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Thinning degrees of freedom in lattice field theories

J. B. Bronzan*
Institute for Advanced Study, Princeton, New Jersey 08540

R. L. Sugar
Department of Physics, University of California, Santa Barbara, California 93106
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We discuss the calculation of critical behavior by thinning degrees of freedom in lattice field theories. We point out the tendency of simple approximations to give incorrect critical indices, even for free-field theory. We introduce a precise definition of the thinning transformation and analyze how it induces interactions between distant lattice points. We introduce three procedures for calculating the parameters of the Hamiltonian in the thinned space, all of which give correct free-field-theory indices. For one of them, we study the interacting theory and obtain the critical indices of the large-n approximation, where n is the number of field components.

I. INTRODUCTION

In this paper we study the calculation of critical behavior of lattice field theory by the thinning of degrees of freedom. Except for free-field theories, the thinning procedure cannot be carried through in closed form, and the standard approximation techniques lead to a significant distortion of the density of states.[1,2] In the most straightforward calculations one finds an error of a factor of 2 in the correlation-length critical index, even for free-field theory.[3] We will show how to avoid this pitfall, and in the course of our discussion we will be able to clarify some other features of the thinning process. In particular, in lattice field theory with a nearest-neighbor gradient interaction, the thinning transformation induces couplings between all pairs of lattice sites. We will follow this through the iteration of the thinning process, and show under which circumstances the proliferation of couplings is harmless.

Kadanoff invented the idea of thinning degrees of freedom.[4] His procedure has three steps.

(1) Take a system with N degrees of freedom on a lattice, and in some way "thin out" every other degree of freedom.

(2) Under the thinning process we pass from a system described by a Hamiltonian $H(\alpha, N)$ to a system having a Hamiltonian $H(\alpha', N/2)$. Here α and α' are constants—masses, couplings, etc.,—parametrizing the Hamiltonian.

(3) Large-scale features of the system are not affected by the thinning. For example, the *physical* correlation length of systems having the two Hamiltonians in (2) must be the same, provided the correlation length greatly exceeds the lattice spacing. Let $\xi(\alpha)$ be the correlation length *measured in terms of spin sites*. The physical correlation length is unaffected by thinning when

$$\xi(\alpha') = \tfrac{1}{2}\xi(\alpha). \tag{1.1}$$

Of course, the assumption that ξ is large is well satisfied near the critical point of a second-order phase transition, where ξ diverges. Equation (1.1) can be used to calculate the index controlling the critical behavior of ξ. We assume that if we vary one of the parameters β keeping the others, γ, fixed, then near the critical value $\beta_c(\gamma)$, ξ has the behavior

$$\xi(\beta) \sim [\beta - \beta_c(\gamma)]^{-\nu}. \tag{1.2}$$

(Renormalization-group arguments can be used to justify this ansatz.) Next define a fixed point of the renormalization-group (or thinning) transformation by $\alpha' = \alpha \equiv \alpha^*$. We assume the nonsingular behavior near α^*,

$$\beta' = \beta^* + K(\beta - \beta^*). \tag{1.3}$$

Now choose $\gamma = \gamma^*$. From the equations above we deduce

$$\begin{aligned} &\beta_c(\gamma^*) = \beta_c'(\gamma'^*) = \beta^*, \\ &[K(\beta - \beta^*)]^{-\nu} = \tfrac{1}{2}(\beta - \beta^*)^{-\nu}. \end{aligned} \tag{1.4}$$

The critical index is determined by the renormalization-group transformation through the formula

$$\nu = \frac{\ln 2}{\ln K}. \tag{1.5}$$

Now apply these formulas to free-field theory with nearest-neighbor gradient couplings:

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$$H=\sum_{i=0}^{N-1}[p_i^2+m^2\chi_i^2+\Delta(\chi_i-\chi_{i+1})^2] \quad (\chi_N\equiv\chi_0). \tag{1.6}$$

This and most of the following equations are written for field theory in $1+1$ spacetime dimensions. Much of the generalization to $[(D-1)+1]$-dimensional field theory is obvious; for example, the renormalization-group transformation thins out the degrees of freedom by a factor 2^{D-1}. For the moment we ignore the fact that after thinning, the system is described by a Hamiltonian with non-nearest-neighbor interactions. Then it is physically obvious that the renormalization-group transformation ought to be

$$\Delta'=\tfrac{1}{4}\Delta, \quad m'^2=m^2. \tag{1.7}$$

The reason is that after thinning we have a system whose lattice spacing has been doubled. However, the lattice spacing appears in H only as a factor in Δ, where it occurs to the -2 power.

Equation (1.7) gives the correct index ν. First note that ξ can depend only on the dimensionless ratio m^2/Δ. The transformation $(m^2/\Delta)'=4(m^2/\Delta)$ has two fixed points, $m^2/\Delta=0$ or ∞. The former fixed point results in an infinite correlation length and critical behavior. Then $K=4$ and $\nu=\frac{1}{2}$. This is correct, as we can see by calculating the propagator at equal times and large site separation. For $m^2/\Delta\ll 1$,

$$\langle\chi_l\chi_q\rangle\sim(32\pi m|l-q|\sqrt{\Delta})^{-1/2}e^{-|l-q|/\xi}, \qquad \xi=(m^2/\Delta)^{-1/2}. \tag{1.8}$$

Next consider a field theory with interactions. For example, one could add a term $\lambda\sum_i(\chi_i)^4$ to the right side of Eq. (1.6). One generally formulates field theory on lattice when λ is large, taking the zero-order approximation to be an ensemble of uncoupled anharmonic oscillators. The intersite coupling is then introduced perturbatively. This method was used in Ref. 1 to calculate the renormalization-group transformation. It can be applied to free-field theory, where it gives an incorrect result:

$$\Delta'=\tfrac{1}{2}\Delta, \quad m'^2=m^2, \quad \nu=1. \tag{1.9}$$

In Ref. 2 the renormalization-group transformation was calculated variationally, using a trial wave function separable in the χ's. The separability is motivated by the strongly nonlinear coupling at each site. When applied to free-field theory, the variational calculation again gives Eq. (1.9). In Sec. II we will see that Eq. (1.9) appears even when a large class of intersite correlations is allowed in the trial wave function.

Why do these calculations fail? For the perturbative treatment the reason is clear. Note that the critical fixed point is at $m^2/\Delta=0$, whereas perturbation theory works in the neighborhood of $m^2/\Delta=\infty$. In Sec. III we will see that the true renormalization-group transformation for free-field theory approaches Eq. (1.7) at $m^2/\Delta=0$ and Eq. (1.9) at $m^2/\Delta=\infty$. All is consistent, and we learn that one must do much better than treat the gradient coupling perturbatively.

In the varitional calculation, the site-to-site correlations induced by the gradient term apparently are not taken into account adequately. The correlations are especially important in the small-m^2/Δ limit. Altogether, we see there is a dilemma. Since λ is large, one wants to treat the single-site terms in H accurately. On the other hand, critical phenomena are long-range phenomena, which requires an accurate treatment of the gradient term.

The layout of the remainder of the paper is as follows. In Sec. II we adopt and explain the thinning transformation of Ref. 1. Then we give both the perturbative and variational derivations of the erroneous transformation (1.9). In Sec. III we present a variational procedure for calculating the parameters α'. We then introduce a new trial wave function which must give the renormalization-group transformation for free-field theory because it includes the true ground-state wave function within its parameter space. We discuss the properties of the transformation, including the appearance of non-nearest-neighbor interactions. In Sec. IV we apply our new trial wave function to lattice n-component $\lambda\varphi^4$ field theory, and compare our critical indices to known results. We also introduce an alternative trial function which more fully takes into account nonlinear effects. Finally in Sec. V we study directly the transformation properties of the correlation function under the thinning process, and show how they can be used to calculate the parameters α'.

II. RENORMALIZATION-GROUP TRANSFORMATION

We start with a lattice with N sites, N even. We introduce new canonical coordinates and momenta

$$\chi_{l+}=\frac{1}{\sqrt{2}}(\chi_{2l+1}+\chi_{2l}), \quad p_{l+}=\frac{1}{\sqrt{2}}(p_{2l+1}+p_{2l}),$$
$$\chi_{l-}=\frac{1}{\sqrt{2}}(\chi_{2l+1}-\chi_{2l}), \quad p_{l-}=\frac{1}{\sqrt{2}}(p_{2l+1}-p_{2l}), \tag{2.1}$$

where $l=0,\ldots,N/2-1$. Since the erroneous factor in (1.9) follows from the normalization of these coordinates, it is worth emphasizing that the $\sqrt{2}$ is uniquely determined by canonical commutation rules and our convention that the square of any canonical momentum shall appear in H with co-

efficient one. Denote the ground-state wave functions of $H(\alpha, N)$ and $H(\alpha', N/2)$ by $\psi_0(\chi_{l+}, \chi_{l-})$ and $\psi_0'(\chi_l)$, $l=0,\ldots,N/2-1$. Then α' is determined in terms of α by Friedman's constraint[1]

$$|\psi_0'(\chi_{l+})|^2=\int_{-\infty}^{\infty}\prod_{l'=0}^{N/2-1}d\chi_{l'-}|\psi_0(\chi_{l+},\chi_{l'-})|^2. \quad (2.2)$$

This ensures that

$$\langle 0|A(\chi_{l+})|0\rangle_N=\left\langle 0\left|A\left(\frac{\chi_{2l}+\chi_{2l+1}}{\sqrt{2}}\right)\right|0\right\rangle_N$$
$$=\langle 0|A(\chi_l)|0\rangle_{N/2}, \quad (2.3)$$

where A is any operator and the suffix specifies which Hamiltonian has been used. Equation (1.1) is a consequence of this relation. It is clear from Eq. (2.2) that the determination of the renormalization-group transformation is tantamount to solving the theory, since if we knew ψ_0, we could calculate all Green's functions directly. However, Eq. (2.2) sets the stage for any approximations we may wish to make.

Following Friedman, let us calculate perturbatively, treating Δ as the small parameter.[1] We use the free-field Hamiltonian (1.6), which can be written in terms of the new coordinates as

$$H=H_0+H_{1a}+H_{1b}+H_{1c},$$
$$H_0=\sum_{l=0}^{N/2-1}(p_{l+}^2+m^2\chi_{l+}^2+p_{l-}^2+m^2\chi_{l-}^2),$$
$$H_{1a}=\sum_{l=0}^{N/2-1}(\chi_{l+1-}+\chi_{l-})(\chi_{l+}-\chi_{l+1+}), \quad (2.4)$$
$$H_{1b}=\sum_{l=0}^{N/2-1}[2\chi_{l-}^2+\tfrac{1}{2}(\chi_{l-}+\chi_{l+1-})^2],$$
$$H_{1c}=\tfrac{1}{2}\Delta\sum_{l=0}^{N/2-1}(\chi_{l+1+}-\chi_{l+})^2.$$

To zeroth order in Δ, the ground-state wave function Φ_0 is a product of N mass-m oscillator states. The first-order correction is

$$\psi_0=\Phi_0+\sum_{n\neq 0}\frac{|n\rangle\langle n|H_1|\Phi_0\rangle}{E_0-E_n}. \quad (2.5)$$

In evaluating (2.2) through first order in Δ, only those terms in (2.5) contribute where $|n\rangle$ has all minus oscillators in the ground state. This allows us to drop H_{1a}, which always changes the parity of a minus oscillator. Since $|n\rangle\neq 0$ and all minus oscillators are in the ground state, at least one plus oscillator must be excited. However H_{1b} does not depend on plus coordinates, cannot excite a plus oscillator, and so can be dropped. This leaves H_{1c}, which depends only on plus coordinates. We can now perform the integral in (2.2) mentally. The resulting probability density is what we would calculate perturbatively from a Hamiltonian having $N/2$ degrees of freedom and the primed parameters given in Eq. (1.9).

In Ref. 2, the renormalization-group transformation is determined by using Φ_0 as ground states in Eq. (2.2). This ansatz is a special case of a variational calculation. Let us determine ψ_0 and ψ_0' by variation of the functional

$$\mathcal{E}=\langle\psi_0|H|\psi_0\rangle-\tau(\langle\psi_0|\psi_0\rangle-1). \quad (2.6)$$

The renormalization-group transformation can be obtained if we choose ψ_0 to be separable in the plus and minus coordinates:

$$\psi_0=\psi_+(\chi_{l+})\psi_-(\chi_{l-}). \quad (2.7)$$

Φ_0 belongs to this class of functions. Arbitrary correlations are allowed within the plus and minus sets, but we take ψ_0 and ψ_+ to have even parity because they approximate ground states of $H(\alpha, N)$ and $H(\alpha', N/2)$. It follows that ψ_- has even parity. For this calculation it is convenient to write Eq. (2.4) as

$$H=H_++H_-+\Delta H,$$
$$H_+=\sum_{l=0}^{N/2-1}[p_{l+}^2+m^2\chi_{l+}^2+\tfrac{1}{2}\Delta(\chi_{l+1+}-\chi_{l+})^2],$$
$$H_-=\sum_{l=0}^{N/2-1}[p_{l-}^2+(m^2+2\Delta)\chi_{l-}^2 \quad (2.8)$$
$$+\tfrac{1}{2}\Delta(\chi_{l-}+\chi_{l+1-})^2],$$
$$\Delta H=\Delta\sum_{l=0}^{N/2-1}(\chi_{l+1-}+\chi_{l-})(\chi_{l+}-\chi_{l+1+}).$$

When we substitute Eqs. (2.7) and (2.8) into (2.6), we immediately see that the matrix element of ΔH vanishes by parity. Varying $\mathcal{E}$ with respect to $\psi_\pm$ yields

$$H_\pm|\psi_\pm\rangle=\left(\tau-\frac{\langle\psi_\mp|H_\mp|\psi_\mp\rangle}{\langle\psi_\mp|\psi_\mp\rangle}\right)|\psi_\pm\rangle. \quad (2.9)$$

This shows that the optimum ψ_+ is the ground state of the operator H_+. The parameters Δ' and m'^2 of this Hamiltonian are given by Eq. (1.9).

III. VARIATIONAL CALCULATION—GAUSSIAN TRIAL FUNCTION

Since we cannot hope to carry out the renormalization-group transformation $H(\alpha, N)\to H(\alpha', N/2)$ exactly, we proceed by means of a variational calculation. We first introduce a trial function for the ground state of $H(\alpha, N)$, $\psi_0(\chi_{l+}, \chi_{l-}; \beta)$. The parameters β are determined from the Rayleigh-Ritz variational principle, and are therefore functions of α. Substituting $\psi_0(\chi_l, \beta)$ into Eq. (2.2) determines $\psi_0'(\chi_{l+}, \beta)$. Finally we determine the α' by requiring that β are the optimal variational

parameters when ψ_0' is used as the trial function for the Hamiltonian $H(\alpha', N/2)$.

We illustrate this approach by a Gaussian trial function which must give the correct transformation for free-field theory because it includes the true ground-state wave function within its parameter space. Our ansatz is

$$\psi_0 = C\exp(-A)\,,$$
$$A = \sum_{l,m=0}^{N-1} \alpha_{lm}\chi_l\chi_m\,. \tag{3.1}$$

We want to use this trial function for both free- and interacting-field theories, so we will first show how to integrate out the minus coordinates before relating the matrix α to the Hamiltonian. Interacting lattice φ^4 field theory will be discussed in Sec. IV.

Since ψ_0 is to approximate a ground-state wave function, we choose α to be real, symmetric, and translation invariant:

$$\alpha_{lm} = \alpha_{l+d,m+d}\,. \tag{3.2}$$

d is an arbitrary integer, and indices are evaluated modulo N. It follows that

$$\alpha_{lm} = \alpha_{0,m-l} = \alpha_{0,l-m} = \alpha_{0,m-l+kN}\,, \tag{3.3}$$

where k is an integer. We see that ψ_0 contains $N/2+1$ variational parameters $\alpha_{0,l}$, $l=0,1,\ldots,N/2$. Eigenfunctions of α are

$$V_m(p) = \frac{1}{\sqrt{N}}\exp(2\pi ipm/N)\,,\quad p=0,1,\ldots,N-1\,. \tag{3.4}$$

Corresponding eigenvalues are

$$\begin{aligned}\lambda(p) &= \sum_{l=0}^{N-1}\alpha_{0,l}\exp(2\pi ipl/N)\\ &= \alpha_{0,0} + 2\sum_{l=1}^{N/2}\alpha_{0,l}\cos(2\pi pl/N)\\ &= \lambda(N-p)\,.\end{aligned} \tag{3.5}$$

The eigenvalues $\lambda(p)$, $p=0,1,\ldots,N/2$ can be taken equally well as variational parameters, and henceforth we will do so. Since eigenvalues with quantum numbers p and $N-p$ are equal, we can form real eigenvectors

$$V_m(p,1) = \left(\frac{\epsilon_p}{N}\right)^{1/2}\cos(2\pi pm/N)\,,\quad p=0,\ldots,N/2 \tag{3.6}$$

$$V_m(p,2) = \left(\frac{\epsilon_p}{N}\right)^{1/2}\sin(2\pi pm/N)\,,\quad p=1,\ldots,N/2-1\,.$$

With the choice

$$\epsilon_p = \begin{cases}1, & p=0,N/2\\ 2, & p=1,2,\ldots,N/2-1\end{cases} \tag{3.7}$$

the eigenvectors are orthonormal:

$$\sum_{m=0}^{N-1} V_m(p,j)V_m(p',j') = \delta_{pp'}\delta_{jj'}\,. \tag{3.8}$$

Let us expand the coordinates in terms of normal coordinates

$$\begin{aligned}\chi_m = {}& \frac{q^{0,1}}{\sqrt{N}} + \frac{q^{N/2,1}}{\sqrt{N}}(-1)^m\\ &+\left(\frac{2}{N}\right)^{1/2}\sum_{p=1}^{N/2-1}\left[q^{p,1}\cos(2\pi pm/N)\right.\\ &\left.+q^{p,2}\sin(2\pi pm/N)\right]\,.\end{aligned} \tag{3.9}$$

This diagonalizes the exponent A in the trial wave function:

$$\begin{aligned}A = {}& \lambda(0)(q^{0,1})^2 + \lambda(N/2)(q^{N/2,1})^2\\ &+\sum_{p=1}^{N/2-1}\lambda(p)\left[(q^{p,1})^2 + (q^{p,2})^2\right]\,.\end{aligned} \tag{3.10}$$

Finally, using the inverse of Eq. (3.9),

$$\begin{aligned}q^{p,1} &= \left(\frac{\epsilon_p}{N}\right)^{1/2}\sum_{l=0}^{N-1}\chi_l\cos(2\pi pl/N)\,,\\ q^{p,2} &= \left(\frac{2}{N}\right)^{1/2}\sum_{l=0}^{N-1}\chi_l\sin(2\pi pl/N)\,,\end{aligned} \tag{3.11}$$

we can use Eq. (3.10) to parametrize α in terms of its eigenvalues:

$$\begin{aligned}\alpha_{lm} = {}& \frac{\lambda(0)}{N} + \frac{\lambda(N/2)}{N}(-1)^{l-m}\\ &+\frac{2}{N}\sum_{p=1}^{N/2-1}\lambda(p)\cos\left[2\pi p(m-l)/N\right]\,.\end{aligned} \tag{3.12}$$

The next step is to introduce the plus and minus coordinates of Eq. (2.1):

$$\begin{aligned}q^{p,1} = {}& \left(\frac{2\epsilon_p}{N}\right)^{1/2}\sum_{l=0}^{N/2-1}\left[\chi_{l+}\cos\frac{p\pi}{N}\cos\left(\frac{4\pi pl}{N}+\frac{p\pi}{N}\right)\right.\\ &\left.-\chi_{l-}\sin\frac{p\pi}{N}\sin\left(\frac{4\pi pl}{N}+\frac{p\pi}{N}\right)\right]\,,\end{aligned} \tag{3.13}$$

$$\begin{aligned}q^{p,2} = {}& \frac{2}{\sqrt{N}}\sum_{l=0}^{N/2-1}\left[\chi_{l+}\cos\frac{p\pi}{N}\sin\left(\frac{4\pi pl}{N}+\frac{p\pi}{N}\right)\right.\\ &\left.+\chi_{l-}\sin\frac{p\pi}{N}\cos\left(\frac{4\pi pl}{N}+\frac{p\pi}{N}\right)\right]\,.\end{aligned}$$

Substituting these expressions into Eq. (3.10), we can write

$$\begin{aligned}A = \sum_{l,m=0}^{N/2-1}{}& (\alpha^{++}_{lm}\chi_{l+}\chi_{m+} + \alpha^{+-}_{lm}\chi_{l+}\chi_{m-}\\ &+\alpha^{-+}_{lm}\chi_{l-}\chi_{m+} + \alpha^{--}_{lm}\chi_{l-}\chi_{m-})\,,\end{aligned} \tag{3.14}$$

where

$$\alpha^{++}_{lm}=\frac{2\lambda(0)}{N}+\frac{2\lambda(N/4)}{N}(-1)^{l-m}+\frac{4}{N}\sum_{p=1}^{N/4-1}\left[\lambda(p)\cos^2\frac{p\pi}{N}+\lambda\left(\frac{N}{2}-p\right)\sin^2\frac{p\pi}{N}\right]\cos[4\pi p(l-m)/N]\,,$$

$$\alpha^{+-}_{lm}=\alpha^{-+}_{ml}=\frac{2}{N}\sum_{p=1}^{N/4-1}\left[\lambda(p)-\lambda\left(\frac{N}{2}-p\right)\right]\sin(2\pi p/N)\sin[4\pi p(l-m)/N]\,, \tag{3.15}$$

$$\alpha^{--}_{lm}=\frac{2\lambda(N/2)}{N}+\frac{2\lambda(N/4)}{N}(-1)^{l-m}+\frac{4}{N}\sum_{p=1}^{N/4-1}\left[\lambda(p)\sin^2\frac{p\pi}{N}+\lambda\left(\frac{N}{2}-p\right)\cos^2\frac{p\pi}{N}\right]\cos[4\pi p(l-m)/N]\,.$$

We can now evaluate the integral over minus coordinates on the right side of Eq. (2.2). We use Eq. (3.14), and by completion of squares we find

$$\int_{-\infty}^{\infty}\sum_{i'=0}^{N/2-1}d\chi_{i'-}\,|\psi_0(\chi_{l+},\chi_{i'-})|^2=(C')^2\exp\left[-2\sum_{l,m=0}^{N/2-1}(\alpha')_{lm}\chi_{l+}\chi_{m+}\right], \tag{3.16}$$

where

$$\alpha'=\alpha^{++}-(\alpha^{+-})(\alpha^{--})^{-1}(\alpha^{-+})\,. \tag{3.17}$$

The inverse of α^{--} is

$$(\alpha^{--})^{-1}_{lm}=\frac{2}{N\lambda(N/2)}+\frac{2(-1)^{l-m}}{N\lambda(N/4)}+\frac{4}{N}\sum_{p=1}^{N/4-1}\frac{\cos[4\pi p(l-m)/N]}{[\lambda(p)\sin^2(\pi p/N)+\lambda(N/2-p)\cos^2(\pi p/N)]}\,. \tag{3.18}$$

We then find that α' has the form (3.12), but with $N\to N/2$ and $\lambda(p)\to\lambda'(p)$, where

$$\frac{1}{\lambda'(p)}=\frac{\cos^2(\pi p/N)}{\lambda(p)}+\frac{\sin^2(\pi p/N)}{\lambda(N/2-p)}\,. \tag{3.19}$$

The constraint $\lambda'(p)=\lambda'(N/2-p)$ is assured by this mapping. We see that Gaussian trial functions are automatically mapped into Gaussian functions when the minus coordinates are integrated out. In Sec. V we obtain a result analogous to Eq. (3.19) for the Green's function in the interacting theory.

Equation (3.19) becomes the renormalization-group transformation when the eigenvalues $\lambda(p)$ are related to the Hamiltonian. In the remainder of this section we will consider the general free-field-theory lattice Hamiltonian

$$H=\sum_{l=0}^{N-1}\left[p_l{}^2+m^2\chi_l{}^2+\sum_{p=1}^{N/2}d_p(\chi_l-\chi_{l+p})^2\right]$$
$$(\chi_{l+N}\equiv\chi_l)\,. \tag{3.20}$$

This Hamiltonian is diagonal when written in terms of normal coordinates (3.9) and normal momenta $\pi^{p,j}$:

$$p_m=\frac{\pi^{0,1}}{\sqrt{N}}+\frac{\pi^{N/2,1}}{\sqrt{N}}(-1)^m+\left(\frac{2}{N}\right)^{1/2}\sum_{p=1}^{N/2-1}\left[\pi^{p,1}\cos(2\pi pm/N)+\pi^{p,2}\sin(2\pi pm/N)\right]\,. \tag{3.21}$$

The q's and π's are Hermitian canonical variables:

$$[q^{p,j},\pi^{p',j'}]=i\delta_{pp'}\delta_{jj'}\,. \tag{3.22}$$

The Hamiltonian is

$$H=\sum_{p=0}^{N/2}[(\pi^{p,1})^2+\omega^2(p)(q^{p,1})^2]+\sum_{p=1}^{N/2-1}[(\pi^{p,2})^2+\omega^2(p)(q^{p,2})^2]\,, \tag{3.23}$$

where

$$\omega(p)=\left[m^2+2\sum_{j=1}^{N/2}d_j(1-\cos 2\pi pj/N)\right]^{1/2}. \tag{3.24}$$

We can verify that Eqs. (3.1) and (3.10) give the ground state of this Hamiltonian, where

$$C^{-2}=\prod_{p=0}^{N/2}\left[\frac{\pi}{2\lambda(p)}\right]^{s_p/2}\,,$$
$$\lambda(p)=\tfrac{1}{2}\omega(p)\,. \tag{3.25}$$

Alternatively, we can evaluate C and $\lambda(p)$ by variation of the functional (2.6). This method will be used in Sec. IV, where ψ_0 is not the exact ground state of interacting-field theory. When applied to free-field theory, the variational method reproduces Eq. (3.28). It follows that Eq. (3.19) is a mapping of $\omega(p)\to\omega'(p)$, and this mapping in combination with Eq. (3.24) gives the renormalization-group transformation.

A number of properties of this transformation can now be seen. Since $\omega'(0)=\omega(0)$,

$$m'^2=m^2\,. \tag{3.26}$$

The transformed intersite couplings can be obtained from the original ones by the projection equation

$$d_j' = -\frac{\epsilon_j'}{N}\sum_{p=0}^{N/2-1}[\omega'(p)]^2\cos\frac{4\pi pj}{N}$$
$$= -\frac{\epsilon_j'}{N}\left\{[\omega'(0)]^2+\left[\omega'\left(\frac{N}{4}\right)\right]^2(-1)^j + 2\sum_{p=1}^{N/4-1}[\omega'(p)]^2\cos\frac{4\pi pj}{N}\right\}$$
$$(j=1,\ldots,N/4), \quad (3.27)$$

where

$$\epsilon_j' = \begin{cases} 1, & j=0, N/4 \\ 2, & j=1,2,\ldots,N/4-1. \end{cases} \quad (3.28)$$

In general, all intersite couplings are nonzero, even if one starts with a Hamiltonian having only nearest-neighbor gradient terms. A simple way to see this is to start with $\omega(p)^2 = m^2 + 4d_1\sin^2(\pi p/N)$ and develop $[\omega'(p)]^2$ as a power series in d_1. We find

$$[\omega'(p)]^2 = m^2 + 4\left(\frac{d_1}{2}\right)\sin^2\frac{2\pi p}{N} + 4\left(-\frac{3d_1^2}{16m^2}\right)\sin^2\frac{4\pi p}{N} + O(d_1^3). \quad (3.29)$$

We then read off

$$d_1' = \tfrac{1}{2}d_1 + O(d_1^3), \qquad d_2' = -\frac{3d_1^2}{16m^2} + O(d_1^3). \quad (3.30)$$

The first of these equations reproduces what we discovered by perturbation theory in $\Delta = d_1$ in Sec. II; the second shows the nonzero second nearest-neighbor interaction.

We are particularly interested in lattice field theories in which the d_j's are large, because the lattice field theory then approximates continuum field theory. We intuitively expect that in this limit the lattice field theory ought to depend only on the sum

$$\Delta = \sum_{j=1}^{N/2} d_j(j)^2, \quad (3.31)$$

and not on the individual d_j's. The reason is that there are many ways of approximating a derivative by difference operators, and all sums of difference operators having the same $\Delta \propto$ (lattice spacing)$^{-2}$ are approximants to $(\partial\varphi/\partial\chi)^2$. Thus when N and Δ are large, all lattice field theories having the same m^2 and Δ ought to have the same Green's functions. If this is so, then we do not have to keep track of the renormalization-group transformation of all the d_j's, but only of the combination Δ.

It is easy to justify these ideas for free-field theory. Consider the equal-time propagator

$$\langle \chi_l \chi_m\rangle = \frac{1}{2N}\sum_{p=0}^{N/2}\frac{\cos[2\pi p(l-m)/N]}{\omega(p)}. \quad (3.32)$$

Introduce the momentum variables $k = 2\pi p/N$ and $(\delta k) = 2\pi/N$. Then

$$\langle \chi_l \chi_m\rangle = \frac{1}{4\pi}\sum_{k=0}^{\pi}(\delta k)\frac{\cos k(l-m)}{\left(m^2+4\sum_{j=1}^{N/2}d_j\sin^2 kj/2\right)^{1/2}}. \quad (3.33)$$

When $N\to\infty$, the sum becomes an integral. In addition, when the d_j's are large, the integral is dominated by the region near $k=0$. Then we have the approximate formula

$$\langle \chi_l \chi_m\rangle \sim \frac{1}{4\pi}\int_0^{\pi}\frac{dk\cos k(l-m)}{(m^2+\Delta k^2)^{1/2}} \quad \left(\frac{\Delta}{m^2}\gg 1\right). \quad (3.34)$$

Equation (1.8) was obtained by extending the upper limit of the integral to infinity and finding the leading behavior for large $(l-m)$.

The renormalization-group transformation for Δ is easily found by noting that

$$\Delta = \frac{m}{2}\left(\frac{N}{\pi}\right)^2\left.\frac{\partial\omega(p)}{\partial p^2}\right|_{p^2=0}, \qquad \Delta' = \frac{m}{8}\left(\frac{N}{\pi}\right)^2\left.\frac{\partial\omega'(p)}{\partial p^2}\right|_{p^2=0}. \quad (3.35)$$

We find for $\Delta \gg m^2$,

$$\Delta' = \tfrac{1}{4}\Delta. \quad (3.36)$$

Equations (3.26) and (3.36) together comprise Eq. (1.7), which we introduced on physical grounds, and which gives the correct critical exponent for the correlation length.

Looking back, we can see more precisely what went wrong with the elementary calculations in Sec. II. The perturbative calculation failed on several counts. As we noted in Sec. I, the fundamental difficulty with the perturbative calculation is that it would apply in the domain $\Delta/m^2 \ll 1$, not the domain $\Delta/m^2 \gg 1$ relevant to the critical fixed point. It also makes the naive identification $\Delta = d_1$. This is correct to first order in d_1, where the renormalization-group transformation does not induce non-nearest-neighbor couplings. However, we have seen that such couplings are generated beyond first order, and then one should define Δ as in Eq. (3.31). The variational trial function in Sec. II effectively omits the α^{+-} and α^{-+} matrices of Eq. (3.14), which introduce correlations between plus and minus coordinates. This drastically affects the relation between $\lambda'(p)$ and $\lambda(p)$. It re-

places α' by α^{++}, and Eq. (3.19) by

$$\lambda'(p)=\lambda(p)\cos^2\frac{p\pi}{N}+\lambda\left(\frac{N}{2}-p\right)\sin^2\frac{p\pi}{N}. \quad (3.37)$$

This incorrect transformation differs from Eq. (3.19) by having the reciprocal of the eigenvalue take the place of the eigenvalue.

IV. INTERACTING-FIELD THEORY

We can apply the Gaussian trial function of Sec. III to an interacting-field theory; our choice is the n-component lattice field theory given by the Hamiltonian

$$H=\sum_{l=0}^{N-1}\sum_{\alpha=1}^{n}\left[p_{l,\alpha}{}^2+m^2\chi_{l,\alpha}+\sum_{j=1}^{N/2}d_j(\chi_{l,\alpha}-\chi_{l+j,\alpha})^2\right]+\sum_{l=0}^{N-1}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}g(\chi_{l,\alpha})^2(\chi_{l,\beta})^2. \quad (4.1)$$

Before calculating the exponent ν, let us review what we expect to find. The critical behavior of our system is well understood.[5] When the number D of spacetime dimensions exceeds 4, g vanishes at the fixed point and we revert to a free-field theory having $\nu=\frac{1}{2}$. At $D<4$, the relevant fixed point has a nonzero g^*, and at least at first this fixed point coupling increases as D is decreased. Our Gaussian trial function is therefore exact at $D\geq 4$, and we expect to find $\nu=\frac{1}{2}$ there. Of course, this limiting value would *not* be given correctly by the approximate renormalization-group transformations we discussed in Sec. II. For $D<4$, we expect to see a smooth change in ν away from $\nu=\frac{1}{2}$. In general we expect ν to become less accurate as D is reduced because the Gaussian wave function does not take account of interactions except through the form of $\lambda(p)$.

The calculation proceeds by evaluation of the functional $\mathcal{S}$ of Eq. (2.6), using the trial function (3.1) and Hamiltonian (4.1). We use normal coordinates (3.9) and find

$$\mathcal{S}=-\tau+\left\{C^2\prod_{p=0}^{N/2}\left[\frac{\pi}{2\lambda(p)}\right]^{n\epsilon_p/2}\right\}\left\{\tau+n\sum_{l=0}^{N-1}\epsilon_l\left[\lambda(l)+\frac{\omega^2(l)}{4\lambda(l)}+\frac{(3n+5)gN}{64}\left(\frac{1}{N}\sum_{p=0}^{N/2}\frac{\epsilon_p}{\lambda(p)}\right)^2\right]\right\}. \quad (4.2)$$

The term proportional to g is not the full contribution of the quartic term, but it is all that survives when the number of lattice sites N tends to infinity, provided we stay in the disordered phase where $\langle\chi_i\rangle=0$. We vary parameters so that $\mathcal{S}$ is stationary. This leads to the identification of λ.

$$\lambda(p)=\tfrac{1}{2}\bar{\omega}(p),$$
$$\bar{\omega}^2(p)=\bar{m}^2+4\sum_{j=1}^{N/2}d_n\sin^2\frac{\pi jp}{N}, \quad (4.3)$$
$$\bar{m}^2=m^2+\frac{(3n+5)g}{4}\frac{1}{N}\sum_{q=0}^{N/2}\frac{\epsilon_q}{\bar{\omega}(q)}.$$

The only effect of interactions is to renormalize the mass. The renormalization-group transformation is given by (3.19). We will assume $\bar{m}^2/\Delta$ is small, where Δ is given by (3.31). The renormalization-group transformation reduces to (1.7), with $\bar{m}^2$ replacing m^2. The fixed point is still $\bar{m}^2/\Delta=0$, and the critical behavior of the correlation length is

$$\xi=\left(\frac{\bar{m}^2}{\Delta}\right)^{-1/2}. \quad (4.4)$$

Note that we do not obtain renormalization-group transformation formulas for m^2 and g separately, but only for the combination $\bar{m}^2$. This oversimplification is due to our crude trial function.

In order to calculate ν, we must replace $\bar{m}^2$ by m^2 in Eq. (4.4). The last formula in (4.3) relates these parameters. The sum can be converted to an integral as in Eq. (3.34):

$$\bar{m}^2=m^2+\frac{(3n+5)}{4\pi}g\int_0^{\pi}\frac{d^{D-1}k}{(\bar{m}^2+\Delta\vec{k}^2)^{1/2}}. \quad (4.5)$$

We have generalized from 2 to D spacetime dimensions; $\vec{k}$ is now a $(D-1)$-dimensional vector. At this point we must recognize that Eq. (4.5) lacks the oscillating factor in (3.34), so the region of small k dominates the integral only for $D<2$. However, small k dominates the formula for the derivative for $D<4$:

$$\frac{\partial m^2}{\partial\bar{m}^2}=1+\frac{(3n+5)g}{8\pi}\int_0^{\pi}\frac{d^{D-1}k}{(\bar{m}^2+\Delta\vec{k}^2)^{3/2}}. \quad (4.6)$$

We now extend the integration to $\pm\infty$ and evaluate the integral

$$\frac{\partial m^2}{\partial\bar{m}^2}=1+\frac{(3n+5)g(\pi)^{(D-3)/2}\Gamma(4-D/2)}{2^{D+3}(\Delta)^{(D-1)/2}}(\bar{m}^2)^{(D-4)/2}, \quad (4.7)$$

$$\frac{m^2-m_c{}^2}{\Delta}=-\frac{(3n+5)g(\pi)^{(D-3)/2}\Gamma(2-D/2)}{2^{D+3}\Delta^{3/2}}\times\left(\frac{\bar{m}^2}{\Delta}\right)^{(D-2)/2},$$

where $\bar{m}^2$ small, $2<D<4$. Equation (4.7) can be inverted

$$\frac{\bar{m}^2}{\Delta} = (\text{const})\left(\frac{m^2 - m_c^2}{\Delta}\right)^{2/(D-2)} \quad (m^2 \sim m_c^2,\ 2 < D < 4). \tag{4.8}$$

For $0 < D < 2$, $m_c^2 = -\infty$, and we see directly from Eq. (4.5) that

$$\bar{m}^2/\Delta \underset{m^2 \to -\infty}{\sim} \text{const}(-m^2/\Delta)^{2/(D-2)}.$$

We then have

$$\nu = \frac{1}{D-2} \quad (0 < D < 4). \tag{4.9}$$

For $D \geq 4$, we must differentiate (4.5) a second time to assure dominances of the integral by the region near $k = 0$. We learn

$$\frac{\bar{m}^2}{\Delta} = (\text{const})\left(\frac{m^2 - m_c^2}{\Delta}\right) \quad (m^2 \sim m_c^2,\ 4 \leq D), \qquad \nu = \tfrac{1}{2} \quad (4 \leq D). \tag{4.10}$$

Equation (4.9) is the correct critical exponent in the limit $n \to \infty$ (Ref. 5). This is not surprising: if we solve Eq. (4.5) for $\bar{m}^2$, and express the answer as a power series in g, we generate the self-energy Feynman diagrams which are dominant in the large-n limit. But Eq. (4.9) can be quite misleading when D and n are small. At $D = 2$ and $n = 1$, the correct critical exponent is $\nu = \frac{1}{8}$, the exponent of the two-dimensional Ising model.[5]

For finite values of n and $D \leq 4$, the Gaussian trial function is inadequate for the interacting theory. It is important to take into account both the strong quartic single-site interaction and the small momentum fluctuations.

One possible approach is to take advantage of our ability to solve the single-site problem.[6] If we set $n = 1$ for simplicity, then the single-site Hamiltonian is

$$H_l = p_l^2 + m^2 \chi_l^2 + g \chi_l^4. \tag{4.11}$$

If all the parameters d_j were zero, then the ground state of H would simply be

$$|0\rangle = \prod_{l=0}^{N-1} |0, l\rangle, \tag{4.12}$$

where $|0, l\rangle$ is the ground state of H_l. For $d_j \neq 0$ we can introduce intersite correlations by writing the ground-state trial function in the form

$$|\psi_0\rangle = f(\chi_0, \ldots, \chi_{N-1})|0\rangle. \tag{4.13}$$

The trial function f is to be determined by the variational principle for the ground-state wave function.

To illustrate this approach let us return to the case of free-field theory and take

$$f = 1 + 2m \sum_{p=0}^{p_0} B(p) \tilde{\chi}_p \tilde{\chi}_{N-p}, \tag{4.14}$$

where

$$\tilde{\chi}_p = \sum_m V_m(p) \chi_m \tag{4.15}$$

and $V_m(p)$ is defined in Eq. (3.4). Equation (4.14) is the only form quadratic in the χ_m that is consistent with translational invariance. Since we are interested in large-distance correlations, we have cut off the sum over p at $p_0 \ll N$. In fact f has the functional form (4.15) only for small p, but any nonzero range of p suffices to deduce the renormalization-group transformation on the Hamiltonian parameters. Using $|\psi_0\rangle$ in the Rayleigh-Ritz variational principle, we find that

$$B(p) = -\frac{\Delta}{m^2} \sin^2 \frac{\pi p}{N} \simeq -\left(\frac{\Delta}{m^2}\right)\left(\frac{\pi p}{N}\right)^2. \tag{4.16}$$

Equation (2.2) now becomes

$$|\psi_0'(\chi_{0+}, \ldots, \chi_{N/2-1+})|^2 = \int_{-\infty}^{\infty} \prod_{l=0}^{N/2-1} d\chi_{l-} \prod_{k=0}^{N-1} (m/\pi)^{1/2} e^{-m\chi_k^2} \times \left[1 + 4m \sum_{p=0}^{p_0} B(p) \tilde{\chi}_p \tilde{\chi}_{N-p}\right], \tag{4.17}$$

since it is sufficient to work to first-order $B(p)$. Now

$$\chi_p = e^{i\pi p/N}[\cos(\pi p/N)\tilde{\chi}_{p+} + i \sin(\pi p/N)\tilde{\chi}_{p-}], \tag{4.18}$$

where

$$\tilde{\chi}_{p\pm} = \sum_{l=0}^{N/2-1} V_l'(p) \chi_{l\pm} \tag{4.19}$$

and

$$V_l'(p) = \frac{e^{2\pi i l p/(N/2)}}{(N/2)^{1/2}}.$$

So, we can do the χ_{l-} integration in Eq. (4.17) and obtain

$$|\psi_0'(\chi_{0+}, \ldots, \chi_{N/2-1+})|^2 = \prod_{k=0}^{N/2-1} (m/\pi)^{1/2} e^{-m\chi_{k+}^2} \times \left[1 + 4m \sum_{p=0}^{p_0} B(p) \tilde{\chi}_{p+} \tilde{\chi}_{N-p+}\right]. \tag{4.20}$$

Comparing this result with that which we would have obtained by using a trial function of the same form for a Hamiltonian with parameters m' and Δ' on a lattice with $N/2$ sites, we see that

$$m' = m, \qquad \Delta' = \Delta/4, \tag{4.21}$$

the exact results.

The reader should note the close resemblance between the wave function of Eqs. (4.13) and (4.15) on the one hand, and that of Eq. (2.5) on the other. They are identical except that in Eq. (2.5), $p_0 = N/2$. The reason that the wave function in Eq. (2.5) leads to the incorrect renormalization-group transformation is that after we thin coordinates, we find the sum in p ranges twice over the same operators in Eq. (4.20). We can reduce p_0 to $N/4$, but we pick up the notorious factor of 2. In fact Eq. (4.21) results only for p_0 independent of N and less than $N/4$. We have already given other reasons for choosing p_0 small.

One can use the same procedure to study the interacting theory, and we hope to do so in the future. The point which we wish to emphasize here is that there exists a straightforward variational approach which is guaranteed to reduce to the correct answer in the weak-coupling limit.

V. TRANSFORMATION PROPERTIES OF THE CORRELATION FUNCTION

If one defines the renormalization-group transformation by Eq. (2.2), one needs information about both the large- and small-distance behavior of the ground-state wave function. We will now sketch an alternative approach in which the transformation properties of the nonlocal term in the Hamiltonian are explicitly taken into account. This approach appears to require less detailed information about the large-distance behavior of the wave function.

Our starting point is the Lehmann representation for the correlation function. It can be written in the form

$$F_{jk} = \langle \chi_j(t)\chi_k(t)\rangle = \int_0^\infty d\sigma\, \rho(\sigma) \frac{1}{N} \sum_{p=0}^{N-1} \frac{e^{2\pi i (j-k)p/N}}{2\omega(p,\sigma)}, \tag{5.1}$$

where $\omega(p,\sigma) = \omega(N-p,\sigma)$ is the energy of a single excitation with mass $\sigma^{1/2}$ and momentum $\pi p/N$:

$$\rho(\sigma) = z_3 \delta(\sigma - m^2) + \theta(\sigma - 9m^2)\bar{\rho}(\sigma). \tag{5.2}$$

For a free-field theory with nearest-neighbor couplings, $z_3 = 1$, $\bar{\rho} = 0$, and $\omega(p,\sigma) = [\sigma + 4\Delta \sin^2(\pi p/N)]^{1/2}$.

Now consider the Green's function on the thinned lattice defined by

$$G'_{jk} = \tfrac{1}{2}(G_{2j,2k} + G_{2j+1,2k} + G_{2j,2k+1} + G_{2j+1,2k+1}) = \int_0^\infty d\sigma\, \rho'(\sigma) \frac{1}{N/2} \sum_{p=0}^{N/2-1} \frac{e^{2\pi i (j-k)/(N/2)}}{2\omega'(p,\sigma)}. \tag{5.3}$$

After a brief calculation we find that

$$\rho'(\sigma) = \rho(\sigma), \qquad \frac{1}{\omega'(p,\sigma)} = \frac{\cos^2(\pi p/N)}{\omega(p,\sigma)} + \frac{\sin^2(\pi p/N)}{\omega(N/2-p,\sigma)}. \tag{5.4}$$

Equation (5.4) is the generalization of Eq. (3.19) for the interacting theory.

Our program is to use the Hamiltonians $H(\alpha)$ and $H'(\alpha')$ to make the best possible calculations of $\rho(\sigma)$ and $\rho'(\sigma)$. We then determine the parameters α' as a function of α from the requirement that $\rho(\sigma) = \rho'(\sigma)$. In order to calculate $\rho(\sigma)$ we need the matrix elements $\langle \psi_p | \chi_i | \psi_0 \rangle$, where $|\psi_p\rangle$ is a general odd-parity eigenstate of H. Since χ_i is a local operator we expect these matrix elements to be less sensitive to the large-distance behavior of the wave functions. On the other hand, we must now estimate the excited states of H instead of just the ground state as in the previous approach.

To illustrate this approach let us again consider free-field theory. We make the very crude approximation that terms in H and H' proportional to the d_j can be neglected in calculating $\rho(\sigma)$. We then immediately find that $\rho(\sigma) = \delta(m^2 - \sigma)$ and $\rho'(\sigma) = \delta(m'^2 - \sigma)$, so

$$m' = m, \qquad \Delta' = \Delta/4. \tag{5.5}$$

Again, corresponding calculations can be carried through for the interacting theory.

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*Permanent address: Serin Physics Laboratory, Rutgers University, Piscataway, N. J. 08854.

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