\overleftrightarrow{D}_-)^n\psi) | \pi^+, 2p\rangle = 4 p_-{}^{n+1} f_\pi^{(n)}, \tag{4.18}$$

where $f_\pi^{(0)} \equiv f_\pi$ is the usual pion decay constant ($f_\pi \simeq 96$ MeV). The factorization formula (2.16) then yields in this case $[\alpha_{\mathrm{eff}}(Q^2) \equiv g_{\mathrm{eff}}{}^2(Q^2)/4\pi]$

$$F(Q^2) \underset{Q^2\to\infty}{\sim} 16\pi \frac{C_2(F)}{N_c} \frac{\alpha_{\mathrm{eff}}(Q^2)}{Q^2} \times \left(\sum_{N, n\ \mathrm{even}} f_\pi^{(n)} e_n^{(N)} \frac{(2N+3)!!}{(N+2)!} t^{-C_N} \right)^2$$

$$+ \text{ terms down by inverse powers of } t. \tag{4.19}$$

The normalization of the leading term in (4.19) (corresponding to $N = n = 0$) is completely determined in terms of f_π:

$$F(Q^2) \underset{Q^2\to\infty}{\sim} 36\pi \frac{C_2(f)}{N_c} f_\pi{}^2 \frac{\alpha_{\mathrm{eff}}(Q^2)}{Q^2}$$

$$+ \text{ terms down by inverse fractional powers of } t. \tag{4.20}$$

V. NUCLEON FORM FACTORS AND THE RENORMALIZATION GROUP

In this section we wish to describe the results of applying an analysis along the lines of Secs. II and III to the nucleon electromagnetic form factors $G_E(Q^2)$ and $G_M(Q^2)$. Let the momentum and spin of the initial (final) nucleon be P, σ (P', σ'), with $(P-P')^2 \equiv -Q^2$. Then these form factors are defined in terms of the matrix element of the electromagnetic current as follows:

$$\langle P'\sigma' | j_\mu(0) | P\sigma\rangle = \bar{u}_{\sigma'}(P')\left[\left[\frac{G_E(Q^2) + \frac{Q^2}{4M^2} G_M(Q^2)}{1 + \frac{Q^2}{4M^2}}\right]\gamma_\mu + i\frac{(P'-P)^\nu}{2M} \frac{G_M(Q^2) - G_E(Q^2)}{1 + \frac{Q^2}{4M^2}} \sigma_{\mu\nu}\right] u_\sigma(P). \tag{5.1}$$

Once again, the analysis is considerably simplified in the Breit frame, where

$$P_- = P'_+ = \frac{Q}{\sqrt{2}} + O\left(\frac{M^2}{Q}\right), \quad P_+, P'_- \sim \frac{M^2}{Q}.$$

In this frame one may easily check that $G_E(Q^2)$, $G_M(Q^2)$ are given directly by insertions of the longitudinal and transverse current respectively:

$$\langle P'\sigma' | j_-(0) | P\sigma\rangle = \frac{1}{\sqrt{2}} \delta_{\sigma\sigma'} G_E(Q^2), \tag{5.2}$$

$$\langle P'\sigma' | j_i(0) | P\sigma\rangle = i\frac{Q}{2M} \epsilon_{ij} \chi^\dagger_{\sigma'} \sigma_j \chi_\sigma G_M(Q^2) \tag{5.3}$$

where $\chi_{\pm 1/2}$ are unit normalized spinors satisfying $\sigma_3 \chi_{\pm 1/2} = \pm\chi_{\pm 1/2}$, and ϵ_{ij} is the totally antisymmetric tensor in two dimensions ($\epsilon_{12} = +1$).

Consider first the electric form factor $G_E(Q^2)$.

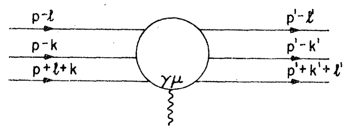

FIG. 14. Seven-point function Γ leading to baryon form factor.

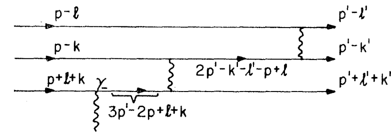

FIG. 15. A Born contribution to Γ.

In this case we cannot even hope for a factorization of the amplitude for large Q^2. The relevant quantity here is the amputated seven-point function shown in Fig. 14, with $\mu = -$ at the current insertion (we find it convenient to set $P = 3p$, $P' = 3p'$). Just as in the pion case, we consider the tensor structures which survive application of γ_+ on the incoming quark lines, and γ_- on the outgoing quark lines. One of the Born diagrams for this process is shown in Fig. 15. The reader may easily verify that the dominant asymptotic terms contributing to such tensor structures are obtained by keeping the mass or transverse momentum of either the $(2\not{p}' - \not{k}' - \not{l}' + \not{p} - \not{l} + m)$ numerator or the $(3\not{p}' + \not{k} + \not{l} + m)$ numerator. The result is an asymptotic contribution of order m/Q^7. We have, then, a situation very analogous to that of the scalar form factor of mesons, described briefly at the end of Sec. III. When the Born graph of Fig. 15 appears as the hard subprocess (the "T" subdiagram) in our subtraction procedure, it will necessarily be coupled to the soft part of the diagram by transverse-momentum integrations. Such coupling clearly destroys the factorization.

In the case of the magnetic form factor $G_M(Q^2)$, one obtains a purely dimensional contribution for large Q^2 from the Born graphs—there are no explicit quark masses in the asymptotic form, and small components ($\ll Q$) of the external momenta are ignorable as only momentum components of order Q enter in the internal lines. In fact, we expect the form factor to factorize in this case. If σ, A refer to the initial nucleon spin and isospin, then (with a_1, a_2, a_3 denoting quark color indices) the leading operator appearing across the initial nucleon leg may be written

$$\epsilon_{a_1 a_2 a_3} \epsilon_{A_1 A_2} [\bar{\psi}^c_{a_1 A_1}(0) \gamma_5 \gamma_- \psi_{a_2 A_2}(0)] [\bar{u}_\sigma(P) \psi_{a_3 A}(0)],$$

where ψ^c is the charge-conjugated fermion field. Note that this operator is separately antisymmetric in color and spin-isospin.

If one were confident that the renormalization group controlled the behavior of the coefficient functions multiplying such operators in the factorized expression for the form factor, one could proceed to the calculation[11] of anomalous dimensions for the operator cited above (and its higher-derivative relatives). Unfortunately, the presence of double-flow momentum configurations makes it impossible to derive a Callan-Symanzik equation for this process. More precisely, consider the graph shown in Fig. 16. [Figure 16(a) labels the momenta for the graph while Fig. 16(b) gives the relevant γ matrices appearing at the vertices and on the Feynman propagators of the fermions.] Call the contribution of the graph shown in Fig. 16 $\Gamma_{abc,def}$. Then define ($C \equiv$ charge conjugation matrix)

$$(\gamma_i)_{fc} \Gamma = \tfrac{1}{4} \sum_{\substack{ab\\de}} (\gamma_+ \gamma_5 C)_{ab} (C \gamma_- \gamma_5)_{ed} \Gamma_{abc,def}\,.$$

Γ has a single $\ln Q^2$ coming from a region of integration where $\underline{k}_1, \underline{k}_2 = O(m)$ with $k_{i+} \ll p'_+$, $k_{i-} \ll p_-$. (There is a second region of integration where a single $\ln Q^2$ can arise, however this $\ln Q^2$ comes about from a region where $\underline{k}_1{}^2$ and $\underline{k}_2{}^2$ are large and thus can be separated uniquely from the above soft region.) In the above momentum range ($x \equiv l_-/p_-$, $x' \equiv l'_+/p'_+$, etc.)

$$\Gamma = \frac{g^8}{4(2\pi)^8} \frac{1}{(1-x)(1-x')(2-x-y)^2(2-x'-y')^2(p\cdot p')^3}$$
$$\times \int_R \frac{d^4k_1 d^4k_2 (\underline{k}_1 \cdot \underline{k}_2 + m^2)}{(k_1{}^2 - m^2 + i\epsilon)(k_2{}^2 - m^2 + i\epsilon)} \frac{1}{(k_{1+} - i\epsilon)(k_{1-} - i\epsilon)[(k_1 + k_2)_+ - i\epsilon]^2 [(k_1 + k_2)_- - i\epsilon]^2}, \qquad (5.4)$$

where R denote the restricted range of $d^4k_1 d^4k_2$. (5.4) is easily evaluated by doing the k_{1+} and k_{2+} contour integrations first. The result of doing these integrations is

$$\Gamma = -\frac{g^8}{8(2\pi)^4 (p\cdot p')^3} \frac{m^2}{(1-x)(1-x')(2-x-y)^2(2-x'-y')^2} \int_R \frac{d\underline{k}_1{}^2 d\underline{k}_2{}^2 dk_{1-} dk_{2-} k_{1-} k_{2-}}{(\underline{k}_1{}^2 + m^2)[k_{2-}(\underline{k}_1{}^2 + m^2) + k_{1-}(\underline{k}_2{}^2 + m^2)]^2 (k_1 + k_2)_-{}^2} \qquad (5.5)$$

After doing the transverse-momentum integration one obtains

$$\Gamma = -\frac{g^8}{8(2\pi)^4(p\cdot p')^3}$$
$$\times\frac{1}{(1-x)(1-x')(2-x-y)^2(2-x'-y')^2}$$
$$\times\int_R \frac{dk_{1-}dk_{2-}}{(k_1+k_2)_-^{\,2}}\frac{k_{2-}}{k_{1-}}\ln\left(1+\frac{k_{1-}}{k_{2-}}\right). \qquad (5.6)$$

Let us now look more closely at the integration range in Eq. (5.6). The original limits dictated by R in (5.4) were say, for k_2,

$$\frac{\Delta_2^{\,2}}{2p'_+} \ll k_{2-} \ll p_-,$$

$$\frac{\Delta_1^{\,2}}{2p_-} \ll k_{2+} \ll p'_+,$$

where

$$\Delta_2^{\,2} = \max\{m^2, (p-l)^2, (2p'-l-k)^2, (3p)^2\}$$

and with a similar formula for $\Delta_1^{\,2}$. Now

$$k_{2+} = \frac{k_2^{\,2}+m^2}{2k_{2-}} = O\left(\frac{m^2}{2k_{2-}}\right).$$

Thus in terms of k_{2-} the limits become

$$\frac{\Delta_2^{\,2}}{2p'_+} \ll k_{2-} \ll p_-\frac{m^2}{\Delta_1^{\,2}}.$$

An identical limit holds for k_{1-}. One then can easily evaluate (5.6) to get

$$\Gamma = \frac{-g^8}{2^7\pi^4}\frac{1}{(p\cdot p')^3(1-x)(1-x')(2-x-y)^2(2-x'-y')^2}$$
$$\times\ln\frac{Q^2m^2}{\Delta_1^{\,2}\Delta_2^{\,2}}. \qquad (5.7)$$

The peculiar thing about the logarithm in (5.7) is that the internal mass m^2 appears even when the external particles are off-shell. In the purely zero-mass theory the above logarithm would not even appear. This clearly means that the logarithm appearing in (5.7) cannot arise in any straightforward way from a Callan-Symanzik equation. We believe that m^2 in (5.7) should be identified with a constituent rather than a current quark mass, but of course we do not know how to go about proving this.

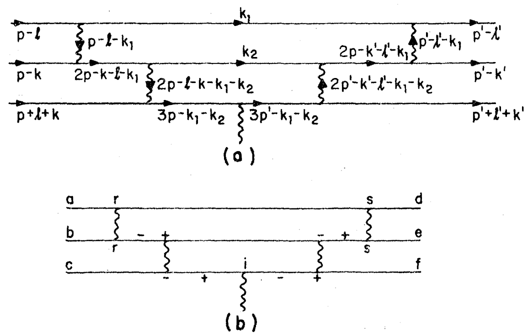

FIG. 16. (a) Momentum routing for a two-loop contribution to Γ. (b) γ-matrix structure for dominant contribution to graph of (a).

The graph shown in Fig. 16 does indeed factorize even though there are momentum flows where $p\cdot k_i$, $p'\cdot k_i$ are both large. The factorization of (5.7) is apparent if one writes

$$\ln\frac{Q^2m^2}{\Delta_1^{\,2}\Delta_2^{\,2}} = \ln\frac{Q^2}{m^2} - \ln\frac{\Delta_1^{\,2}}{m^2} - \ln\frac{\Delta_2^{\,2}}{m^2},$$

in which case the $\ln(\Delta_1^{\,2}/m^2)$ term represents the flow where $p'\cdot k_i = O(Q^2)$, the $\ln(\Delta_2^{\,2}/m^2)$ term represents the flow where $p\cdot k_i = O(Q^2)$, and the $\ln(Q^2/m^2)$ term represents the double momentum flow $(p\cdot k_i)(p'\cdot k_i) = O(Q^2m^2)$. The mass-inserted graph, with the mass insertion on the lines labeled by k_1 and k_2, does not factorize in any way consistent with dimensional counting. This appears to be characteristic of any graph which has a large momentum flow around a region where the momentum is soft.

In higher orders it is very likely that the graphs describing G_M continue to factorize into a part which depends on Q^2 times a part which is wave-function dependent, much as in the pion case. We have not attempted to prove this with any degree of thoroughness, however.

Our conclusion then is that there is no straightforward use of factorization and the Callan-Symanzik equation which extends the Brodsky-Lepage result to all subleading logarithms in an asymptotically free theory.

ACKNOWLEDGMENT

This research was supported in part by the U. S. Department of Energy.

[1]H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); D. J. Gross and F. Wilczek, *ibid*. 30, 1323 (1973).

[2]N. Christ, B. Hasslacher, and A. Mueller, Phys. Rev. D 6, 3543 (1972); D. J. Gross and F. Wilczek, *ibid*. 9, 980 (1974); H. Georgi and H. D. Politzer, *ibid*. 9, 416 (1974).

[3]R. K. Ellis, H. Georgi, M. Machacek, H. D. Politzer, and G. Ross, Nucl. Phys. B152, 285 (1979); A. Mueller,

Phys. Rev. D 18, 3705 (1978); D. Amati, R. Petronzio, and G. Veneziano, Nucl. Phys. B146, 29 (1978); S. Libby and G. Sterman, Phys. Rev. D 18, 3252 (1978); 18, 4737 (1978).

[4]H. D. Politzer, Nucl. Phys. B129, 301 (1977); C. Sacrajda, Phys. Lett. 73B, 185 (1978); 76B, 100 (1978); Yu. L. Dokshitser, D. I. Dyakanov, and S. I. Troyan, SLAC Report No. SLAC-Trans-183, 1978 (unpublished); D. Amati, R. Petronzio, and G. Veneziano, Nucl. Phys. B146, 29 (1978); R. K. Ellis, H. Georgi, M. Machacek, H. D. Politzer, and G. Ross, *ibid.* B152, 285 (1979); S. Gupta and A. Mueller, Phys. Rev. D 20, 118 (1979).

[5]C. Callan and D. J. Gross, Phys. Rev. D 11, 2905 (1975).

[6]P. Menotti, Phys. Rev. D 13, 1778 (1976).

[7]M. L. Goldberger, A. Guth, and D. Soper, Phys. Rev. D 14, 1117 (1976).

[8]A. Guth and D. Soper, Phys. Rev. D 12, 1143 (1975).

[9]T. Applequist and E. Poggio, Phys. Rev. D 10, 3280 (1974).

[10]G. Farrar and D. Jackson, Phys. Rev. Lett. 43, 246 (1979); A. Efremov and A. Radyushkin, Dubna Report No. E2-11983, 1979 (unpublished).

[11]S. Brodsky and G. P. Lepage (unpublished) and SLAC Report No. SLAC-PUB-2294, 1979 (unpublished).

[12]S. Brodsky has informed us that he, in collaboration with Y. Frishman, G. P. Lepage, and C. Sacrajda, has also made an operator-product analysis of form factors. See S. Brodsky, Y. Frishman, G. P. Lepage, and C. Sacrajda (unpublished).

[13]A. Polyakov, in *Proceedings of the 1975 International Symposium on Lepton and Photon Interactions at High Energies*, edited by W. T. Kirk (SLAC, Stanford, California, 1976); G. Parisi, Phys. Lett. 84B, 225 (1979).

[14]W. Zimmermann, in *Elementary Particles in Quantum Field Theory* (MIT Press, Cambridge, Mass., 1971).

[15]Our notation for these operators is justified by the fact that their matrix elements only require a single subtraction.

[16]C. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970); 23, 49 (1971).

[17]See also A. Efremov and A. Radyushkin (Ref. 10).