Confinement in SU(N) lattice gauge theories

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The behavior of the Wilson and 't Hooft loops, and of the electric- and magnetic-flux free energies, is examined in weak-coupling SU(N) lattice gauge theories. The ability of thin tubes of magnetic flux to spread into thick flux tubes is emphasized as the key feature which is required for confinement. While control over the energetics of these fluctuations sufficient to prove confinement is still lacking, we are able to prove that if constraints are inserted which prevent this spreading of $Z(N)$ magnetic flux then nonconfining behavior results.

I. INTRODUCTION

Hopefully, quantum chromodynamics (QCD) may be defined nonperturbatively as the limit of a lattice gauge theory in which the lattice spacing is taken to zero.[1-3] In order to recover an asymptotically free continuum theory the bare coupling must vanish as a function of the lattice spacing in such a way that physical masses are held fixed. For such an approach to succeed, one must prove that the confining phase of the theory, known to be present for strong coupling,[1-3] persists all the way down to zero coupling. Unfortunately, there exists very little direct evidence showing this to be the case. (Numerical calculations at best support the assumption of no phase transition.[2,4])

We assume that confinement is a property which characterizes the pure gauge theory since vacuum polarization effects due to quarks appear to be unimportant in physical hadrons. Hence, we will study the SU(N) pure gauge theory without matter fields. However, having eliminated the quarks, one obviously cannot see quark confinement directly and must instead introduce a pure gauge criterion for confinement. Wilson proposed the first such criterion, now called the Wilson loop.[1] It is easily related to the heavy-quark potential governing the effective force between nonrelativistic heavy quarks. The Wilson loop is the most natural order parameter to use for characterizing the phases of a pure gauge theory.

Later, 't Hooft introduced a topological disorder parameter which we will call the 't Hooft loop.[5] The Wilson and 't Hooft loops satisfy a remarkable commutation relation, and in a sense they may be regarded as conjugate order and disorder variables. 't Hooft argued that his disorder parameter could be considered as an alternative confinement criterion which might be calculationally more convenient than the Wilson loop. However, the behavior of the Wilson loop may not be directly deduced from corresponding knowledge of the 't Hooft loop.

Recently, 't Hooft also introduced a different set of observables which we will call the electric- and magnetic-flux free energies.[6] They are defined by imposing twisted periodic boundary conditions on the theory in a finite volume in such a way that configurations with a specified electric or magnetic flux flowing through the periodic box are projected out. The electric and magnetic free energies satisfy an exact duality relation which essentially shows that they are Fourier transforms of each other.

After reviewing our notation in Sec. II, we explicitly define each of these observables in the lattice gauge theory in Sec. III. We discuss the expected behavior of the observables and their interpretation as equivalent confinement criteria. In Sec. IV we consider the weak-coupling limit of the theory and argue that the expected confining behavior of all our observables may be understood as a consequence of a single mechanism, the spreading of magnetic flux possible in an SU(N) theory. While we do not have sufficient control over these fluctuations to prove confinement, we are able to prove that if confinement occurs then it must be due to this mechanism. Specifically, in Sec. V we insert constraints into the SU(N) theory which prevent this flux spreading and then show that nonconfining behavior of our observables results. A few concluding remarks are contained in Sec. VI, which is followed by three appendices. Appendices A and B present technical bounds needed in Sec. V. Appendix C briefly considers the strong-coupling limit and shows how the same methods used earlier may be employed to provide a simple proof of the convergence of the strong-coupling cluster expansion. The equivalence of the Wilson loop and the electric-flux free energy is demonstrated in this limit.

This paper was largely motivated by recent work of Mack and Petkova[7-9] and may be considered as an extension and explanation of their results. They considered a modified SU(2) theory and proved that the 't Hooft loop has area-law (i.e., nonconfining) behavior.[7] We consider the magnetic-flux free energy as well as the 't Hooft loop and extend the analysis to any SU(N) gauge group. We explain at some length why the constraints which modify the theory are expected to produce such nonconfining behavior and then rigorously prove this result. In their second paper, Mack and Petkova found a simple bound on the expectation of the Wilson loop based on the behavior of "thick vortices."[8] We use essentially the same procedure in Sec. IV in order to relate the behavior of the Wilson loop to the properties of magnetic flux. Finally, in a third paper Mack and Petkova discussed how the standard SU(2) theory could be interpreted as a $Z(2)$ gauge theory with fluctuating coupling constants in the presence of magnetic monopoles produced by the SU(2)/$Z(2)$ dynamics.[9] (In their modified model these dynamical monopoles are eliminated.) This result, which emerges naturally from the $Z(2)$ dual transformation used by Mack and Petkova, may be easily generalized to any SU(N) theory. However, in the present paper we will not discuss this $Z(N)$ dual transformation, simply because we have not found it particularly helpful for understanding the SU(N) theory. [If $N>2$ then the dual measure is complex which makes it rather inconvenient for proving rigorous bounds. Furthermore, we have found it useful to emphasize the spreading of magnetic-flux sheets which is possible in an SU(N) theory, instead of stressing the dynamical monopoles which simply give the boundaries of these flux sheets. In this way all of our observables may be discussed from a unified viewpoint.] See Ref. 10 for an application of a $Z(N)$ dual transformation to the SU(N) theory which does not explicitly separate the SU(N)/$Z(N)$ magnetic monopoles.

II. NOTATION

We will consider the standard SU(N) lattice gauge theory defined on a d-dimensional simple cubic lattice Λ of size $A_0\times\cdots\times A_{d-1}$, $d>2$. For convenience we will choose $A_\mu=2^{\alpha_\mu}$, for integral $\{\alpha_\mu\}$. The size of the lattice will always be increased in a "power law" fashion: Let $\alpha_\mu=l\delta_\mu+\alpha_\mu^0$ for a fixed choice of integer exponents $\delta_\mu>0$, and $l=0,1,2,\ldots$. Then $\Lambda\to Z^d$ as $l\to\infty$. We will make frequent use of boundary and coboundary operators and so begin by introducing the appropriate notation.[11]

The lattice Λ contains sites $[s]$, bonds $[b]$, plaquettes $[p]$, cubes $[c]$, etc., generically referred to as r-cells $[c_r]$, $r=0,\ldots,d$. Each r-cell is assigned a standard orientation; $-c_r$ will be understood to label an r-cell with the opposite orientation. When necessary, sites will be labeled by their lattice position n (n_μ = integer). This implies the partial ordering $s(n)<s(n')$ iff $n_\mu\leq n'_\mu\ \forall\,\mu$. Higher r-cells may then be labeled by the smallest site they contain plus the positive directions they extend in: $b_\mu(n)$, $p_{\mu\nu}(n)$, $c_{\mu\nu\lambda}(n)$, etc., $\mu<\nu<\lambda<\cdots$.

We will not distinguish between sets of r-cells and the characteristic functions defined for each set. For example, if S is a set of plaquettes, then the function $S[p]$ equals ±1 if $\pm p\in S$, 0 otherwise.

Each r-cell has a boundary of $(r-1)$-cells, ∂c_r, defined with appropriate orientation so that $\partial^2=0$. Explicitly,

$$\partial s(n)=0\,,$$

$$\partial b_\mu(n)=(s(n)-s(n+\hat e_\mu))\,,$$

$$\partial p_{\mu\nu}(n)=(b_\mu(n)-b_\mu(n+\hat e_\nu))-(b_\nu(n)-b_\nu(n+\hat e_\mu))\,,$$

$$\begin{aligned}\partial c_{\mu\nu\lambda}(n)=&(p_{\mu\nu}(n)-p_{\mu\nu}(n+\hat e_\lambda))\\&-(p_{\mu\lambda}(n)-p_{\mu\lambda}(n+\hat e_\nu))\\&+(p_{\nu\lambda}(n)-p_{\nu\lambda}(n+\hat e_\mu))\,.\end{aligned}$$

Each r-cell has a coboundary of $(r+1)$-cells, $\hat\partial c_r$, defined as those $(r+1)$-cells whose boundaries contain c_r, that is, $\pm c_{r+1}\in\hat\partial c_r$ iff $\pm c_r\in\partial c_{r+1}$. Explicitly,

$$\hat\partial s(n)=\sum_\mu\,(b_\mu(n)-b_\mu(n-\hat e_\mu))\,,$$

$$\hat\partial b_\mu(n)=\sum_{\nu\neq\mu}\,(p_{\mu\nu}(n)-p_{\mu\nu}(n-\hat e_\nu))\,,$$

$$\hat\partial p_{\mu\nu}(n)=\sum_{\lambda\neq\mu,\nu}\,(c_{\mu\nu\lambda}(n)-c_{\mu\nu\lambda}(n-\hat e_\lambda))\,.$$

Note that $\hat\partial c_d=0$, and $\hat\partial^2=0$. The boundary and coboundary operators extend to arbitrary sums of r-cells (r-chains) by linearity. To illustrate these definitions observe that a set of plaquettes S is closed, that is, $\partial S=0$, if no bond is contained in the boundary of an odd number of plaquettes. Similarly, S is coclosed, $\hat\partial S=0$, if no cube has a boundary which contains an odd number of plaquettes of S.

We will introduce numerous variables taking values in $Z(N)$, the center of SU(N). We will consider $Z(N)$ as the multiplicative group with elements $\{1,\iota,\iota^2,\ldots,\iota^{N-1}\}$, where $\iota\equiv\exp(2\pi i/N)$. This group is obviously isomorphic to the mod$N$ additive group $z(N)$ with elements $\{0,1,2,\ldots,N-1\}$

under $\iota^k \in Z(N) \sim k \in z(N)$. In order to freely use this isomorphism without comment in what follows, we will standardize our choice of symbols and use the greek letters $(\epsilon, \zeta, \eta, \sigma, \tau, \omega, \ldots)$ to label elements of $Z(N)$ and similar latin letters $(e, z, n, s, t, w, \ldots)$ to label corresponding elements of $z(N)$. Hence, either element of an equivalent pair, such as (τ, t), will be used when convenient. The normalized Haar measure on $Z(N)$ is given by

$$\int_{Z(N)} d\omega f(\omega) = \frac{1}{N} \sum_{w=0}^{N-1} f(\iota^w) .$$

If $\omega[c_r]$ is a $Z(N)$ lattice field defined on r-cells, then $\omega[-c_r] \equiv \omega[c_r]^{-1}$; if S is any set of r-cells, then we define $\omega[S] = \prod_{c_r \in S} \omega[c_r]$.

The basic variables of the theory are the bond variables $U[b] \in \mathrm{SU}(N)$, $U[-b] \equiv U[b]^{-1}$. If C is an oriented path consisting of bonds $b_1, \ldots, b_n$, then $U[C] \equiv U[b_1] \cdots U[b_n]$. The standard SU(N) lattice gauge theory is defined by the (Euclidean) action

$$L(U) = \sum_{p \in \Lambda} -\mathcal{L}(U[\partial p]) , \tag{1}$$

where

$$\mathcal{L}(U) = \beta \operatorname{Re} \operatorname{tr}(U)$$

and $\beta = 1/g^2$ with g the standard gauge coupling constant. The plaquette variable $U[\partial p]$ is the product of bond variables around the perimeter of the plaquette p. We will find it convenient to define $\eta[p]$ as the $Z(N)$ part of $U[\partial p]$, $U[\partial p] \equiv \hat{U}[\partial p]\eta[p]$, where $-\pi/N < \arg \operatorname{tr}\hat{U}[\partial p] \leq \pi/N$. The partition function Z is given by

$$Z = \int \prod_{b \in \Lambda} dU[b] \exp[-L(U)] , \tag{2}$$

where $dU[b]$ is the normalized Haar measure on SU(N) (i.e., $\int_{\mathrm{SU}(N)} dU[b] = 1$). Boundary conditions for $U[b]$ will be discussed in Sec. III. For later convenience define $d\nu(U) = \prod_{b \in \Lambda} dU[b]$. Observables are functions $F(U)$ of the bond variables $U[b]$. Their expectation value is given by

$$\langle F \rangle = \int d\mu(U) F(U) ,$$
$$d\mu(U) = \frac{1}{Z} e^{-L(U)} \prod_b dU[b] . \tag{3}$$

III. OBSERVABLES

In order to clearly define our observables we will first rephrase the description of the standard lattice gauge theory in terms appropriate to quantum mechanics.

The Hilbert space of states consists of wave functions $\Psi(\{U[b]\})$. They depend on variables $U[b]$ in an $n_0 = t =$ constant "spacelike" plane Σ. A gauge transformation $V[s] \in \mathrm{SU}(N)$, $s \in \Sigma$ is implemented by the operator $\Omega(V)$, defined by

$$(\Omega(V)\Psi)(\{U[b]\}) = \Psi(\{U'[b]\}) , \tag{4}$$

where $U'[b] = V[s]U[b]V[s']^\dagger$ if $\partial b = s - s'$. The scalar product is given by

$$(\Psi_1, \Psi_2) = \int \prod_{b \in \Sigma} dU[b] \bar{\Psi}_1(\{U[b]\}) \Psi_2(\{U[b]\}) \times \exp\left(-\sum_{p \in \Sigma} \mathcal{L}(U[\partial p])\right) . \tag{5}$$

(The factor $\exp\{-\sum_{p \in \Sigma} \mathcal{L}(U[\partial p])\}$ could be absorbed in the wave functions. However, the choice (5) is more convenient for our purposes.) The partition function Z is defined as $Z = \operatorname{Tr}(e^{-TH})$ where T is the length of the lattice in the "time" direction (i.e., $T = A_0$), and the transfer matrix e^{-H} is given by

$$(\Psi_1, e^{-H}\Psi_2) = \int \prod_{b \in \Sigma} dU[b] \prod_{b \in \Sigma'} dU'[b] \prod_{b \in \Xi} dU''[b] \bar{\Psi}_1(\{U[b]\}) \Psi_2(\{U'[b]\}) \exp\left(\sum_{p \in \Xi} \mathcal{L}(U[\partial p])\right) . \tag{6}$$

Here Σ' is the spacelike plane $n_0 = t - 1$, one unit before Σ, and Ξ is the open region $t - 1 < n_0 < t$ between Σ' and Σ. (So Ξ contains only timelike plaquettes.) Note that e^{-H} projects out gauge-invariant physical states. Any gauge transformation acting on the initial or final states may be absorbed by a change of variables in (6). This definition of Z agrees with the previous expression (2) supplemented by the further condition coming from the Hilbert-space trace which requires us to impose periodic boundary conditions in the time direction.

The Wilson loop[1] $A^E[C]$ is defined for a closed curve of bonds C lying in Σ as the multiplication operator

$$(A^E[C]\Psi)(\{U[b]\}) = \chi^E(U[C])\Psi(\{U[b]\}) . \tag{7}$$

Here E labels a representation of SU(N) with n-ality e, that is $\chi^E(\iota) = \iota^e$. Defining the quantum expectation $\langle A^E[C]\rangle = \operatorname{Tr}(e^{-TH}A^E[C])/Z$ and comparing with Eq. (3), we clearly have

$$\langle A^E[C]\rangle = \langle \chi^E(U[C])\rangle . \tag{8}$$

This expression may be extended to any loop $C \subset \Lambda$ not necessarily lying in Σ for which the operator form (7) may not be defined. We will always take C to be a simple closed curve so that $C = \partial S$ for some set of plaquettes S. (S is not unique.)

The expectation of the Wilson loop obviously does not depend on which lattice direction is chosen as "time." To interpret its effect it is convenient to choose the plaquettes $p \in S$ to be timelike. As a result, the timelike legs of the Wilson loop modify the transfer matrix so that at times which cut the Wilson loop, non-gauge-invariant states are in fact propagating. In particular, if we perform a gauge transformation V^t, which is equal to 1 at the position of the downward leg of the loop but equals $\tau = \iota^t$, some element of the center $Z(N)$, at the other leg, then intermediate states acquire a phase,

$$\Omega(V^t)\Psi = \iota^{(et)}\Psi\,. \tag{9}$$

This defines what we mean by saying that this leg of the Wilson loop is a source of e units of electric flux. The other leg is similarly a flux sink. In general, a Wilson loop may be regarded as creating an electric loop which acts as a source of electric flux spanning the loop.

The 't Hooft operator[5] $B^\tau[C^*]$, $\tau \in Z(N)$, is defined for a set of bonds L^* lying in Σ as the operator

$$(B^\tau[C^*]\Psi)(\{U[b]\}) = \Psi(\{\tau^{L^*[b]}U[b]\})\,. \tag{10}$$

Remember that $\tau^{L^*[b]} \equiv \{\tau \text{ if } b \in L^*,\ 1 \text{ if } b \notin L^*\}$. The relation between C^* and L^* is as follows: Let Q^* be the coboundary of L^* restricted to Σ, $Q^* = \hat{\partial}L^* \cap \Sigma$. Note that Q^* is coclosed in Σ. Let S^* be the set of plaquettes protruding from L^* one unit in the positive time direction, $S^* = \{p_{0\mu}(n) | b_\mu(n) \in L^*\}$, and similarly let C^* be the set of cubes directly above Q^*. Then $C^* = \hat{\partial}S^*$ is a closed set of cubes in Λ (see Fig. 1). Defining the quantum expectation $\langle B^\tau[C^*]\rangle = \mathrm{Tr}(e^{-TH}B^\tau[C^*])/Z$, one finds that[7,10]

$$\langle B^\tau[C^*]\rangle = \left\langle \exp\left(\sum_{p\in S^*}\{\mathcal{L}(\tau U[\partial p]) - \mathcal{L}(U[\partial p])\}\right)\right\rangle \tag{11}$$

since $B^\tau[C^*]$ inserts the twist τ in each of the timelike plaquettes of S^* extending from L^*. This expression is in fact independent of the particular choice of S^*; it only depends on the coboundary C^*. [If $\hat{\partial}S^* = C^*$ and $\hat{\partial}\bar{S}^* = C^*$, then $\hat{\partial}(\bar{S}^* - S^*) = 0$ so there exists a set of links K^* such that $\hat{\partial}K^* = \bar{S}^* - S^*$. The change of variables $U[b] \to \tau U[b]$ for $b \in K^*$ moves S^* to $\bar{S}^*$ in the right-hand side of (11).] Clearly, expression (11) may also be extended to any coclosed set of cubes $C^* = \hat{\partial}S^*$ for some set $S^* \subset \Lambda$.

As a consequence of the definitions (7) and (10), the following commutation relation ('t Hooft algebra) holds for $C, L^* \subset \Sigma$:

$$B^\tau[C^*]A^E[C] = \tau^{eL^*[C]}A^E[C]B^\tau[C^*]\,, \tag{12}$$

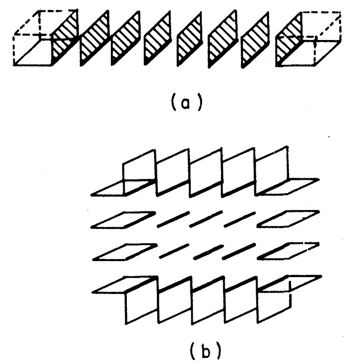

FIG. 1. (a) The 't Hooft loop in three dimensions. Heavy lines are links L^*, light lines are plaquettes Q^*. Plaquettes S^* are shaded while the cubes C^* are shown at either end. (b) $t=0$ plane of the 't Hooft loop in four dimensions. Links L^* and plaquettes Q^* are shown. A cross section cutting through the loop will appear for all t, as in (a).

where $L^*[C] \equiv \sum_{b\in C} L^*[b]$.

We will be primarily interested in sets C^* of the form shown in Fig. 1. In four dimensions, C^* is a loop of cubes (or a closed curve in the dual lattice) and $L^*[C]$ is simply the linking number of the two loops. In three dimensions, C^* is just two separated cubes (of opposite orientation) and $L^*[C]$ counts the number of cubes of C^* (or rather plaquettes of Q^*) inside C. For convenience we will refer to $B^\tau[C^*]$ as the 't Hooft loop regardless of dimension even though this name is only really appropriate in four dimensions.

The 't Hooft operator may be regarded as creating a magnetic loop which acts as a source of magnetic flux. The action for plaquettes $p \in S^*$ is changed from $\exp\{\mathcal{L}(U[\partial p])\}$ to $\exp\{\mathcal{L}(\tau U[\partial p])\}$. Thus, instead of favoring the classical vacuum configuration $U[\partial p] = 1$, the action is now minimized for configurations with magnetic flux $U[\partial p] = \tau^{-1}$ for $p \in S^*$. So the twist introduced by $B^\tau[C^*]$ acts as a Lagrange multiplier for magnetic flux analogous to the conventional sources used in, e.g., symmetry breaking. Because the plaquette variables $U[\partial p]$ are not independent, $\langle B^\tau[C^*]\rangle$ does not depend on the precise location of S^* except for its coboundary C^*. Hence C^* may be thought of as a magnetic source, with S^* a Dirac sheet of magnetic flux. (In three dimensions, S^* is a Dirac string.)

Using the Wilson (or 't Hooft) loop to explicitly insert a flux source is not the only way to satisfy Gauss's law (or the Bianchi indentities) in order to study the energy of electric (or magnetic) flux. To avoid the introduction of a source coupled to the boundary of a flux sheet, we may instead connect the ends of the flux sheet by imposing periodic boundary conditions. This allows the formation of topologically stable closed flux lines which wind about the lattice. This is the approach recently suggested by 't Hooft.[6]

Specifically, we will modify the transfer matrix [Eq. (6)] by requiring the timelike bond variables on opposite sides of the cubic spatial lattice to be equal. Physical states, that is, states propagated by the transfer matrix, are now invariant under "proper" gauge transformations, ones where $V[s]$ satisfies the periodic boundary conditions. However, we may now in addition consider the behavior of states under "improper" gauge transformations, those with nonperiodic $V[s]$. In particular consider a set of gauge transformations $V^{\vec{t}}$, $\vec{t}=(t_1,\ldots,t_{d-1})$, $t_j\in z(N)$ given by

$$V^{\vec{t}}[s(n)]=\iota^{(\vec{t}\cdot\vec{N}^*[s])},$$

where $\vec{N}^*=(N_1^*,\ldots,N_{d-1}^*)$, and $N_j^*=\{s(n)\in\Sigma\,|\,s(n-\hat{e}_j)\notin\Sigma\}$. In words, N_j^* is the set of sites in the first plane normal to e_j in the spatial lattice Σ. $V^{\vec{t}}[s]$ is equal to one except on N_j^* where $V^{\vec{t}}=\tau_j$ (except on $N_j^*\cap N_k^*$ where $V^{\vec{t}}=\tau_j\tau_k$, etc.).

Thus, $V^{\vec{t}}$ varies by τ_j from one side of the lattice (N_j^*) to the other. These gauge transformations commute with each other and consequently may be simultaneously diagonalized. Physical states need not be invariant under $\Omega(V^{\vec{t}})$. Therefore, if a state Ψ transforms as

$$\Omega(V^{\vec{t}})\Psi=\iota^{(\vec{e}\cdot\vec{t})}\Psi\,,\quad \vec{e}\in z(N)\,, \tag{13}$$

then Ψ has e_j units of electric flux flowing through the periodic lattice in the jth direction. [Compare with Eq. (9).] We may separate the contribution of states of a given global flux to the partition function by introducing the projection operators

$$P(\vec{e})=\int d\vec{\tau}\,\iota^{-(\vec{e}\cdot\vec{t})}\Omega(V^{\vec{t}})\,. \tag{14}$$

Clearly $\sum_{\vec{e}}P(\vec{e})=1$. Notice now that

$$(\Omega(V^{\vec{t}})\Psi)(\{U[b]\})=\Psi(\{\iota^{(\vec{t}\cdot\vec{L}^*[b])}U[b]\})\,, \tag{15}$$

where L_j^* is the set of bonds in the jth direction which begin at N_j^*, that is, $L_j^*=\{b_j(n)\,|\,s(n)\in N_j^*\}$. But, comparing with Eq. (10) we see that $\Omega(V^{\vec{t}})$ is nothing but a product of 't Hooft operators (one for each spatial direction j) defined on the sets of links L_j^*. Since L_j^* is coclosed in Σ (it completely winds about the periodic spatial lattice once), $\Omega(V^{\vec{t}})$ may be thought of as creating magnetic flux which winds about the lattice without sources or sinks. (Once again, we emphasize that while $V^{\vec{t}}$ is simply a gauge transformation, it is not a proper gauge transformation so that it need not have trivial physical consequences.) Equation (15) says that $P(\vec{e})$, which projects out states with a given electric flux, is simply the Fourier transform of $\Omega(V^{\vec{t}})$, which creates magnetic flux.[6] This is the precise connection between electric and magnetic flux that we need.

Repeating the steps leading to Eq. (11), we find

$$\langle\Omega(V^{\vec{t}})\rangle=\left\langle\exp\left(\sum_{p\in S_j^*}\{\mathfrak{L}(\tau_jU[\partial p])-\mathfrak{L}(U[\partial p])\}\right)\right\rangle,$$

where S_j^* is any coclosed set of plaquettes which winds once through each $[0j]$ plane of Λ. Finally, we note that such a twist may be defined for each plane direction $[\mu\nu]$. Thus, we define the magnetic-flux free energy $F_m(t_{\mu\nu})$, $t_{[\mu\nu]}\in z(N)$ by

$$e^{-F_m(t_{\mu\nu})}=\left\langle\exp\left(\sum_{p\in S_{\mu\nu}^*}\{\mathfrak{L}(\tau_{\mu\nu}U[\partial p])-\mathfrak{L}(U[\partial p])\}\right)\right\rangle \tag{16}$$

and the electric-flux free energy $F_e(e_{\mu\nu})$, $e_{[\mu\nu]}\in z(N)$ by the Fourier transform

$$e^{-F_e(e_{\mu\nu})}=\int d\tau_{[\mu\nu]}\,\iota^{-(e\cdot t)}e^{-F_m(t_{\mu\nu})}\,. \tag{17}$$

Here $(e\cdot t)=\sum_{\mu<\nu}e_{\mu\nu}t_{\mu\nu}$ and $S_{\mu\nu}^*$ is any coclosed set of plaquettes which winds once through each $[\mu\nu]$ plane. (For example, $S_{\mu\nu}^*=\{p_{\mu\nu}(n)\,|\,n_\mu=n_\nu=0\}$.)

Henceforth, the lattice Λ is to be regarded as periodic in all directions. Equation (17) may of course be inverted to read

$$e^{-F_m(t_{\mu\nu})}=\sum_{e[\mu\nu]}\iota^{(e\cdot t)}e^{-F_e(e_{\mu\nu})}\,. \tag{18}$$

These formulas exhibit the dual symmetry between electric and magnetic flux. See Ref. 6 for various applications of this duality.

We would now like to discuss the interpretation of each of our observables as a confinement criterion. Consider the Wilson loop for a large, rectangular contour of length T and width R. By examining the propagation of arbitrarily heavy quarks in the representation E, one finds that[1] $\langle A^E[C]\rangle\sim\exp[-TV^E(R)]$ as $T\to\infty$, where $V^E(R)$ is the nonrelativistic heavy-quark potential. If $V^E(R)\to\infty$ as $R\to\infty$ then (heavy) quarks in the representation E are confined. This is the basis for the use of the Wilson loop as the standard quark-confinement criterion. For large T, the Wilson loop is sensitive to the states of lowest energy which have electric flux connecting a point source and sink a distance R apart. If the electric flux is focused into a physical flux sheet, then the Wilson loop will obey the area law

$$\langle A^E[C]\rangle\sim\exp[-\rho(e)A] \tag{19}$$

for large loops C spanned by a surface of minimal area A. This implies a linear confining potential with string tension $\rho(e)$. [One expects ρ to only depend on the n-ality e since any representation of $SU(N)/Z(N)$ may be screened. This produces only perimeter terms.[9]]

Similarly, the electric-flux free energy probes the states of lowest energy with a specified global flux. Therefore, if electric flux forms physical flux sheets, then one expects that

$$\exp[-F_e(e_{\mu\nu})] \sim \prod_{\substack{[\mu\nu]\\ e_{\mu\nu}\neq 0}} A^*_{\mu\nu} \exp[-\rho(e_{\mu\nu})A_{\mu\nu}] \quad \text{as } |\Lambda| \to \infty . \tag{20}$$

Here $A_{\mu\nu}=A_\mu A_\nu$ is the minimal size of the global flux sheet, and the factor $A^*_{\mu\nu} \equiv \prod_{\lambda\neq\mu,\nu} A_\lambda$ is present because the flux sheet may be located anywhere in the transverse directions. This predicts that $F_e(e_{\mu\nu}) \to \infty$ for $e_{\mu\nu}\neq 0$ as $|\Lambda| \to \infty$ in any power-law fashion. For a finite-size lattice, contributions from multiple flux sheets will essentially exponentiate the above expression. Furthermore, flux sheets in different directions may interfere giving rise to additional contributions.

Standard folklore suggests that the focusing of electric flux needed for linear confinement (19) will be the result of a dual Meissner effect if magnetic monopoles or solitons have condensed in the vacuum. In such a state the magnetic flux produced by the 't Hooft loop should be completely defocused, so one expects that

$$\langle B^\tau[C^*]\rangle \sim \exp[-\alpha(\tau)|C^*|] . \tag{21}$$

(That is, $\langle B^\tau[C^*]\rangle \sim$ const if $d=3$, and $\langle B^\tau[C^*]\rangle \sim \exp[-(\text{perimeter})]$ if $d=4$. We will call this perimeter-law behavior of the 't Hooft loop regardless of dimension.)

Finally, the duality relation (18) and electric confinement (20) imply that the magnetic-flux free energy behaves for any power-law growth of Λ as

$$\exp[-F_m(t_{\mu\nu})] \sim 1 \quad \text{as } |\Lambda| \to \infty \tag{22}$$

with exponentially small corrections.

If in place of a confining phase we have a Higgs phase such as would be produced by a condensation of electric objects, then electric flux will be free to wander through the lattice and magnetic flux will be strongly focused. Then one would expect

$$\langle A^E[C]\rangle \sim \exp[-\alpha(E)|C|] , \tag{23}$$

$$e^{-F_e(e_{\mu\nu})} \sim N^{-d(d-1)/2} \quad \text{as } |\Lambda| \to \infty , \tag{24}$$

$$\langle B^\tau[C^*]\rangle \sim \exp[-\rho(\tau)A^*] , \tag{25}$$

$$e^{-F_m(t_{\mu\nu})} \sim \prod_{\substack{[\mu\nu]\\ t_{\mu\nu}\neq 0}} A_{\mu\nu} \exp[-\rho(t_{\mu\nu})A^*_{\mu\nu}] \quad \text{as } |\Lambda| \to \infty . \tag{26}$$

Here A^* is the minimal size of the set S^* (for fixed coboundary C^*). The constant in (24) is an artifact of our normalization condition $\sum_{e_{\mu\nu}} e^{-F_e(e_{\mu\nu})}=1$. Note that (23) implies a mere mass shift for heavy quarks. For more discussion of this intuitive picture, see Refs. 5, 10, and 12, and references therein.

This view of dual confining and Higgs phases is very appealing. However, it obviously does not prove that the 't Hooft loop or the flux free energies are equivalent to the Wilson loop as confinement criteria.

One may argue that the electric-flux free energy and the Wilson loop will exhibit equivalent behavior in any phase with a mass gap since both probe the properties of electric-flux sheets which are large compared to the coherence length. Underlying this argument is the assumption that the dynamics is local so that the flux sheets produced by the free energy and the Wilson loop are physically equivalent. If the behavior of the Wilson loop depends critically on the pinching of electric flux at the loop itself, then obviously this behavior will not be reflected in the electric-flux free energy. Such effects may lead to nonlinear confinement, that is, $V(R) \to \infty$, but $V(R)/R \to 0$ as $R \to \infty$, which presumably is possible only in the absence of a mass gap. Free Abelian theories in three dimensions provide an example of such a phase. Note also that $V(R)$ cannot rise faster than linearly.[13]

In his original paper,[5] 't Hooft tried to relate $\langle A^E[C]\rangle$ and $\langle B^\tau[C^*]\rangle$ by considering the consequences of the commutation relation (12). He used cluster decomposition of $\langle A^E[C]B^\tau[C^*]\rangle$ to argue that in the presence of a mass gap, perimeter-law behavior for both the Wilson and 't Hooft loops is incompatible with the commutation relation. He therefore concluded that one or both types of loops must have area-law behavior with the consequent formation of physical flux sheets. He suggested that perimeter-law behavior (21) for the 't Hooft loop could be used as an alternate confinement criterion. Unfortunately, there are a number of potential difficulties with this proposal. First, since the separation of two interlocked loops cannot be greater than their perimeters, it is not obvious that exponentially small corrections to clustering are less than the exponentially small product of expectation values. Second, the assumption of a mass gap is required. Thus, perimeter-law behavior of the 't Hooft loop is, by itself, insufficient to prove confinement. And third, since the possibility of simultaneous area-law behavior of both Wilson and 't Hooft loops could not be eliminated, area-law behavior for the 't Hooft loop is insufficient to disprove confinement.

We will argue in the next section that the equivalence between the behavior of the 't Hooft loop or flux free energies and the Wilson loop may be selectively destroyed by inserting certain constraints in the theory. Thus any model-independent argument relating the various observables is bound to leave certain loopholes.

IV. FLUX SPREADING

In this section we would like to understand qualitatively how it can be that the standard SU(N) lattice gauge theory confines for weak coupling. In particular, we want to determine what sort of fluctuations can give rise to the expected behavior (19)–(22) for our observables.

Consider first the expectation of the 't Hooft operator

$$\langle B^\tau[C^*]\rangle = \frac{1}{Z}\int d\nu(U)\exp\Big(\sum_p \mathcal{L}(\tau^{S^*[p]}U[\partial p])\Big). \tag{27}$$

It may be regarded as creating a thin string ($d=3$) or sheet ($d=4$) of magnetic flux on the plaquettes S^*. Consequently $\langle B^\tau[C^*]\rangle$ may be thought of as probing the response of the theory to the external introduction of magnetic flux. This situation is analogous to the use of the operator $\bar\psi(x)\exp[\int_x^y A\cdot dx]\psi(y)$ for studying the Coulomb potential in QED. There one introduces a thin string of electric flux and subsequently finds that the dominant fluctuations are those which exactly cancel the thin string and replace it with a spread-out dipole field. Similarly, we need to find the dominant fluctuations in the presence of the 't Hooft loop.

Consider for the moment a $Z(N)$ gauge theory, where $U[b]\in Z(N)$. As $\beta\to\infty$ (weak coupling), $\exp[\mathcal{L}(U)]$ becomes highly peaked about $U=1$. Therefore, the integrand for the 't Hooft loop (27) is maximized when

$$U[\partial p] = \tau^{-S^*[p]}. \tag{28}$$

However, no fluctuations exist which satisfy this condition in a $Z(N)$ theory. To see this, note that the identity

$$\eta[S] = \prod_{p\in S}\eta[p] = 1 \tag{29}$$

is automatically satisfied in the $Z(N)$ theory for any closed surface S. (This follows since $\eta[p]=U[\partial p]=\prod_{b\in\partial p}U[b]$ and each bond appears twice (with opposite orientations) in the product in (29).) If we consider any closed surface S which wraps around the 't Hooft loop C^*, then (28) would imply $\eta[S]=\tau^{-1}$, in contradiction with (29). (S wrapping around C^* means $S[S^*]=1$. In three dimensions this implies that S surrounds one of the cubes of C^*, whereas in four dimensions a two-surface S may wrap around the closed loop C^*.) Hence, in a $Z(N)$ theory no fluctuation can remove the magnetic flux introduced by the 't Hooft operator. In particular, every possible configuration must violate (28) on some set of plaquettes $\bar S^*$ whose coboundary is C^*. Since each such plaquette contributes a finite action (at least $\beta[1-\cos(2\pi/N)]$ above the minimum) we see that $\langle B^\tau[S^*]\rangle$ will have the area-law behavior (25) instead of the perimeter behavior we need. (This argument, while correct if $d>2$, is not really complete since we have not considered entropy effects. We will show how to prove the result rigorously in the next section.)

So we see that $Z(N)$ fluctuations are unable to cancel the localized magnetic flux introduced by the 't Hooft loop, and because localized flux possesses a large energy this leads to the wrong weak-coupling behavior. Fortunately, new fluctuations are available in the SU(N) theory due to the continuous nature of the group. In particular, $\eta[S]$ need not equal 1 so that many configurations exist which cancel the $Z(N)$ flux sheet produced by $B^\tau[C^*]$ and replace it with spread-out magnetic flux. (We will be more explicit about how to construct such configurations later.) Such flux spreading will lower the action of the dominant fluctuations since $\exp\{\mathcal{L}(U[\partial p]\tau^{S^*[p]})\}$ will be close to its maximum for nearly all plaquettes. If the average action per unit length (or area) of S^* of such configurations is reduced to zero, then, at most, end-point effects will contribute so that the 't Hooft loop will have the desired perimeter-law behavior (21).

The magnetic-flux free energy

$$e^{-F_m(t_{\mu\nu})} = \frac{1}{Z}\int d\nu(U)\,\exp\Big(\sum_p \mathcal{L}(\tau_{\mu\nu}{}^{S^*_{\mu\nu}[p]}U[\partial p])\Big) \tag{30}$$

may be discussed in exactly the same terms. It creates thin strings (or sheets) of magnetic flux on the sets $S^*_{\mu\nu}$. This sourceless flux is topologically stable since it runs completely around the periodic lattice. If only $Z(N)$ fluctuations are considered, then these flux sheets cannot spread out. To see this consider any closed surface $S_{\alpha\beta}$ which wraps once around the lattice in the $[\alpha\beta]$ direction. (For example, $S_{\alpha\beta}=\{p_{\alpha\beta}(n)\,|\,n_\lambda=0, \lambda\neq\alpha,\beta\}$.) Note that $S_{\alpha\beta}[S^*_{\mu\nu}]=\delta^{\alpha\beta}_{\mu\nu}$ since the coclosed surface $S^*_{\mu\nu}$ runs once through every closed surface $S_{\mu\nu}$. Since $\eta[S_{\alpha\beta}]$ is identically equal to one for $Z(N)$ fluctuations, every surface $S_{\alpha\beta}$ will contain at least one plaquette with a localized $Z(N)$ flux (i.e., $\eta[p]\tau_{\mu\nu}{}^{S^*_{\mu\nu}[p]}\neq 1$) when $t_{\alpha\beta}\neq 0$. Hence, if only $Z(N)$ fluctuations are considered then the magnetic-flux free energy will exhibit the area-law behavior (26).

Therefore, in order for magnetic flux to be "light," that is, $e^{-F_m(t_{\mu\nu})}\sim 1$ as $|\Lambda|\to\infty$, it is essential that the magnetic flux be able to spread. Specifically, the dominant fluctuations will satisfy $\eta[p]=\tau_{\mu\nu}{}^{-S^*_{\mu\nu}[p]}$ and thus exactly cancel the $Z(N)$ magnetic flux. If the flux can spread sufficiently rapidly so that the average action per unit length (or area) of $S^*_{\mu\nu}$ drops to zero, then $e^{-F_m(t_{\mu\nu})}$ will approach 1 as $|\Lambda|\to\infty$. Notice, however, that in order to produce area-law behavior for the electric-flux free energy (20), $e^{-F_m(t_{\mu\nu})}$ must approach 1 exponentially rapidly. In other words, the average action of a magnetic-flux sheet must decrease exponentially as the transverse area through which the flux can spread increases.

Finally, we would like to show that the behavior of the Wilson loop $\langle A^E[C]\rangle$ may also be understood as a consequence of the behavior of magnetic flux. This discussion closely follows the treatment by Mack and Petkova.[8]

Let us consider a coclosed set of plaquettes P^* which winds once about the Wilson loop C. Since $\hat{\partial}P^*=0$, there exists a set of links L^* such that $P^*=\hat{\partial}L^*$. Note that $L^*[C]=1$ since P^* winds around C. Let us insert $1=\int d\sigma$, where $\sigma\in Z(N)$ and then make the change of variables $U[b]\to\sigma^{L^*[b]}U[b]$. The Wilson loop becomes

$$\langle A^E[C]\rangle=\int d\sigma\,\sigma^e\left[\frac{1}{Z}\int d\nu(U)\chi^E(U[C])\times\exp\left(\sum_p \mathcal{L}(\sigma^{P^*[p]}U[\partial p])\right)\right]. \tag{31}$$

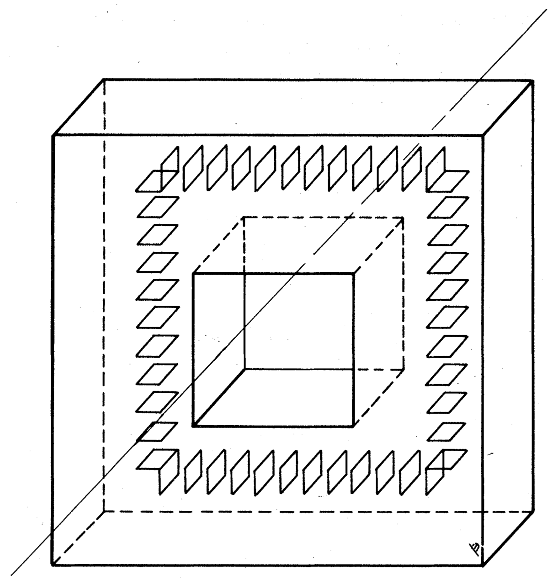

FIG. 2. A vortex container in three dimensions. Shown is a segment of the Wilson loop C threading the center of the vortex container Λ', inside of which is the thin vortex at P^*.

Here, the twist σ has been inserted in the action for each of the plaquettes of P^*. In other words, a thin vortex or loop of magnetic flux has been created at P^*. This sourceless loop of flux is topologically trivial and therefore may be removed (as it was inserted) by a mere change of variables. We are interested in effectively preventing this possibility and therefore we proceed as follows: Let us choose a vortex container, that is, some sublattice $\Lambda'\subset\Lambda$ which contains P^* and which wraps around the loop C (see Fig. 2). In other words C is contained in $\tilde{\Lambda}$ the closure of the complement of Λ'. We will first integrate over bond variables on bonds in the interior of Λ' and then integrate over the remaining outside variables. Thus

$$\begin{aligned}\langle A^E[C]\rangle&=\frac{1}{Z}\int\prod_{b\in\tilde{\Lambda}}dU[b]\chi^E(U[C])\prod_{p\in\tilde{\Lambda}}\exp\{\mathcal{L}(U[\partial p])\}\left(\int d\sigma\,\sigma^e\int\prod_{b\notin\tilde{\Lambda}}dU[b]\prod_{p\notin\tilde{\Lambda}}\exp\{\mathcal{L}(\sigma^{P^*[p]}U[\partial p])\}\right)\\&=\frac{1}{Z}\int d\nu(U)\chi^E(U[C]f_{\Lambda',U}(e)\exp\left\{\sum_{p\in\Lambda}\mathcal{L}(U[\partial p])\right\},\end{aligned} \tag{32}$$

where

$$f_{\Lambda',U}(e)\equiv\int d\sigma\,\sigma^e\left(\int\prod_{b\notin\tilde{\Lambda}}dU[b]\prod_{p\notin\tilde{\Lambda}}\exp\{\mathcal{L}(\sigma^{P^*[p]}U[\partial p])\}\Big/\int\prod_{b\notin\tilde{\Lambda}}dU[b]\prod_{p\notin\tilde{\Lambda}}\exp\{\mathcal{L}(U[\partial p])\}\right). \tag{33}$$

The subscripts indicate that $f_{\Lambda',U}(e)$ depends on the vortex container Λ' chosen and on the fixed values of the bond variables on the boundary of Λ'. However, $f_{\Lambda',U}(e)$ is independent of the location of the thin flux tube; P^* may be any coclosed set of plaquettes which winds once around the interior of Λ'. $f_{\Lambda',U}(e)$ is completely analogous to the electric-flux free energy $e^{-F_e(e_{\mu\nu})}$, except that instead of being defined on a cubic lattice with periodic boundary conditions it is defined on a nonsimply connected lattice Λ' with fixed boundary conditions.

The interpretation of (33) should, by now, be clear. We choose a vortex container Λ' and consider the change in free energy upon the addition of a thin tube of magnetic flux running through the container. If the free energy is nearly independent of the flux σ, then its Fourier transform $f_{\Lambda',U}(e)$ will be very small when $e\neq 0$ and will suppress the Wilson loop (32). As before, $Z(N)$ fluctuations are unable to accomplish this since they obey the identity $U[C]\eta[S]^{-1}=1$ for any surface S with boundary $\partial S = C$. SU(N) fluctuations, however, may violate this condition and are therefore able to spread the external flux throughout Λ'.

To obtain the most control over the behavior of the Wilson loop, we may introduce an arbitrary collection of nonintersecting vortex containers $\{\Lambda_i'\}$ and repeating the above steps find

$$\langle A^E[C]\rangle = \Big\langle \chi^E(U[C]) \prod_i f_{\Lambda_i',U}(e)\Big\rangle . \qquad (34)$$

From this, one may trivially derive the bound

$$|\langle A^E[C]\rangle| \leq \chi^E(1) \prod_i \left\{ \max_U |f_{\Lambda_i',U}(e)| \right\} . \qquad (35)$$

As noted by Mack and Petkova,[8] if $f_{\Lambda_i',U}(e)$ decreases exponentially with the transverse size of the vortex container Λ_i' for any set of boundary conditions U, then one may choose a set of vortex containers for which (35) yields the nearly linear bound $V^E(R) \geq (\text{const})R/(\ln R)^2$ on the heavy-quark potential. This behavior of $f_{\Lambda_i',U}(e)$ is characteristic of the presence of a mass gap.

We have seen how the behavior of any of our observables may be understood in terms of a single mechanism, the spreading of magnetic flux possible in an SU(N) theory. Unfortunately, the very features which make confinement possible also make standard weak-coupling approximations inapplicable. Consider, for example, the magnetic-flux free energy (30). One might try to compute its weak-coupling behavior by expanding about the minimal action configuration in the presence of the twists $\tau_{\mu\nu}$. This configuration may be easily found as follows. Choose any one parameter subgroup which interpolates between 1 and $\iota = e^{2\pi i/N}$. This defines a particular choice for ι^α, $-\frac{1}{2}N \leq \alpha \leq \frac{1}{2}N$, and therefore for $(\tau)^\beta$, $\tau \in Z(N)$, $0 \leq \beta \leq 1$. Consider now any particular plane surface $X_{\mu\nu} = \{p_{\mu\nu}(n) \,|\, n_\lambda = \text{const}, \lambda \neq \mu,\nu\}$ and begin with the configuration $U[b]=1$ for all bonds in $X_{\mu\nu}$. This configuration has a flux of $\tau_{\mu\nu}$ concentrated on the single plaquette $\bar{p} = S^*_{\mu\nu} \cap X_{\mu\nu}$. This flux may be spread out by the following procedure: Choose a bond in the boundary of $\bar{p}$ and set $U[b] = (\tau_{\mu\nu}^{-1})^{1-1/A_{\mu\nu}}$ on this bond. This reduces the flux on $\bar{p}$ to $(\tau_{\mu\nu})^{1/A_{\mu\nu}}$ and moves the remaining flux in $X_{\mu\nu}$ onto a plaquette next to $\bar{p}$. Pick another bond in the boundary of this plaquette and set $U[b] = (\tau_{\mu\nu}^{-1})^{1-2/A_{\mu\nu}}$. Proceeding in this manner one may move the remaining lump of flux completely through the surface $X_{\mu\nu}$ in such a way that every plaquette in $X_{\mu\nu}$ acquires the flux $(\tau_{\mu\nu})^{1/A_{\mu\nu}}$. This configuration is now replicated on every plane parallel to $X_{\mu\nu}$, and the same procedure is applied to each direction $[\mu\nu]$. We obtain a configuration (which satisfies the periodic boundary conditions) in which the external fluxes $(\tau_{\mu\nu})$ have been spread completely through the lattice. The action of this configuration (above the minimum) is given by

$$\sum_{[\mu\nu]} |\Lambda| \frac{1}{g^2} \operatorname{Re}\operatorname{tr}(1-\tau_{\mu\nu}^{1/A_{\mu\nu}}) \underset{|\Lambda|\to\infty}{\sim} \sum_{[\mu\nu]} (\text{const}) A^*_{\mu\nu}/A_{\mu\nu} .$$

This is naturally the same result as one would obtain in any free Abelian theory. If we attempt to compute the free energy by expanding around this configuration, then we will obtain the leading semiclassical approximation

$$e^{-F_m(t_{\mu\nu})} \underset{\text{semiclassical}}{\sim} \exp\Big(\sum_{\substack{[\mu\nu]\\ t_{\mu\nu}\neq 0}} -\alpha(t_{\mu\nu}) A^*_{\mu\nu}/A_{\mu\nu}\Big) .$$

This illustrates how the magnetic flux can spread and thereby lower the action; however, no indication of the development of a mass gap is seen. This is hardly surprising since the above approach is nothing more than the usual massless perturbative expansion. Notice that this result indicates that the average action per unit length ($d=3$) or area ($d=4$) of magnetic flux drops to zero as the transverse area increases. We will argue in a moment that the fluctuations which have been neglected simply further decrease the magnetic free energy. Consequently it seems clear that the SU(N) theory will not have a Higgs-type weak-coupling phase where magnetic flux possesses a finite action per unit length (or area). Therefore the bound (35) on the Wilson loop shows that the existence of a mass gap is actually sufficient to prove confinement.

The basic problem we are confronted with is the following: For weak coupling, the action becomes highly peaked about $U[\partial p] \approx 1$. In a $Z(N)$ theory the only configurations which satisfy this are in fact pure gauge configurations. Therefore semiclassical methods are perfectly adequate and correctly predict a Higgs phase.[10] However, in an SU(N) theory $U[\partial p] \approx 1$ does not restrict the configurations to be small fluctuations about pure gauge fields. A configuration may have arbitrarily small flux on every plaquette ($|U[\partial p]-1|\leq\delta$) without being a small deviation from a pure gauge field ($|U[b]-V[\partial b]|\nleq\epsilon$, for any $V[s]$) if the tiny amounts of flux add coherently over a sufficiently large distance. Such configurations can disorder arbitrarily large Wilson loops, spread magnetic flux, and invalidate the use of semiclassical methods for large distance scales.

Since we do not have sufficient control over these fluctuations we are unable to determine whether or not they provide a mass gap and thereby cause confinement. [Except in the exactly soluble two-dimensional theory. There semiclassical expansions predict, for example, the power-law behavior $e^{-F_m(t)}\sim 1+O(1/|\Lambda|)$ while the correct inclusion of all fluctuations yields $e^{-F_m(t)}=1+O(e^{-m|\Lambda|})$ and thus indicates the presence of a mass gap.] We may, however, proceed (somewhat circuitously) as follows.

We have argued that the important SU(N) fluctuations which may cause confinement are precisely those which violate the $Z(N)$ identities preventing the spread of magnetic flux. Therefore, if constraints are inserted in the SU(N) theory which enforce these identities, then we expect nonconfining behavior to result. This is what we will prove to be the case. (This may be thought of as justifying the qualitative arguments presented in this section.)

It is important to note that while all our observables may be thought of as probing the behavior of magnetic flux, each one does so in a slightly different and distinguishable manner. Specifically, the 't Hooft operator creates magnetic flux with a source, while the flux free energies examine the behavior of flux which runs completely through the lattice, and the Wilson loop probes the behavior of flux loops which circle the Wilson loop.

If we impose the constraint

$$\eta[\partial c]=1 \quad \text{for all cubes } c\,, \tag{36}$$

then this enforces the conservation of $Z(N)$ flux and eliminates all configurations which cancel the external $Z(N)$ flux created by the 't Hooft loop. This constraint implies $\eta[S]=1$ for all topologically trivial closed surfaces S. Notice, however, that this constraint does not affect the conserved flux loops probed by the flux free energies or by the Wilson loop. Consequently, we expect this constraint to change the behavior of the 't Hooft loop from the perimeter-law form (21) to the area law (25) appropriate for a Higgs phase. However, the behavior of the flux free energies and of the Wilson loop should be qualitatively unaffected.

If we then add the constraint

$$\eta[S_{\mu\nu}]=1 \tag{37}$$

for any particular set of closed surfaces $\{S_{\mu\nu}\}$ running through the lattice in each $[\mu\nu]$ direction, then all configurations with $Z(N)$ flux running around the lattice will be eliminated. Equations (36) and (37) together imply that the total $Z(N)$ flux flowing through any closed surface in the lattice is zero. The external flux introduced by the flux free energies will now be unable to spread out so that we would expect the nonconfining behavior (24) and (26). However, the behavior of the Wilson loop remains unchanged.

Lastly, (36) plus the new constraint

$$|\arg \operatorname{tr} U[C]\eta[S]^{-1}|\leq\pi/N\,, \tag{38}$$

for a particular loop $C=\partial S$ eliminate the spreading of flux vortices winding around the loop C. This is finally expected to destroy the confining behavior of the Wilson loop $A^E[C]$ and to produce instead the perimeter law (23). To truly create a nonconfining phase the constraint (38) would have to be introduced for all possible loops C.

This illustrates how the equivalence between our various confinement criteria may be selectively destroyed by inserting constraints which in effect couple directly to a particular observable. Notice how this mechanism exploits the loopholes mentioned in the general discussion in Sec. III.

In the next section we examine the SU(N) theory in the presence of the constraints (36) and (37) and prove that the 't Hooft loop and the magnetic-flux free energy exhibit the nonconfining behavior (25) and (26). We discuss the effect of the additional constraint (38); however, due to technical reasons (such as the need for a lower bound instead of an upper bound) we are unable to prove rigorously that the Wilson loop $\langle A^E[C]\rangle$ exhibits the nonconfining behavior (23).

To close this section we would like to explain one final point. One may wonder why we have placed so much emphasis on $Z(N)$ magnetic flux instead of considering general SU(N) flux. The basic answer is simply that it is not necessary to do so. $Z(N)$ flux does not have a particularly distinguished position in the SU(N) theory; we could, for example, define a generalization of the 't Hooft operator which would insert an arbitrary SU(N) twist. However, it is much more convenient to deal with $Z(N)$ variables since they commute with all elements of the group.

A more deductive argument for emphasizing $Z(N)$ flux may also be given. Consider evaluating the expectation of the Wilson loop by first inserting $1=\int dg\,\delta(gU[C])$ and then integrating out the bond variables. This yields

$$\langle A^E[C]\rangle = \int dg\, \mu(g)\chi^E(g^{-1}),$$

where $\mu(g)$ is the probability that $U[C]=g^{-1}$. μ has a character expansion in terms of representations of SU(N), $\mu = \sum_{\text{rep}E'} c^{E'} X^{E'}(g)$. In order for the Wilson loop to show area-law behavior, we must have $c^E \sim O(e^{-(\text{area})})$. However, we expect all representations of nonzero n-ality to be confined. This implies

$$\mu(\gamma g) = \mu(g) + O(e^{-(\text{area})}), \quad \gamma \in Z(N).$$

However, by the same change of variables used in (31) we see that $\mu(\gamma g)/\mu(g)$ simply compares the Wilson loop with and without an extra thin $Z(N)$ flux vortex circling the loop. Thus we are automatically led to the introduction of $Z(N)$ flux in exactly the manner described previously.

V. CONSTRAINED THEORY

The behavior of magnetic flux will now be studied in a modified theory in which the constraints (36) and (37) are imposed. To implement these constraints we simply change from the bare measure $d\nu(U) = \Pi_b dU[b]$ to the constrained measure

$$d\nu'(U) = \prod_{b\in\Lambda} dU[b] \prod_{c\in\Lambda} \delta(\eta[\partial c]) \prod_{[\mu\nu]} \delta(\eta[S_{\mu\nu}])$$
$$(\delta(\eta) \equiv \{1, \eta = 1; 0, \eta \neq 1\}). \quad (39)$$

We will use the following strategy to prove that the magnetic-flux free energy and the 't Hooft loop exhibit area-law behavior (25) and (26) in this constrained theory. First we introduce a weak-coupling cluster expansion based on the decomposition

$$e^{\mathcal{L}(U)} = e^{\mathcal{L}(U)}[\theta(\pi/N - |\arg \operatorname{tr} U|) + \theta(|\arg \operatorname{tr} U| - \pi/N)]. \quad (40)$$

Thus, the contribution of U's which are not in the $Z(N)$ sector about 1 will be treated as a small perturbation. This will clearly generate a reasonable expansion as $\beta \to \infty$ since $\exp[\mathcal{L}(U)]$ becomes highly peaked about $U=1$.

We next resum the disconnected portion of the cluster expansion and then apply the chessboard estimates described in Appendix A. This automatically extracts the volume dependence in a uniform manner so that each term of the expansion may then be bounded in a very simple fashion. Next we bound the number of terms of a given order in the cluster expansion. This results in a proof that the cluster expansion converges for sufficiently weak coupling. Finally, a determination of the first nonzero term in the expansion (which is actually a by-product of the first steps) yields a bound proving the expected magnetic confinement.

We will first detail the treatment of $F_m(t_{\mu\nu})$ and afterwards sketch the equivalent analysis of $\langle B^\tau[C^*]\rangle$. The magnetic-flux free energy for the constrained model is given by

$$e^{-F_m(t_{\mu\nu})} = Z^{-1} \int d\nu'(U) \exp\left(\sum_p \mathcal{L}(U[\partial p]\tau_{\mu\nu}{}^{S^*_{\mu\nu}[p]})\right),$$
$$Z = \int d\nu'(U) \exp\left(\sum_p \mathcal{L}(U[\partial p])\right). \quad (41)$$

Inserting the decomposition (40) produces

$$e^{-F_m(t_{\mu\nu})} = \sum_{Q\subset\Lambda} Z^{-1} \int d\nu'(U) \exp\left(\sum_p \mathcal{L}(U[\partial p]\tau_{\mu\nu}{}^{S^*_{\mu\nu}[p]})\right)$$
$$\times \prod_{p\notin Q} \delta(n[p] + t_{\mu\nu}S^*_{\mu\nu}[p])$$
$$\times \prod_{p\in Q} \{1 - \delta(n[p] + t_{\mu\nu}S^*_{\mu\nu}[p])\}, \quad (42)$$

where Q is any set of plaquettes in the lattice. Each set Q consists of a number of connected components, where two plaquettes p and p' are defined to be connected if $\hat{\partial}p \cap \hat{\partial}p' \neq 0$, i.e., they are in the boundary of a single cube. A connected component is topologically nontrivial if it contains any coclosed surface $S^*_{\alpha\beta}$ which winds through each $[\alpha\beta]$ plane. Thus we may decompose Q as $\bar{Q} \cup Q'$, where $\bar{Q}$ is the union of all topologically nontrivial components and Q' is the topologically trivial remainder. Let $I^*_{\alpha\beta}$ be the number of distinct nonoverlapping surfaces $S^*_{\alpha\beta}$ contained in $\bar{Q}$.

We will first show that if $t_{\alpha\beta} \neq 0$ and $I^*_{\alpha\beta} = 0$ then the contribution vanishes. If Q contains no coclosed surface $S^*_{\alpha\beta}$, then a closed surface $S_{\alpha\beta}$ always exists which winds about the lattice but does not contain any plaquette of Q. The δ functions of (42) require $n[p] = -t_{\mu\nu}S^*_{\mu\nu}[p]$ on this surface. Therefore $n[S_{\alpha\beta}] = -t_{\mu\nu}S^*_{\mu\nu}[S_{\alpha\beta}] = -t_{\alpha\beta} \neq 0$ and so the constraints in $d\nu'(U)$ are violated. Consequently, if $t_{\mu\nu} \neq 0$ we may always, by a change of variables, deform the particular surface $S^*_{\mu\nu}$ in (42) so that it lies within $\bar{Q}$.

We may now resum over all topologically trivial sets Q' for a fixed $\bar{Q}$. Note that

$$\sum_{Q'} \prod_{p\notin Q} \delta(n[p]) \prod_{p\in Q'} \{1 - \delta(n[p])\}$$
$$\leq \prod_{p\notin\bar{Q}} (\delta(n[p]) + \{1 - \delta(n[p])\}) = 1$$

since the right-hand side, when expanded in products δ's and $(1-\delta)$'s, contains the left-hand side plus contributions from components which connect to $\bar{Q}$ or are topologically nontrivial. Therefore,

$$e^{-F_m(t_{\mu\nu})}=\sum_{\overline{Q}}\hat{I}_{\overline{Q}}\,,$$

where

$$\hat{I}_{\overline{Q}}\le Z^{-1}\int d\nu'(U)\exp\Big(\sum_p \mathcal{L}(U[\partial p]\tau_{\mu\nu}{}^{S^*_{\mu\nu}[p]})\Big)\times\prod_{p\in\overline{Q}}\{1-\delta(n[p]+t_{\mu\nu}S^*_{\mu\nu}[p])\}$$
$$=\Big\langle\prod_{p\in\overline{Q}}\hat{g}_p(U[\partial p])\Big\rangle \tag{43}$$

with

$$\hat{g}_{p'}(U[\partial p])\equiv\{1-\delta(n[p]+t_{\mu\nu}S^*_{\mu\nu}[p'])\}\times\exp\{\mathcal{L}(U[\partial p]\tau_{\mu\nu}{}^{S^*_{\mu\nu}[p']})-\mathcal{L}(U[\partial p])\}.$$

For later convenience define $\hat{g}_{p'}(U[\partial p])=1$ if $p'\notin\overline{Q}$. Note that

$$e^{\mathcal{L}(U)}\hat{g}_p(U)\le\{e^{\beta N\cos(\pi/N)}\text{ if }p\in\overline{Q}\,,\quad e^{\beta N}\text{ if }p\notin\overline{Q}\}. \tag{44}$$

To bound this expectation uniformly in $|\Lambda|$ we may use the chessboard estimates discussed in Appendix A. Using (A6) and (A7) we find

$$\Big\langle\prod_{p\in\overline{Q}}\hat{g}_p(U[\partial p])\Big\rangle\le\prod_{s^*}\Big\langle\prod_p\hat{g}_{p'(s^*)}(U[\partial p])\Big\rangle^{1/|\Lambda|}. \tag{45}$$

Here s^* runs over all dual sites (i.e., d-cells) which contain at least one plaquette in $\overline{Q}$. $p'(s^*)$ is that plaquette in s^* which may be obtained from p by reflections in planes of the lattice. Let δ be the number of plaquettes in s^* contained in $\overline{Q}$; $1\le\delta\le\Gamma_{2,d}$, where $\Gamma_{r,s}=2^{(s-r)}\binom{s}{r}$ is the number of r-cells contained in a single s-cell, $r\le s$. Now

$$\Big\langle\prod_p\hat{g}_{p'(s^*)}(U[\partial p])\Big\rangle=Z^{-1}\int d\nu'(U)\prod_p e^{\mathcal{L}(U[\partial p])}\times\hat{g}_{p(s^*)}(U[\partial p])$$
$$\le Z^{-1}[e^{\beta N|P|}e^{\beta N[\cos(\pi/N)-1]\alpha|P|}], \tag{46}$$

where $\alpha=\delta/\Gamma_{2,d}$ is the fraction of plaquettes of s^* contained in $\overline{Q}$; $|P|\equiv\sum_{p\in\Lambda}1$. Here we simply bounded the integrand by its maximum using (44). To bound the partition function

$$Z=\int d\nu'(U)\prod_p\exp\{\mathcal{L}(U[\partial p])\}$$

from below, we may restrict each integration $\int dU[b]$ to a sufficiently small region about $U=1$ so that $|\mathrm{tr}U[\partial p]|\ge Ne^{-\theta}$ and $|\arg\mathrm{tr}U[\partial p]|\le\pi/N$. Let τ_θ be the volume of this region. Since the constraints in $d\nu'(U)$ are now automatically satisfied, we find the bound

$$Z\ge\tau_\theta^{|L|}e^{\beta Ne^{-\theta}|P|}. \tag{47}$$

$|L|\equiv\sum_{b\in\Lambda}1$. Thus combining (45)–(47) we find

$$\Big\langle\prod_{p\in\overline{Q}}\hat{g}_p(U[\partial p])\Big\rangle\le\prod_{s^*,\delta>0}[\tau_\theta^{-d}\exp(\beta N\tfrac{1}{2}d(d-1)\{(1-e^{-\theta})+\alpha[\cos(\pi/N)-1]\})]$$
$$=\{\tau_\theta^{-d}\exp[\beta N\tfrac{1}{2}d(d-1)(1-e^{-\theta})]\}^{|\overline{Q}^*|}(\exp\{\beta N\tfrac{1}{2}d(d-1)[\cos(\pi/N)-1]\})^{|\overline{Q}|\Gamma_{d,2}/\Gamma_{2,d}},$$

where $|\overline{Q}^*|$ is the number of dual sites which contain plaquettes of $\overline{Q}$, and $\Gamma_{s,r}=2^{(s-r)}\binom{d-r}{d-s}$ is the number of s-cells which contain a given r-cell, $s\ge r$. Since $|\overline{Q}^*|\le|\overline{Q}|\Gamma_{d,2}$, we find

$$\hat{I}_{\overline{Q}}\le\hat{c}(\beta)^{|\overline{Q}|},$$

where

$$\hat{c}(\beta)=\min_\theta\exp\{-\beta N[1-\cos(\pi/N)-(1-e^{-\theta})\Gamma_{2,d}]\}/(\sqrt{\tau_\theta})^{\Gamma_{1,d}} \tag{48}$$

and $\hat{c}(\beta)\to 0$ as $\beta\to\infty$.

We must now sum over all sets $\overline{Q}$. If $\overline{Q}$ contains $I^*_{\mu\nu}$ surfaces $S^*_{\mu\nu}$, then $\overline{Q}$ must contain at least $I^*_{\mu\nu}$ plaquettes of any closed surface $S_{\mu\nu}$. Let $\{X_{\mu\nu}\}$ be a standard set of such closed surfaces of minimal size $A_{\mu\nu}$. We will first sum over all sets $\overline{Q}$ which contain a specific choice of plaquettes $x^i_{\mu\nu}\in X_{\mu\nu}$, $i=1,\ldots,I^*_{\mu\nu}$. Each set $\overline{Q}$ contains some number J of connected components, $1\le J\le I^*\equiv\sum_{[\mu\nu]}I^*_{\mu\nu}$. Each component $\overline{Q}_j$ contains a particular subset $\{{}^jx^i_{\mu\nu},\ i=1,\ldots,I^j_{\mu\nu}\}$ of the plaquettes $\{x^i_{\mu\nu}\}$. $I^j_{\mu\nu}$ is some partition of $I^*_{\mu\nu}$, $\sum_j I^j_{\mu\nu}=I^*_{\mu\nu}$. We show in Appendix B that the number of connected surfaces of size q which contain a given plaquette is bounded by $\hat{b}^q$, where $\hat{b}=[10(d-2)]^{10(d-2)}$. Thus the contribution of surfaces $\overline{Q}$ with J components each of which contains the plaquettes $\{{}^jx^i_{\mu\nu}\}$ is bounded by

$$\prod_{j=1}^{J}\Big(\sum_{q_j=I^j_{\mu\nu}A^*_{\mu\nu}}^{\infty}(\hat{c}(\beta)\hat{b})^{q_j}\Big)\le(1-\hat{c}(\beta)\hat{b})^{-I}\times(\hat{c}(\beta)\hat{b})^{I^*_{\mu\nu}A^*_{\mu\nu}}$$

since $I^j_{\mu\nu}A^*_{\mu\nu}$ is the minimal size of the component j. This holds for β sufficiently large so that $\hat{c}(\beta)\hat{b}<1$. We must now multiply by the number of ways of partitioning $\{x^i_{\mu\nu}\}$ into any number of connected components. We show in Appendix B that this number is equal to $\chi_{I^*} \equiv (1/e)\sum_{m=0}^{\infty} m^{I^*}/m!$. Then we must multiply by the $(A_{\mu\nu})^{I^*_{\mu\nu}}/I^*_{\mu\nu}!$ different choices for $\{x^i_{\mu\nu}\}$, and finally we sum over all possible $I^*_{\mu\nu}$. We find

$$e^{-F_m(t_{\mu\nu})} \leq \sum_{m=0}^{\infty} \frac{1}{em!} \prod_{[\mu\nu]} \left(\sum_{I^*_{\mu\nu} = j_{\mu\nu}}^{\infty} \frac{1}{I^*_{\mu\nu}!} (m\hat{a}A_{\mu\nu})^{I^*_{\mu\nu}} \times e^{-\hat{\rho} I^*_{\mu\nu} A^*_{\mu\nu}} \right)$$

$$= \sum_{m=0}^{\infty} \frac{1}{em!} \prod_{[\mu\nu]} [\exp(m\hat{a}A_{\mu\nu}e^{-\hat{\rho}A^*_{\mu\nu}}) - j_{\mu\nu}],$$

where

$$e^{-\hat{\rho}} \equiv \hat{c}(\beta)\hat{b}, \quad \hat{a} = (1-\hat{c}(\beta)\hat{b})^{-1}, \text{ and } j_{\mu\nu} = 1 - \delta_{t_{\mu\nu},0}.$$

So finally

$$e^{-F_m(t_{\mu\nu})} \leq \sum_{k_{\alpha\beta}=0,1} \exp\left[\exp\left(\sum_{[\mu\nu]} \hat{a}k_{\mu\nu}A_{\mu\nu}e^{-\hat{\rho}A^*_{\mu\nu}}\right) - 1\right] \times \prod_{[\alpha\beta]} [k_{\alpha\beta} - j_{\alpha\beta}(1-k_{\alpha\beta})]. \tag{49}$$

This expression reduces to the area law (26) as $|\Lambda| \to \infty$ in any power-law fashion. Note that the second exponentiation was essentially a consequence of the interference of flux sheets in different directions.

Let us briefly sketch the steps needed to bound the 't Hooft loop $\langle B^\tau[C^*]\rangle$, Eq. (27), in an analogous fashion. One may begin with the basic expansion (40) and now separate from Q the connected component Q^c (if it exists) which intersects the boundary of an arbitrary single cube of C^*. One must show that the contribution vanishes if Q^c does not contain a surface S^* whose coboundary is C^*. This is a consequence of the fact that if Q contains no surface S^* with $\hat{\partial}S^* = C^*$ then a closed surface S always exists which winds around C^* but does not intersect Q. [Winds about means $S^*(S) = 1$, where S^* is the particular set of plaquettes which defines the 't Hooft loop.] On this surface $n[p] = -tS^*[p]$, so that $n[S] = -t \neq 0$, which violates the constraints in $d\nu'(U)$.

Therefore the surface S^* which defines the 't Hooft loop may always be chosen to lie in Q^c. One may then resum Q' and bound the contribution of a given set Q^c in exactly the manner as above. Summing over all connected sets Q^c of a given size q, one finds that their contribution is bounded by $6(\hat{c}(\beta)\hat{b})^q$. Finally summing over q starting from the minimal size A^*, one finds the bound

$$|\langle B^\tau[C^*]\rangle| \leq \alpha e^{-\hat{\rho}A^*}, \tag{50}$$

where $\alpha = 6/(1-\hat{c}(\beta)\hat{b})$, $e^{-\hat{\rho}} = \hat{c}(\beta)\hat{b}$. This proves the area law (25) for sufficiently weak coupling. We see that the magnetic-flux free energy and the 't Hooft loop exhibit the expected weak-coupling behavior in this constrained model.

Finally, we would like to discuss the behavior of the Wilson loop in the presence of the constraint

$$|\arg \operatorname{tr}(U[C]\eta[S]^{-1})| \leq \pi/N. \tag{51}$$

This constraint is obviously rather unnatural since it explicitly refers to the particular Wilson loop we are considering. If it is included in the measure, then the constraint destroys translation invariance and consequently we lose reflection positivity (see Appendix A). However, if the new constraint is only inserted in the numerator of the expectation, then the previous bounds on the cluster expansion are valid and imply that the first nonzero term of the expansion dominates for sufficiently weak coupling. Therefore consider the leading term for which $\eta[p] = 1$ on all plaquettes. Owing to the constraint (51), only configurations with $|\arg \operatorname{tr}(U[C])| \leq \pi/N$ will contribute to the Wilson loop. In other words $U[C]$ is restricted to lie in the $Z(N)$ sector about 1 and consequently it is clear that the Wilson loop will be highly ordered. However, in order to prove that the Wilson loop exhibits nonconfining behavior we would need to find a lower bound on this term. We are unable to do this rigorously with the present methods.

VI. CONCLUDING REMARKS

We have seen how confinement may result from the spreading of magnetic flux which is possible in an SU(N) theory. However, in order to prove confinement one must be able to show that the average action of magnetic flux decreases exponentially as the transverse area through which the flux may spread increases. This requires far greater control over the energetics of these SU(N) fluctuations than we currently possess. It is the case that nearly all of the analytical methods used here and elsewhere to study SU(N) theories have been based on exploiting the effects of the easily controlled $Z(N)$ part of the dynamics. It appears that these methods have reached the limit of their utility. The Abelian $Z(N)$ dynamics is definitely insufficient to produce the expected weak-coupling behavior. In the future it seems clear that inherently non-Abelian features of the dynamics, such as flux spreading and thick vortices, should be the focus of attention.

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APPENDIX A: REFLECTION POSITIVITY AND CHESSBOARD ESTIMATES

Reflection positivity (Osterwalder-Schrader positivity, physical positivity, etc.) is a fundamental property of Euclidean lattice theories.[3,14] It guarantees the existence of a positive-metric Hilbert space and allows one to construct the transfer matrix. We will use it in the following form.

Consider a reflection θ in a plane containing sites of the periodic lattice. Specifically, let us position the sites $s(n)$ at $x_\mu = n_\mu$, $-2^{\alpha_\mu -1} < n_\mu \leq 2^{\alpha_\mu -1}$ (mod 2^{α_μ}) and reflect in the $(d-1)$-dimensional hyperplane(s) Λ_0 given by $x_0 = 0$ or $x_0 = 2^{\alpha_0 -1}$: $\theta x = (-x_0, x_1, \ldots, x_{d-1})$. (Since the lattice is periodic any reflection has two fixed planes opposite each other.) If $F(U[b])$ is any function of bond variables, then

$$\theta F(U[b]) \equiv F^*(\theta U[b]), \tag{A1}$$

where $\theta U[b_{x,y}] = U[b_{\theta x, \theta y}]$ and $b_{x,y}$ is the bond with boundary $s(x) - s(y)$. Partition the lattice into the three regions $\Lambda_0 = \{x_\mu \mid x_0 = 0 \text{ or } x_0 = 2^{\alpha_0 -1}\}$, $\Lambda_\pm = \{x_\mu \mid 0 < \pm x_0 < 2^{\alpha_0 -1}\}$. Let $U_\theta^\pm$ be the space of functions $F(U[b])$ which depend only on bonds in $\Lambda_0 \cup \Lambda_\pm$. Reflection positivity is the statement that

$$\langle (\theta F) F \rangle \geq 0 \quad \text{for all } F \in U_\theta^+ . \tag{A2}$$

Let us show that this is true for the lattice gauge theory (3):

$$\begin{aligned}
\langle (\theta F)F\rangle &= Z^{-1}\int \prod_b dU[b] \exp\Big(\sum_p \mathcal{L}(U[\partial p])\Big) F^*(\theta U[b]) F(U[b]) \\
&= Z^{-1}\int \prod_{b\in\Lambda_0} dU[b] \prod_{p\in\Lambda_0} e^{\mathcal{L}(U)} \Big(\prod_{b\in\Lambda_+} dU[b] \prod_{p\in\Lambda_+} e^{\mathcal{L}(U)} F(U[b\in\Lambda_0], U[b\in\Lambda_+])\Big) \\
&\quad\times \Big(\prod_{b\in\Lambda_-} dU[b] \prod_{p\in\Lambda_-} e^{\mathcal{L}(U)} F^*(U[b\in\Lambda_0], \theta U[b\in\Lambda_+])\Big) \\
&= Z^{-1}\int \prod_{b\in\Lambda_0} dU[b] \prod_{p\in\Lambda_0} e^{\mathcal{L}(U)} \Big| \prod_{b\in\Lambda_+} dU[b] \prod_{p\in\Lambda_+} e^{\mathcal{L}(U)} F(U[b\in\Lambda_0], U[b\in\Lambda_+])\Big|^2 \geq 0 .
\end{aligned} \tag{A3}$$

Reflection positivity may be proven for the modified model (40) in exactly the same manner. (The constraint on cubes is local and causes no difficulties. The spacelike surfaces S_{ij}, $0<i<j$ may be chosen to lie in Λ_0 so that $\eta[S_{ij}]$ is reflection invariant. The timelike surfaces S_{0j} may be chosen so that $\theta(S_{0j}\cap\Lambda_+) = -(S_{0j}\cap\Lambda_-)$. (Note that θ reverses the orientation of timelike plaquettes.) The timelike constraints may then be represented as

$$\frac{1}{2\pi}\int_0^{2\pi} d\lambda_j \exp(i\lambda_j\{n[(S_{0j}\cap\Lambda_+)] + n[(S_{0j}\cap\Lambda_-)]\})$$

and one easily sees that these nonlocal constraints do not destroy the reflection positivity.)

Since the periodic lattice has translational and cubic symmetry, reflection positivity holds for reflections in any cubic plane containing sites. (It also holds for reflections between sites,[3] but we will not need to use these.)

The positivity (A2) enables one to use the Schwarz inequality

$$|\langle(\theta F)G\rangle| \leq \langle(\theta F)F\rangle^{1/2}\langle(\theta G)G\rangle^{1/2}, \quad \text{if } F, G \in U_\theta^+ . \tag{A4}$$

This allows one to prove the following chessboard estimates[15]: Let

$$A = \prod_{s^*\in\Lambda} f_{s^*}(U[b(s^*)]), \tag{A5}$$

where s^* is a dual site, that is a d-cell of the lattice, and $b(s^*)$ is any bond contained in s^*. f_{s^*} is an arbitrary function of the variables in a single d-cell. We will show that

$$|\langle A\rangle| \leq \prod_{s^{*\prime}} \Big\langle \prod_{s^*\in\Lambda} T[s^*] f_{s^{*\prime}}(U[b(s^*)])\Big\rangle^{1/|\Lambda|}, \tag{A6}$$

where $T[s^*(n)] = \prod_\mu (I_\mu)^{n_\mu}$ and I_μ is an operator which complex conjugates f and reflects the interior of s^* through the plane normal to $\hat{e}_\mu$. The remarkable utility of chessboard estimates derives from the fact that they allow one to prove bounds on local expectations which are uniform in the lattice volume $|\Lambda|$.

Any product of the form (A5) may clearly be written as $(\theta F)G$ with $F, G \in U_\theta^+$ for any reflection θ. To prove (A6) one simply uses (A4) repeatedly for different choices of θ. Concentrate for the moment on a particular factor f_{s^*} in the product (A5). If we reflect in a hyperplane which intersects the boundary of s^*, then in the RHS of (A4) we find two new expectations each of the form (A5). One of these does not contain f_{s^*} while the other contains f_{s^*} twice for the adjoining sites s^* and θs^*. We may now reflect in a plane through the boundary of this two-site complex. Continuing in this manner one may reflect f_{s^*} onto every dual site of the lattice (since $A_\mu = 2^{\alpha_\mu}$). Repeating this process for each factor of the original product [Eq. (A5)] one arrives at the desired bound (A6). [In fact (A6) is valid for any lattice of size $2l_0 \times \cdots \times 2l_{d-1}$.[15]]

Finally, note that if $B = \prod_{c_r \in \Lambda} g_{c_r}(U[b(c_r)])$ for any size r-cell ($r \geq 1$), then we may define

$$f_{s^*}(U[b(s^*)]) = \prod_{c_r \subset s^*} g_{c_r}(U[b(c_r)])^{1/\Gamma_{d,r}}, \qquad \text{(A7)}$$

where $\Gamma_{d,r} = 2^{(d-r)}$ is the number of d-cells containing a given r-cell, and then use (A6).

APPENDIX B: CONNECTED SURFACES

In this appendix we wish to bound the number of connected surfaces of a given size. The result is essentially a corollary of the Königsberg bridge problem.[16] For an arbitrary connected set Q of size q consider each plaquette $p \in Q$ as an "island." Define two "bridges" between every pair of connected plaquettes in Q. The number of bridges is bounded by $\gamma \cdot q$ where, if connectedness is defined by $\partial p \cap \partial p' \neq 0$, then $\gamma = 4(2d-3)$, while if connectedness is defined by $\hat{\partial} p \cap \hat{\partial} p' \neq 0$, then $\gamma = 10(d-2)$. By the solution to the Königsberg bridge problem (or otherwise) one sees that starting on any island a path always exists which crosses each bridge exactly once. Therefore $N(q)$, the number of connected surfaces of size q containing a given plaquette, is bounded by the number of paths on the lattice consisting of $\gamma \cdot q$ steps. A step moves between two connected plaquettes. Since the number of choices for the path at each step is once again γ, we find the bound

$$N(q) \leq \gamma^{\gamma q}, \qquad \text{(B1)}$$

where

$$\gamma = \begin{cases} 4(2d-3) & \text{if } dp \cap \partial p' \neq 0 \\ 10(d-2) & \text{if } \hat{\partial} p \cap \hat{\partial} p' \neq 0 \,. \end{cases}$$

Lastly, we need to count the number of ways in which any number of connected components of $\overline{Q}$ may attach to a given set of plaquettes $\{x^i_{\mu\nu}\}$. This problem is equivalent to the number of partitions of I distinguishable objects (the plaquettes) into any number of indistinguishable boxes. Let n_k be the number of boxes which will contain k objects; $\sum_{k=1}^\infty n_k \cdot k = I$. The number of partitions yielding a given sequence $\{n_k\}$, $\Gamma(\{n_k\})$ is easily found to equal $I! \prod_{k=1}^\infty 1/n_k!\,(k!)^{n_k}$. Therefore the total number of partitions χ_I is given by

$$\chi_I = \sum_{\{n_k\}} \delta\Big(\sum_k n_k k - I\Big)\Gamma(\{n_k\})$$

$$= I! \int \frac{d\theta}{2\pi} e^{-i\theta I} \prod_{k=1}^\infty \Big(\sum_{n_k=0}^\infty (e^{ik\theta}/k!)^{n_k}/n_k!\Big)$$

$$= \frac{1}{e}\sum_{j=0}^\infty j^I/j!\,. \qquad \text{(B2)}$$

APPENDIX C. STRONG COUPLING

In this appendix we wish to consider the strong-coupling ($\beta \to 0$) limit. Clearly, our observables may still be considered as probing the behavior of magnetic flux; however, in this limit magnetic flux need not spread out in order to have low energy. As $\beta \to 0$, the action $\exp[\mathcal{L}(U)]$ becomes nearly independent of the flux U, so that even thin flux tubes cost little action. In order to show that the magnetic free energy actually decreases exponentially with the transverse area in this "total chaos" limit, one may use the well-known strong-coupling expansion.[1,2] Here one expands the action in powers of β (or rather uses its character expansion) and sums over all possibilities for the set of plaquettes Q on which the $O(\beta)$ (or nontrivial representation) terms are kept. Q must form a collection of closed surfaces in order for the contribution to be nonvanishing.[2] Only if a surface wraps around the whole lattice will the contribution depend on the twist $\tau_{\mu\nu}$. All other terms cancel against identical terms in the expansion of the partition function Z. Consequently, assuming the expansion converges, one finds $e^{-F_m(\tau_{\mu\nu})} = 1 + O(e^{-(\text{area})})$. [The identical result holds for the vortex free energy (33). For the 't Hooft loop, only surfaces which wind around the source C^* depend on the twist τ and this yields the perimeter behavior (21).] This shows how confinement occurs as consequence of the "light" behavior of magnetic flux; obviously one could instead directly examine the behavior of the Wilson loop. In this case one finds that Q must contain a surface S which spans the loop and thus provided the expansion converges one immediately finds the confining area law (19).

Osterwalder and Seiler[3] originally proved the convergence of the strong-coupling expansion and the area-law behavior for the Wilson loop for sufficiently strong coupling. We would like to show that the use of chessboard estimates provides a very simple proof which may be easily extended to nonlocal observables such as the flux free energy. Since the strategy is identical to that used in Sec. V we will be rather brief in presenting the algebraic steps. All of the following expressions will be valid for either the standard model or the constrained model. This shows that non-Abelian flux spreading does not significantly affect the strong-coupling dynamics. This is to be expected since it is known that if the pure $Z(N)$ gauge theory confines (as it does for strong coupling) then any SU(N) theory at the same coupling will also confine.[7,17] In other words, for sufficiently strong coupling, the $Z(N)$ part of the theory alone is sufficient to produce confinement.

Consider, for example, the electric-flux free energy. The strong-coupling cluster expansion is based on writing

$$\exp\{\beta[\mathrm{Re\,tr}(U)+N]\}=1+f(U) \tag{C1}$$

and expanding in products of f's. [$f(U)$ tends to zero as $\beta\to 0$; it is convenient to include the factor $e^{\beta N}$ so that f is positive.] Inserting this decomposition produces a sum over all sets Q. Once again an arbitrary set Q may be decomposed into connected components, where two plaquettes, p and p', are now defined to be connected if $\partial p\cap\partial p'\neq 0$, i.e., they share a bond. A connected component is topologically nontrivial if it contains a closed surface $S_{\mu\nu}$ which winds about the lattice in the $[\mu\nu]$ direction. Decompose Q as $Q=\overline{Q}\cup Q'$, where $\overline{Q}$ is the union of all topologically nontrivial components. Let $I_{\alpha\beta}$ be the number of distinct, nonoverlapping surfaces $S_{\alpha\beta}$ contained in $\overline{Q}$. For each set Q, one may deform $S^*_{\mu\nu}$ (in the region outside $\overline{Q}$) so that $S^*_{\mu\nu}$ does not intersect Q'. Similarily, if $e_{\mu\nu}\neq 0$ but $I_{\mu\nu}=0$, then the contribution is zero since the integral is independent of the twist and therefore the Fourier transform vanishes.

We may now resum over Q' and, using $f\geq 0$, find

$$e^{-F_e(e_{\mu\nu})}=\sum_{\overline{Q}} I_{\overline{Q}}\,,$$

where

$$I_{\overline{Q}}\leq\int d\tau_{\mu\nu}\left\langle\prod_{p\in\overline{Q}} g_p(U[\partial p])\right\rangle, \tag{C2}$$

with

$$g_{p'}(U[\partial p])=f(U[\partial p]\tau_{\mu\nu}{}^{S^*_{\mu\nu}[p']})/1+f(U[\partial p])\,.$$

Applying the chessboard estimates of Appendix A, one finds

$$\left\langle\prod_{p\in\overline{Q}} g_p(U[\partial p])\right\rangle\leq\prod_{s^*}\left\langle\prod_{p} g_{p'(s^*)}(U[\partial p])\right\rangle^{1/|\Lambda|}$$

This may be immediately bounded in the same fashion as before yielding

$$I_{\overline{Q}}\leq c(\beta)^{|\overline{Q}|}\,,$$

where

$$c(\beta)=\min_{\theta}\,(1-e^{-2\beta N})e^{\beta N(1-e^{-\theta})\Gamma_{2,d}}/(\sqrt{\tau_\theta})^{\Gamma_{1,d}} \tag{C3}$$

and $c(\beta)\sim\beta$ as $\beta\to 0$.

Summing over sets $\overline{Q}$ may be performed as in Sec. V. The only modifications are that $\overline{Q}$ must contain $I_{\mu\nu}$ plaquettes in each coclosed surface $S^*_{\mu\nu}$, whose minimal area is $A^*_{\mu\nu}$. Also the number of connected surfaces of sizes q containing a given plaquette is, with the current definition of connectedness, bounded by b^q, where $b=[4(2d-3)]^{4(2d-3)}$ (see Appendix B). Thus, following the previous discussion, one easily finds

$$e^{-F_e(e_{\mu\nu})}\leq\sum_{k_{\alpha\beta}=0,1}\exp\left[\exp\left(\sum_{[\mu\nu]} ak_{\mu\nu}A^*_{\mu\nu}e^{-\rho A_{\mu\nu}}\right)-1\right]\times\prod_{[\alpha\beta]}[k_{\alpha\beta}-l_{\alpha\beta}(1-k_{\alpha\beta})]\,, \tag{C4}$$

where now $e^{-\rho}=c(\beta)b$, $a=(1-c(\beta)b)^{-1}$, and $l_{\alpha\beta}=1-\delta_{e_{\alpha\beta},0}$. This reduces to (20) as $|\Lambda|\to\infty$ in any power-law fashion.

To bound the Wilson loop $\langle A^E[C]\rangle$ using the same method one begins with the expansion (C1) and separates from Q the connected component Q^c (if it exists) which intersects the coboundary of an arbitrary link of C. One extracts the loop itself by the bound $|\chi^E(U)|\leq\chi^E(1)$. Then, resumming the remainder of Q, bounding the contribution of a given Q^c, and summing over all connected surfaces Q^c of a given size q^c, all using exactly the same methods as above, easily yields

$$\langle A^E[C]\rangle=\sum_{q^c=0}^{\infty} a(q^c)\,,$$

$$|a(q^c)|\leq 2(d-1)\chi^E(1)(c(\beta)b)^{q^c}\,.$$

To show that $a(q^c)=0$ if q^c is less than the minimal area A of a surface spanning C, one may proceed as follows. For any set Q, if $q^c<A$, then a coclosed set of plaquettes $P^*=\hat{\partial}L^*$ exists which winds around C but does not intersect Q. Under the change of variables $U[b]\to U[b]\tau^{L^*[b]}$ the Wilson loop changes by τ^e, but by construction all other factors are unaffected. Thus, if the n-ality e is nonzero, then the contribution vanishes and consequently we find

$$|\langle A^E[C]\rangle|\leq\alpha\exp(-\rho A)\,, \tag{C5}$$

with $\alpha=2(d-1)\chi^E(1)/(1-c(\beta)b)$, and $e^{-\rho}=c(\beta)b$. This proves the expected area-law behavior (19) for sufficiently strong coupling. We see that at least in this phase the Wilson loop and the electric-flux free energy are equivalent confinement criteria.

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