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Noncovariant effects in the perturbation theory of two-dimensional gauge theories

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Although the noncovariance of two-dimensional gauge theories in the Coulomb gauge has been established by formal operator methods, no calculations have thus far been presented which display a failure of Lorentz invariance within the framework of perturbation theory. In order to make credible such operator results, the mass of the particle coupled to the gauge field is here computed to order g^4N^2, where g is the coupling constant and the symmetry group is U(N). The calculations are carried out for both the spin-0 and spin-1/2 cases, with the result being that, independent of the spin, the mass is seen to exhibit a dependence upon the spatial momentum incompatible with Lorentz invariance. As a secondary result, some earlier calculations for the spin-1/2 case are corrected.

I. INTRODUCTION

The considerable esteem in which gauge theories lately have come to be held has served amongst other things to stimulate a number of investigations of their more tractable two-dimensional counterparts. Because of the fact that the gauge field in two dimensions becomes a function of the fields to which it is coupled when quantization is carried out in the Coulomb gauge, the latter is perhaps the most physically interesting framework for the discussion of such theories. The present work is consequently concerned exclusively with the non-Abelian gauge theory as formulated in the Coulomb gauge.

Despite the advantages inherent in a two-dimensional calculation, there is at least one aspect in which the four-dimensional theory appears to be much simpler to handle. This is the matter of covariance, which appears to have been resolved satisfactorily by Schwinger[1] in four dimensions, even though the Abelian version with zero fermion mass in two dimensions (i.e., the Schwinger model[2]) is known to violate covariance in the Coulomb gauge.[3] Although the solubility of the model (which is a consequence of the vanishing fermion mass and the Abelian nature of the gauge group) is essential to the rigorous demonstration of noncovariance, it is difficult to see how Poincaré invariance could be restored by giving the fermion a mass and/or by using a non-Abelian gauge group. In the Abelian case the commutator of K, the generator of Lorentz transformations, with the Hamiltonian H yields for a system quantized in the domain $(-L, +L)$ the result

$$[K,H]= iP+\tfrac{1}{2} ig^2LD\partial_0Q, \qquad (1.1)$$

where P is the momentum operator and Q and D are defined in terms of the charge densities $j^0(x)$ by

$$Q=\int j^0dx,$$

$$D=\int xj^0dx.$$

Although the non-Abelian case is somewhat more involved, one finds a similar breakdown of the structure relations of the Poincaré group[4] so that one must expect the failure of covariance in all two-dimensional gauge theories (at least in perturbation theory for nonsinglet sectors).

In view of the fact that no claims for evidence of noncovariant perturbative effects in gauge theories have thus far been advanced, it is of some interest to display such results. In the present paper a gauge field theory with U(N) symmetry is considered in the limit of large N. The mass and wave-function renormalizations are calculated for scalar and spinor fields, respectively, coupled to the gauge field. Since the mass squared is the eigenvalue of the observable $-P_\mu{}^2$, one will have a clear demonstration of the breakdown of Lorentz invariance if the mass renormalization is a function of the spatial momentum. This is, in fact, the result obtained. In Sec. II the structure of the scalar propagator is considered and calculations are carried out to include terms of order g^2N and g^4N^2. Section III extends the analysis to the somewhat more involved spinor case. In each instance the mass renormalization is found to be noncovariant and of very similar form. An earlier calculation in the fermion case by Hanson *et al.*[5] (who made no conclusions concerning covariance) is shown to be in error as a result of an omission of a pole in a contour integral. A brief conclusion summarizes the results and some implications.

II. THE GAUGE FIELD COUPLED TO A SCALAR FIELD

The system to be described consists of a set of scalar fields ϕ which transform according to the fundamental representation of U(N) and are coupled to the gauge fields A_a^μ. An appropriate Lagrangian is

$$\begin{aligned}\mathcal{L} = {}&\phi^{\mu\dagger}\partial_\mu\phi - \phi^\dagger\partial_\mu\phi^\mu + \phi_\mu^\dagger\phi^\mu - \mu^2\phi^\dagger\phi + \tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a \\ &- \tfrac{1}{2}F_a^{\mu\nu}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + igA_\mu t^a A_\nu) \\ &- ig(\phi^{\mu\dagger}T^a\phi - \phi^\dagger T^a\phi^\mu)A_\mu^a \\ &+ \tfrac{1}{2}g^2\lambda\phi^\dagger T^a\phi\phi^\dagger T^a\phi,\end{aligned} \tag{2.1}$$

where the matrices T^a and t^a, are, respectively, the fundamental and adjoint representations of the generators of U(N). In writing (2.1) the usual meson-meson scattering term has been included so that disagreeable divergences can be eliminated by appropriate choice of the dimensionless parameter λ.

The calculation of the meson mass involves the scalar two-point function

$$\mathcal{G}(x-x') = i\,\langle 0|(\phi(x)\phi^\dagger(x'))_+|0\rangle, \tag{2.2}$$

where the plus subscript denotes time ordering. In the case of a manifestly covariant theory it is well known that the Fourier transform of (2.2) admits a Lehmann representation.[6] In the case of a gauge theory which is presumed to be covariant (though not manifestly so), the generalized Lehmann representation is[7]

$$\mathcal{G}(p) = \frac{Z_2(\mathfrak{p}^2)}{p^2+\mu^2-i\epsilon} + \int_{\kappa_0^2}^\infty \frac{B(\kappa,\mathfrak{p}^2)}{p^2+\kappa^2-i\epsilon}d\kappa^2, \tag{2.3}$$

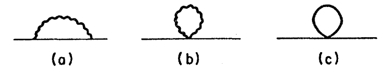

FIG. 1. Second-order contributions to $\mathcal{G}^{-1}(p)$.

where it has been assumed that a stable single-particle pole exists. Thus (2.3), through the requirement of a normal excitation spectrum, imposes a severe restriction on the form of $\mathcal{G}(p)$, namely, that the position of the single-particle pole (if any) should be independent of the Lorentz frame, even though the residue or wave-function renormalization will generally depend on $\mathfrak{p}$, the spatial momentum.

Although the spin-0 covariance problem has not been considered in the literature, it is easy to verify that the techniques of Ref. 4 trivially allow one to verify that covariance fails also in this case. One thus looks for a breakdown of (2.3) in fourth order. It is easy to see that (2.3) implies for the inverse Green's function the structure

$$\mathcal{G}^{-1}(p) = (p^2+\mu^2)Z_2^{-1}(\mathfrak{p}^2) - (p^2+\mu^2)^2\int_{\kappa_0^2}^\infty \frac{\sigma(\kappa,\mathfrak{p}^2)}{p^2+\kappa^2-i\epsilon}d\kappa^2. \tag{2.4}$$

From this point, however, it is possible to omit the integral in (2.4) since in the limit $N\to\infty$, g^2N finite, $\mathcal{G}(p)$ has only a pole term and no continuum.

To second order in g^2 all contributions to $\mathcal{G}^{-1}(p)$ are proportional to N and one consequently includes the standard diagrams as shown in Fig. 1. Defining the mass operator P by

$$P(p) \equiv \mathcal{G}^{-1}(p) - p^2,$$

one obtains for $P(p)$ to second order

$$P^{(2)}(p) = \mu_0^2 - ig^2N\int\frac{dk}{(2\pi)^2}\left(D_\mu^\mu(k) - \frac{(2p-k)^\mu D_{\mu\nu}(k)(2p-k)^\nu}{(p-k)^2+\mu_0^2} - \lambda\frac{1}{k^2+\mu_0^2}\right), \tag{2.5}$$

where $D_{\mu\nu}(k)$ is the photon propagator

$$D_{\mu\nu}(k) = -g_{\mu 0}g_{\nu 0}\frac{1}{k_1^2},$$

with the singularity in $1/k_1^2$ being defined by

$$\frac{1}{k_1^2} = \frac{1}{2}\left[\frac{1}{(k_1+i\epsilon)^2} + \frac{1}{(k_1-i\epsilon)^2}\right].$$

The integral (2.5) can be evaluated by standard techniques[8] to yield

$$P^{(2)} = \mu_0^2 + g^2N\left((1-\lambda)D - \frac{1}{\pi}\right) + \frac{g^2N}{2\pi}\frac{p^2+\mu_0^2}{E^2}\left(1 - \frac{\mathfrak{p}}{2E}\ln\frac{E+\mathfrak{p}}{E-\mathfrak{p}}\right), \tag{2.6}$$

where D is the covariant logarithmically divergent integral

$$D \equiv -i\int\frac{dk}{(2\pi)^2}\frac{1}{k^2+\mu_0^2}$$

and

$E^2 \equiv \mathfrak{p}^2 + \mu_0{}^2$.

By defining

$Z_2(\mathfrak{p}^2) = e^{-w}$,

Eq. (2.6) can be summarized in the two equations

$$\delta\mu^{2(2)} \equiv \mu^{2(2)} - \mu_0{}^2 = g^2N(1-\lambda)D - \frac{g^2N}{\pi},$$

$$w^{(2)} = \frac{g^2N}{2\pi E^2}\left(1 - \frac{\mathfrak{p}}{2E}\ln\frac{E+\mathfrak{p}}{E-\mathfrak{p}}\right).$$

Although all divergences in both second and fourth order can be eliminated by the choice $\lambda = 1$, we choose to retain such terms to the end of the calculation in order to maintain full generality. Regardless of the value of λ, however, one notes that the mass renormalization is a constant and thus no evidence of noncovariance is obtained to this order.

Proceeding next to the fourth-order calculation, one encounters terms which go as N^2, N, and N^0, while the limit of N large has the advantage of retaining only the g^4N^2 terms. Since the most computationally involved graph with crossed photon lines does not contribute to this order, significant simplification results in the large-N limit. The graphs which contribute are given in Fig. 2.

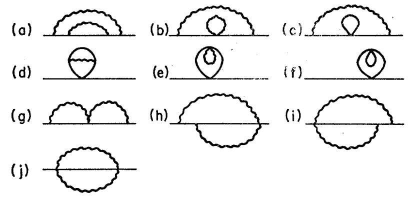

FIG. 2. Order-g^4N^2 contributions to $\mathcal{G}^{-1}(p)$.

Of the ten graphs shown, (a), (b), and (c) represent second-order corrections to the propagator and (d), (e), and (f) are similar second-order propagator corrections to the meson-meson interaction. Including all factors the complete expression for $P(p)$ in order g^4N^2 is thus

$$P^{(4)}(p) = ig^2N\int\frac{dk}{(2\pi)^2}D^{\mu\nu}(k)(2p-k)_\mu(2p-k)_\nu\mathcal{G}^{(2)}(p-k) + i\lambda g^2N\int\frac{dk}{(2\pi)^2}\mathcal{G}^{(2)}(k)$$
$$+ g^4N^2\int\frac{dk}{(2\pi)^2}\frac{dk'}{(2\pi)^2}\left\{D^{\mu\nu}(k)D_{\mu\nu}(k')\frac{1}{(p-k-k')^2+\mu_0{}^2} - D^{\mu\alpha}(k)D_{\mu\beta}(k')\right.$$
$$\left.\times\frac{(2p-k)_\alpha}{(p-k)^2+\mu_0{}^2}\left[\frac{2(2p-2k-k')^\beta}{(p-k-k')^2+\mu_0{}^2} + \frac{(2p-k')^\beta}{(p-k')^2+\mu_0{}^2}\right]\right\}.$$

Following tedious but essentially straightforward manipulation one extracts $\delta\mu^{2(4)}$ and $w^{(4)}$ as

$$\delta\mu^{2(4)} = (\lambda-1)\left[\frac{g^2N\,\delta\mu^{2(2)}}{4\pi\mu^2} + \frac{g^4N^2}{4\pi\mu^2}\right] - g^4N^2\zeta^2E^2$$
$$+ \frac{g^4N^2\zeta}{\pi} - \frac{g^4N^2(1+\lambda)}{8\pi^2\mu^2} - g^4N^2\frac{\mathfrak{p}^2}{\mu^2}I(\mathfrak{p}) \qquad (2.7)$$

and

$$w^{(4)} = \tfrac{1}{2}g^4N^2\zeta^2 + g^2N\,\delta\mu^{2(2)}\frac{\partial}{\partial\mu^2}\zeta$$
$$-g^4N^2\frac{1}{8\pi^2}\frac{\partial}{\partial\mathfrak{p}}\left(\frac{2\mathfrak{p}}{E^2\mu^2} + \frac{1}{E^3}\ln\frac{E+\mathfrak{p}}{E-\mathfrak{p}}\right)$$
$$-\frac{g^4N^2}{2\mu^2}\left(\mathfrak{p}\frac{\partial}{\partial\mathfrak{p}} + 1\right)I(\mathfrak{p}), \qquad (2.8)$$

where we have defined

$\zeta = w^{(2)}/g^2N$

and

$$I(\mathfrak{p}) = \frac{\mu^2}{8\pi^2}\int\frac{d\mathfrak{q}}{E'^4}\frac{P}{\mathfrak{q}-\mathfrak{p}}\ln\frac{E'+\mathfrak{q}}{E'-\mathfrak{q}}$$

where $E'^2 = \mathfrak{q}^2 + \mu^2$.

The results (2.7) and (2.8) have been written in terms of $I(\mathfrak{p})$ inasmuch as the earlier work of Ref. 5 on the spinor case has explicitly computed that nontrivial integral. In the process of evaluating it they reduce it to

$$I(\mathfrak{p} = \mu\sinh\phi) = \frac{i}{16\pi^3\mu^2}\oint\frac{z(z-2\pi i)\,dz}{\cosh^3 z(\sinh z - \sinh\phi)}$$

where the contour is taken over the boundary of the domain $0 \leq \mathrm{Im}z \leq 2\pi$. Unfortunately, Hanson *et al.* have overlooked the pole at $z = -\phi + \pi i$ and consequently obtain an incorrect result. Inclusion of the omitted pole leads to the form

$$I = -\frac{1}{4\pi^2E^2}\left(1 - \frac{\pi^2\mu^2}{4E^2} + \frac{\mu^2}{4E^2}\ln^2\frac{E+\mathfrak{p}}{E-\mathfrak{p}}\right), \qquad (2.9)$$

which upon insertion in (2.7) and (2.8) yields, after a number of cancellations,

$$\delta\mu^{2(4)} = -(1-\lambda)^2\frac{g^4N^2}{4\pi\mu^2}D$$
$$+\frac{g^4N^2}{8\pi^2\mu^2}(1-\lambda) - g^4N^2\frac{\mathfrak{p}^2}{16E^4} \qquad (2.10)$$

and

$$w^{(4)}=g^4N^2(1-\lambda)D\frac{\partial\mathcal{K}}{\partial\mu^2}+\frac{g^4N^2}{8\pi^2E^4}\times\left[\left(\frac{\mu^2}{4E^2}-\frac{1}{2}\frac{\mathfrak{p}^2}{E^2}\right)\ln^2\frac{E+\mathfrak{p}}{E-\mathfrak{p}}+\pi^2\left(\frac{\mathfrak{p}^2}{E^2}-\frac{1}{4}\right)+2-\frac{\mathfrak{p}^2}{\mu^2}\right].$$

Particularly of interest is the case $\lambda=1$ for which the mass renormalization to fourth order is

$$\delta\mu^2=-\frac{g^2N}{\pi}-\frac{g^4N^2\mathfrak{p}^2}{16E^4},$$

a result which, despite its simplicity, displays the lack of covariance of this theory in perturbation theory. This conclusion is, however, independent of the value of λ inasmuch as the additional terms in (2.10) are of covariant form.

III. COUPLING TO A SPINOR FIELD

Having computed the fourth-order results in the scalar case we now take up the somewhat more interesting situation in which the gauge field is coupled to a spin-$\frac{1}{2}$ field ψ. As before, the fermion transforms according to the fundamental representation of U(N) with a Lagrangian description being provided by

$$\mathcal{L}=i\bar{\psi}\gamma^\mu\partial_\mu\psi-m\bar{\psi}\psi+\tfrac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu}-\tfrac{1}{2}F_a^{\mu\nu}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+igA_\mu t^aA_\nu)+g\bar{\psi}\gamma^\mu T_a\psi A^a_\mu.$$

Inasmuch as this case has already been considered by Hanson *et al.*, it is necessary only to correct their result in the light of the comments previously made concerning $I(\mathfrak{p})$ and to prescribe a procedure by which we obtain the mass renormalization.

Although this latter problem has been solved in the literature,[9] it is appropriate to present a brief review of the procedure to be followed in view of the considerable confusion which seems to exist on the subject. As in the case of spin-0, the continuum contribution to the two-point function

$$G(x-x')=i\epsilon(x-x')\langle 0|(\psi(x)\bar{\psi}(x'))_+|0\rangle$$

vanishes and thus we shall not deal with the most general case discussed in Ref. 9 but shall consider only the pole term.

In the case of a free particle, the Fourier transform of G has the form

$$G_0(p)=\frac{1}{\gamma\cdot p+m}=\frac{2m}{p^2+m^2}\left(\frac{m-\gamma\cdot p}{2m}\right).$$

Since the sum over the free-particle spinors is

$$\sum u(p)\bar{u}(p)=\frac{m-\gamma\cdot p}{2m},\qquad(3.1)$$

one sees that the residue at the single-particle pole is essentially just the sum of the products of the single-particle matrix elements of ψ. Couched in these terms it is relatively straightforward to interpret $G(p)$ in the case of a coupling to a gauge field.

Because of the lack of manifest covariance, the single-particle matrix element will not be the free-particle spinor $u(p)$, but rather $\exp(w\vec{\gamma}\cdot\vec{\mathfrak{p}})u(p)$. This means that the sum (3.1) becomes

$$e^{w\vec{\gamma}\cdot\vec{\mathfrak{p}}}\sum u(p)\bar{u}(p)e^{w\vec{\gamma}\cdot\vec{\mathfrak{p}}}=e^{w\vec{\gamma}\cdot\vec{\mathfrak{p}}}\frac{m-\gamma\cdot p}{2m}e^{w\vec{\gamma}\cdot\vec{\mathfrak{p}}}$$

and one thus infers for $G(\vec{\mathfrak{p}})$ the form

$$G(p)=e^{w\vec{\gamma}\cdot\vec{\mathfrak{p}}}\frac{Z_2(\mathfrak{p}^2)}{\gamma\cdot p+m}e^{w\vec{\gamma}\cdot\vec{\mathfrak{p}}}.$$

Although allowance has been made for a residue function $Z_2(\mathfrak{p}^2)$ in addition to $w(\mathfrak{p}^2)$, it will henceforward be dropped since one finds in the model under consideration that $Z_2(\mathfrak{p}^2)=1$. With this observation the task of interpretation then becomes the reconciliation of the perturbative calculation of $G^{-1}(p)$ with the form

$$G^{-1}(p)=e^{-w\vec{\gamma}\cdot\vec{\mathfrak{p}}}(\gamma\cdot p+m_0+\delta m)e^{-w\vec{\gamma}\cdot\vec{\mathfrak{p}}}.$$

To second order, one calculates the diagram of Fig. 1(a) obtaining for the mass operator $M(p)$

$$M(p)\equiv G^{-1}(p)-\gamma\cdot p=m_0-ig^2N\int\frac{dk}{(2\pi)^2}D^{\mu\nu}(k)\gamma_\mu\frac{1}{\gamma\cdot(p-k)+m_0}\gamma_\nu.$$

One readily obtains from this integral the results

$$\delta m^{(2)}=-g^2N/2\pi m$$

and

$$w^{(2)}=\frac{g^2N}{4\pi}\frac{1}{E^2}\left(\frac{1}{m}+\frac{m}{2E\mathfrak{p}}\ln\frac{E+\mathfrak{p}}{E-\mathfrak{p}}\right),\qquad(3.2)$$

where

$$E^2=m^2+\mathfrak{p}^2.$$

It is of interest to compare this with the spinless case and one thus observes that (3.2) implies

$$\delta m^{2(2)}=-g^2N/\pi,$$

a result which is identical to $\delta\mu^{2(2)}$ for the preferred value $\lambda=1$.

The terms of order g^4N^2 are readily obtained from the graph of Fig. 2(a) with the result

$$M(p)^{(4)}=g^4N^2\frac{1}{2m}\left[I+(\mathfrak{p}-m\gamma_1)\frac{\partial}{\partial\mathfrak{p}}I\right],$$

where $I(\mathfrak{p})$ is identical to that of the preceding section provided that μ is replaced by m. Using the corrected value of this integral one obtains

$$\delta m^{2(4)}=\frac{g^4N^2m^2}{16E^4}$$

and

$$w^{(4)}=\left(\frac{g^2N}{4\pi E^2}\right)^2\left(\frac{3m}{4E^2}\ln^2\frac{E+p}{E-p}+\frac{3}{m}+\frac{E^2}{m^3}-\frac{\pi^2 m}{E^2}\right).$$

There are again remarkable similarities to the spin-0 case as the mass renormalizations differ only by a factor of $-(p^2/m^2)$ and in the w functions the single logarithm terms have each dropped out.

IV. CONCLUSION

The mass renormalizations computed here in fourth order have confirmed the predictions of formal operator techniques that covariance should fail in nonsinglet sectors beginning in that order. This is the first perturbative calculation to demonstrate that a gauge theory can suffer a loss of covariance in an appropriate gauge.[10]

One may be tempted to argue that the absence of a "normal" pole (of the type $p^\mu p_\mu+m^2=0$) in the fermion two-point function indicates the absence of a physical fermion in the theory—in other words that the fermion is "confined." The fact, however, is that *there is a pole* and, insofar as poles in two-point functions represent particles, the only interpretation of this would be the existence of a particle whose energy and momentum do not lie on a mass hyperbola. In other words, the theory is noncovariant.

There is at least one other aspect of the results obtained here which should be emphasized. We refer here to the fact that the spin-$\frac{1}{2}$ case when quantized on the light cone is the same as the 't Hooft model.[11] It has been shown by one of us[12] that the latter has an internal inconsistency because of the breakdown of current conservation. Since, however, the usual quantization on a spacelike surface involves no such contradiction, it is clear that the two quantization schemes cannot be equivalent. The work described in this paper provides, however, a more explicit demonstration of this result. Since the mass renormalization of the 't Hooft model was found in Ref. 12 to be $-g^2N/\pi$ *to all orders*, the fourth-order calculation for spacelike quantization presented here gives a direct proof of inequivalence. Finally, note should be made of the fact that similar fourth-order calculations in four dimensions in both Coulomb and axial gauges would be of considerable interest.

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[8]The infrared divergences in this and all subsequent integrations over internal photon momenta are eliminated by noting that the above prescription for the propagator implies $\int_{-\infty}^{\infty} dk_1 k_1^{-2}=0$. This fact allows the substitution

$$\int dk_1 k_1^{-2} f(k_1) \to \int dk_1 k_1^{-2}[f(k_1)-f(k_1=0)];$$

the resulting k_1 integral is clearly free of infrared divergences.

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