Field theory with ϕ^4 and ϕ^6 self-interaction

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A theory for a scalar field with a potential corresponding to three wells is analyzed. Use is made of the lattice approximation and the block-spin renormalization-group method keeping the three lowest levels at each site. A second-order or a first-order phase transition appears according to the relative depths of the wells and the intersite coupling. The same qualitative scheme emerges from a mean-field approximation.

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I. INTRODUCTION

The field theory with ϕ^4 self-interaction has been extensively studied together with its relation to a quantum spin-$\frac{1}{2}$ Ising model with transverse magnetic field.[1] This comparison corresponds to keeping the two lowest-energy levels in each site when the field theory is taken on a lattice. Since there are cases of physical interest in which the two lowest levels are not clearly separated from the rest of the spectrum (e.g., Reggeon field theory[2]), it is interesting to include a third level in each site of a simple theory. This leads to a quantum spin-1 model in d dimensions which is equivalent to a classical spin-1 model in $d+1$ dimensions[3] in the same way as the equivalence for the spin-$\frac{1}{2}$ case is proved.[4] A mean-field approximation of the classical spin-1 model has been applied[5] to the He^3-He^4 mixture.

The purpose of the present work is to consider a theory of a scalar field in 1+1 dimensions with self-interactions up to ϕ^6 which exhibits three lowest levels well separated from the rest and to analyze its lattice version.

In Sec. II we briefly review the quantum mechanics of a three-well potential which corresponds to a single site. When the three wells are of equal depth, one may keep three equally spaced levels, obtaining what we will denote as an Ising model for spin 1.

In Sec. III we show for this equal-depth case that classical, perturbative, and mean-field arguments suggest a second-order phase transition similar to that of the spin-$\frac{1}{2}$ case.

The general case of unequal depths shows already in a mean-field approximation the appearance of an additional first-order transition and a tricritical point. Our mean-field calculation for the quantum model developed in Sec. IV gives for the tricritical point a value in good agreement with the concentration of He^3 at this point obtained with the classical model.[5]

A complete treatment based on the renormalization-group procedure is only possible when the three levels are unequally spaced. This method is developed in Sec. V using the modification of the block-spin technique of Ref. 6. It turns out that keeping one site and one link in each block the form of the Hamiltonian is preserved in the subsequent steps of the renormalization-group procedure. When the separation between the second and third levels (ϵ_S) is at least equal to the separation between the ground and first excited levels (ϵ_I), a second-order phase transition appears with a slight modification of the critical point with respect to the spin-$\frac{1}{2}$ model. When $0.2\,(\epsilon_I+\epsilon_S)<\epsilon_S<\epsilon_I$, a first-order transition appears in addition to the second-order one depending on the value of the intersite coupling. The first-order transition is defined in this region in correspondence to the appearance of a nonzero field expectation value in the first excited state. For $\epsilon_S<0.2(\epsilon_S+\epsilon_I)$ only the first-order transition, this time referred to a ground-state expectation value, is possible. The second-order and first-order critical lines join in a tricritical point again in agreement with the He^3 concentration. Concluding remarks are included in Sec. VI.

II. QUANTUM MECHANICS OF A THREE-WELL POTENTIAL

We are interested in a field theory in 1+1 dimensions described by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-U(\phi)\,, \tag{1}$$

with $U(\phi)=a_1\phi^6+a_2\phi^4+a_3\phi^2$, where a_1 and $a_3>0$, and $a_2<0$.

When we consider its lattice version and define the gradient as difference of fields at neighboring sites, the Hamiltonian takes the form

$$H=\Lambda\sum_j[\tfrac{1}{2}\pi_j^{\,2}+\mathfrak{U}(\phi_j)-\phi_j\phi_{j+1}]\,, \tag{2}$$

where ϕ_j is the field at the site j, π_j is its conju-

gate momentum, Λ^{-1} is the lattice spacing, and $\mathfrak{U}(\phi_j) = U(\phi_j) + \phi_j^2$.

With the above choice for the signs of the coefficients a_1, a_2, and a_3, $\mathfrak{U}(\phi_j)$ exhibits three wells. The quantum mechanics of the single-site Hamiltonian may be approximated by analyzing the square-well potential of Fig. 1. We assume that $V_2 \gg V_1$ and that the width L is sufficiently small to neglect all levels except the lowest one in each well.

We express the Hamiltonian for one site in the basis of the lowest-energy solutions of the problems (Fig. 2). Neglecting the corrections in the diagonal terms and keeping only nearest-neighbor overlapping, the single-site Hamiltonian is

$$H_{SS} = \begin{pmatrix} E_0 + V_1 & V & 0 \\ V & E_0 & V \\ 0 & V & E_0 + V_1 \end{pmatrix}, \tag{3}$$

where E_0 is the lowest energy for the central well and V is a negative overlapping integral, which is exponentially small due to the WKB factor.

The eigenvalues of Eq. (3) are

$$\lambda = V_1, \quad \lambda_\pm = \tfrac{1}{2}[V_1 \pm (V_1^2 + 8V^2)^{1/2}], \tag{4}$$

with the eigenvectors

$$\begin{aligned} |\lambda\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle), \\ |\lambda_\pm\rangle &= \frac{1}{(2V^2 + \lambda_\mp^2)^{1/2}}(V|1\rangle - \lambda_\mp|2\rangle + V|3\rangle), \end{aligned} \tag{5}$$

respectively.

Defining $\Delta^{1/2} = \langle\lambda_+|\phi_j|\lambda\rangle$, since $|2\rangle$ does not affect this matrix element, it follows that

$$\langle\lambda_-|\phi_j|\lambda\rangle = \alpha\Delta^{1/2},$$

where

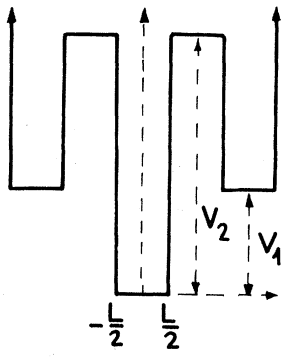

FIG. 1. Simplified three-well potential.

$$\alpha = \left(\frac{\epsilon}{K-\epsilon}\right)^{1/2}, \quad \epsilon = \lambda_+ - \lambda, \text{ and } K = \lambda_+ - \lambda_-.$$

Therefore, the Hamiltonian [Eq. (2)] may be rewritten as

$$\begin{aligned} H &= \sum_j \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & -K \end{pmatrix}_j - \Delta \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix}_j \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix}_{j+1} \right] \\ &= \sum_j (T_{z_j} - \Delta T_{x_j} T_{x_{j+1}}), \end{aligned} \tag{6}$$

where we have taken $\Lambda = 1$.

In the particular case of $V_1 = 0$, the three lowest levels are equally spaced, i.e., $K = 2\epsilon$, and Eq. (6) takes the form

$$H = \sum_j (\epsilon M_{z_j} - 2\Delta M_{x_j} M_{x_{j+1}}), \tag{7}$$

where a constant term has been added and M_z and M_x are angular momentum matrices for spin 1. We shall refer to Eq. (7) as the quantum Ising model for spin 1 because of its similarity with the spin-$\frac{1}{2}$ case.

III. SPIN-1 ISING MODEL

To obtain an equally deep three-well potential for one site of the field theory on the lattice, the potential in the classical theory must be

$$U(\phi) = g\phi^2\left[(\phi^2 - f)^2 - \frac{1}{g}\right], \tag{8}$$

as it is shown in Fig. 3.

As with the ϕ^4 theory, apart from the constant solutions, there are classical x-dependent solutions which interpolate between the minima. These solutions, which are characteristic of a phase transition, in the present case are

$$\begin{aligned} \phi_1(x) &= \pm \frac{\phi_M(\phi_M^2 - 3\phi_S^2)^{1/2} \tanh\{\phi_M[6c(\phi_M^2 - \phi_S^2)]^{1/2}(x - x_0)\}}{(3(\phi_M^2 - \phi_S^2) - 2\phi_M^2 \tanh^2\{\phi_M[6c(\phi_M^2 - \phi_S^2)]^{1/2}(x - x_0)\})^{1/2}}, \\ \phi_2(x) &= \pm|\phi_1(x)|, \end{aligned} \tag{9}$$

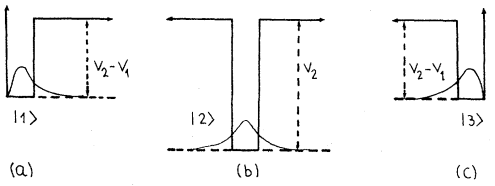

FIG. 2. Basis used for the solution of the three-well problem.

where

$$c=-\frac{U(\phi_M)}{\phi_M{}^4(\phi_M{}^2-3\phi_S{}^2)}.$$

The form of these functions is shown in Fig. 4, where it is apparent that $\phi_1(x)$ are the same kind of solutions as the kinks of the ϕ^4 theory whereas $\phi_2(x)$, which are not acceptable in a continuous theory because of the discontinuity in the derivative, become meaningful on the lattice and are easily understood in the spin-1 approximation.

In fact, taking the model [Eq. (7)], it is clear that in the limit $\Delta\to 0$ the ground state is unique (all spins down) and the degenerate excited states correspond to one spin with zero projection. Therefore, the perturbative treatment gives a band with the dispersion relation

$$E=-(2N+1)\epsilon+\epsilon-2\Delta\cos k \tag{10}$$

for a $(2N+1)$-site chain, which is the same as the one[1] for spin-$\frac{1}{2}$.

On the other side when $\epsilon\to 0$, the ground state is doubly degenerate (all spins aligned along $+x$ or $-x$), and the degenerate first-excited subspace is richer than that for spin $\frac{1}{2}$ due to a larger family of states, i.e., $\rightarrow\rightarrow\leftarrow\leftarrow, \rightarrow\times\leftarrow\leftarrow, \rightarrow\times\rightarrow\rightarrow$. The arrows indicate eigenstates of M_x with eigenvalues ± 1 and $\times$ that of zero eigenvalue. The first two types correspond to solution ϕ_1 and the last to ϕ_2. The perturbation treatment gives rise to an energy band, due to two independent sets of linear combinations of the first two types of states:

$$E=-4N\Delta+2\Delta\pm\sqrt{2}\epsilon\cos\tfrac{1}{2}k\,. \tag{11}$$

Equation (11) is different from the band corresponding to spin $\frac{1}{2}$ which is of the same form as Eq. (10). The common feature is, however, that the energy gap between the ground level and the band decreases for increasing ϵ making easier the production of these topological kink states which tend to disorder the phase. On the other hand, the last type of states, which are not topological kinks, do not couple to the rest and are not easily excited, giving no contribution to the disorder.

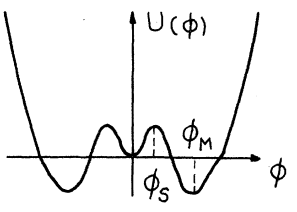

FIG. 3. Potential needed to have equal-depth wells in the lattice version.

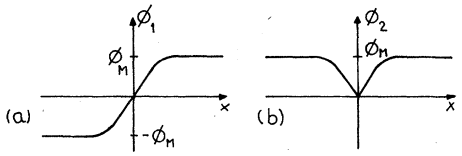

FIG. 4. Kink solutions for the potential of Fig. 3.

It is clear that the two extreme situations $\Delta\to 0$ and $\epsilon\to 0$ correspond to two phases, and a transition must occur for a particular value of ϵ/Δ which from the perturbative argument may be expected to be not far from two. A complete treatment of this phase transition will be given in Sec. V.

We conclude the present section with a mean-field approach which will be applied to the general case in Sec. IV. By this approximation we mean that given a Hamiltonian of the form

$$H=\sum_j(H_{0_j}-\delta O_jO_{j+1})\,, \tag{12}$$

we propose a vector $\prod_j|\Psi\rangle_j$ with the same state $|\Psi\rangle$ for every site and minimize the energy expectation value

$$\frac{\langle\Psi|H_0|\Psi\rangle}{\langle\Psi|\Psi\rangle}-\frac{\delta\langle\Psi|O|\Psi\rangle^2}{\langle\Psi|\Psi\rangle^2}$$

with respect to $|\Psi\rangle$. This gives the Schrödinger-type equation[7]

$$(H_0-2\delta sO)|\Psi\rangle=E|\Psi\rangle \tag{13}$$

with the consistency condition $s=\langle\Psi|O|\Psi\rangle/\langle\Psi|\Psi\rangle$.

In the present case we must replace Eq. (7) by

$$H_{MF}=\epsilon M_z-4s\Delta M_x=(\epsilon^2+16s^2\Delta^2)^{1/2}M_{z'}\,, \tag{14}$$

where $M_{z'}$ is the angular momentum component along an axis z' rotated with respect to the original one. By self-consistency the expectation value of M_x in the ground state, which corresponds to $M_{z'}=-1$, must be equal to s giving the condition

$$s=\frac{4s\Delta}{(\epsilon^2+16s^2\Delta^2)^{1/2}}\,, \tag{15}$$

with the two solutions $s=0$, $(1-\epsilon^2/16\Delta^2)^{1/2}$. For $\epsilon<4\Delta$ the latter is possible and gives rise to the

lowest energy. For $\epsilon > 4\Delta$ only the former is possible. Therefore, there is a critical point at $\epsilon/\Delta = 4$ where the system undergoes a second-order phase transition since s vanishes continuously.

The same critical ratio of parameters is obtained with this method for spin-$\frac{1}{2}$ where the exact result is known[8] to be $\epsilon/\Delta = 2$.

IV. MEAN-FIELD CALCULATION FOR THE GENERAL CASE

Our purpose in this section is to describe the phase transitions of first and second order, which we have anticipated as being contained in the general case of Eq. (1), using the mean-field approximation.

We start performing a unitary transformation which leads the three-level Hamiltonian [Eq. (6)] to the form

$$\tilde{H} = \sum_j \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_j + \tilde{K} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_j - \tilde{\Delta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}_j \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{j+1} \right], \tag{16}$$

where the energy has been rescaled and

$$\tilde{K} = \frac{\sqrt{2}(K - 2\epsilon)}{[\epsilon(K-\epsilon)]^{1/2}}, \quad \tilde{\Delta} = \frac{\sqrt{2}\Delta(1+\alpha^2)}{[\epsilon(K-\epsilon)]^{1/2}}.$$

The mean-field replacement [Eq. (13)] allows us to diagonalize Eq. (16), giving the equation for the eigenvalues γ

$$\gamma(\tilde{K} + 2s\tilde{\Delta} - \gamma)(\tilde{K} - 2s\tilde{\Delta} - \gamma) - 2(\gamma - \tilde{K}) = 0, \tag{17}$$

with the consistency equation for the lowest eigenvalue

$$s = \frac{8(\tilde{K} - \gamma_{\min})\tilde{\Delta}s}{2[(\tilde{K} - \gamma_{\min})^2 + 4\tilde{\Delta}^2 s^2] + [(\tilde{K} - \gamma_{\min})^2 - 4\tilde{\Delta}^2 s^2]^2}. \tag{18}$$

A trivial solution of Eq. (18) is $s = 0$. To look for other solutions one must know $\gamma_{\min}$. Instead of solving Eq. (17) for the general case, we limit ourselves to the region of small s which will correspond to the vicinity of a critical point of second order or a tricritical point, if the first-order transition can also occur.

We expand therefore $\gamma_{\min} = \gamma_0 + as^2 + bs^4$, γ_0 being the lowest solution of Eq. (17) with $s = 0$. Replacing this ansatz in Eq. (17) and solving independently for the s^2 and s^4 terms one gets

$$a = \frac{4\gamma_0\tilde{\Delta}^2}{3\gamma_0^2 - 4\gamma_0\tilde{K} + \tilde{K}^2 - 2},$$
$$b = \frac{\tilde{\Delta}^4}{\gamma_0}\left(\frac{a}{\tilde{\Delta}^2}\right)^2\left(1 + \frac{\tilde{K}a}{2\tilde{\Delta}^2} + \frac{3\gamma_0 a}{4\tilde{\Delta}^2}\right). \tag{19}$$

It can be numerically verified that $a, b < 0$, giving as lowest-energy solution the one with nonvanishing s. For this case Eq. (18) may be put in the general form $Q(s) = A + Bs^2 + Cs^4 = 0$ with $C > 0$. The possibilities for $Q(s)$ are shown in Fig. 5. For case (a) the only possible solution of Eq. (18) is $s = 0$ (disorderd phase). Case (b) gives a nonzero value of s (ordered phase) and the fact that $s \to 0$ continuously with $A \to 0$ indicates a second-order transition, the critical relation among parameters being given by $A = 0$ for $B > 0$, i.e.,

$$\tilde{\Delta} = \frac{\tilde{K} - \gamma_0}{4}\left[1 + \frac{(\tilde{K} - \gamma_0)^2}{2}\right]. \tag{20}$$

Cases (c) and (d), where $B < 0$, correspond to disordered and ordered phases, respectively, with a first-order phase transition between them.

The tricritical point is determined therefore by the simultaneous conditions $A = 0$ [Eq. (20)] and $B = 0$:

$$2\tilde{\Delta}^2[1 - (\tilde{K} - \gamma_0)^2] - (\tilde{K} - \gamma_0)a - (\tilde{K} - \gamma_0)^3 + 2a\tilde{\Delta} = 0. \tag{21}$$

Equations (20) and (21), together with the explicit solution γ_0, allow us to obtain the numerical values of $\tilde{K}$ and $\tilde{\Delta}$. In terms of the original parameters the tricritical point is characterized by

$$\epsilon/K = 0.38, \quad \Delta/K = 0.26. \tag{22}$$

The expectation value of

$$S_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the lowest-energy state for the tricritical parameters gives

$$\langle S_z^2 \rangle = \frac{2}{2 + (\tilde{K} - \gamma_0)^2} = 0.38. \tag{23}$$

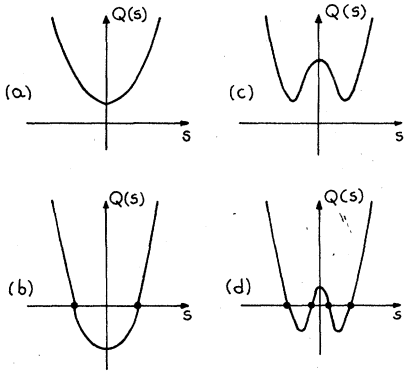

FIG. 5. Possible solutions for the mean field s.

This result is independent on the number of dimensions since $\tilde{\Delta}$ does not appear. Referring to the model of He^3-He^4 mixture, the concentration of He^3 is predicted to be $x = 1 - \langle S_z^2 \rangle = 0.62$, to be compared with the mean-field calculation for a classical model[5] $x = 0.67$ which agrees with the experimental result.[9]

V. RENORMALIZATION-GROUP TECHNIQUE

We wish now to use the renormalization-group method in the block-spin version. There are two procedures to build the block spins in $d = 1$ dimensions. One of them is to take the Hamiltonian of two sites and their link,[1] the other is to include only one site and one link.[6] It has been shown that the latter gives better results for the critical ratio of parameters of the spin-$\frac{1}{2}$ Ising model.

When one applies the method to the spin-1 Hamiltonian [Eq. (7)], it turns out that in keeping the three lowest energy levels of the block, an iterative procedure cannot be built. This is because these three levels are not equally spaced and the interaction between blocks is not given by the angular momentum matrices M_x. An additional problem emerges if one uses the recipe of Ref. 1 since the third and fourth levels exchange their role according to the value of ϵ/Δ; this difficulty is not present with the prescription of Ref. 6.

Therefore, we apply the method of Ref. 6 to the Hamiltonian corresponding to a general three-well potential [Eq. (6)]. The one-site one-link Hamiltonian is (see Fig. 6)

$$H_b = T_{x_2} - \Delta T_{x_1} T_{x_2} . \tag{24}$$

It is convenient to express H_b in the following basis:

$$|1\rangle = \frac{\alpha|\uparrow\uparrow\rangle - |\downarrow\uparrow\rangle}{\eta}, \quad |2\rangle = \frac{|\uparrow\downarrow\rangle - \alpha|\uparrow\uparrow\rangle}{\eta}, \quad |3\rangle = \frac{\alpha|\uparrow 0\rangle - |\downarrow 0\rangle}{\eta},$$
$$|4\rangle = \frac{|\uparrow\uparrow\rangle + \alpha|\downarrow\uparrow\rangle}{\eta}, \quad |5\rangle = \frac{\alpha|\uparrow\downarrow\rangle + |\uparrow\uparrow\rangle}{\eta}, \quad |6\rangle = |00\rangle, \tag{25}$$
$$|7\rangle = \frac{|\downarrow 0\rangle + \alpha|\uparrow 0\rangle}{\eta}, \quad |8\rangle = |0\uparrow\rangle, \quad |9\rangle = |0\downarrow\rangle,$$

where $\eta = (1+\alpha^2)^{1/2}$ and

$$M_z(|\uparrow\rangle, |\downarrow\rangle, |0\rangle) = (|\uparrow\rangle, -|\downarrow\rangle, 0).$$

The states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are eigenvectors of H_b, whereas $|4\rangle$, $|5\rangle$, and $|6\rangle$ form an invariant subspace as well as $|7\rangle$, $|8\rangle$, and $|9\rangle$, giving the same eigenvalue equation

$$E_b^3 + E_b^2(K+\epsilon) + E_b(K\epsilon - \Delta^2\eta^4) - \Delta^2\eta^2 K = 0 . \tag{26}$$

It has been numerically verified for all values of ϵ, Δ, and K that the lowest three levels are the two degenerate states

$$|1'\rangle = N[\Delta\eta(K+E)|4\rangle + \Delta\eta\alpha E|5\rangle - (K+E)E|6\rangle],$$
$$|3'\rangle = N[(K+E)E|7\rangle - \Delta\eta(K+E)|8\rangle - \Delta\eta\alpha E|9\rangle],$$

where

$$N = [(K+E)^2(E^2 + \Delta^2\eta^2) + \Delta^2\eta^2\alpha^2E^2]^{1/2},$$

corresponding to the lowest solution E of Eq. (26) and the eigenstate $|2'\rangle = |2\rangle$ with eigenvalue $-K$.

Diagonalizing the site -1 energy T_{x_1} in this three-dimensional subspace, the above degeneracy is removed giving the new eigenvalues

$$\xi^{\pm} = -\frac{(2K-E)}{2} \pm \tfrac{1}{2}\{[2(K-\epsilon)+E]^2 + 4\beta^2\}^{1/2},$$
$$\xi^0 = -\frac{K\alpha^2}{\eta^2} + E , \tag{27}$$

where $\beta = -NK\alpha^2\Delta E/\eta$ and the eigenvectors

$$|\xi^{\pm}\rangle = N^{\pm}\left[\beta|1'\rangle + \left(\frac{K\alpha^2}{\eta^2} - E + \xi^{\pm}\right)|2'\rangle\right],$$
$$|\xi^0\rangle = |3'\rangle , \tag{28}$$

respectively, where

$$N^{\pm} = \left[\beta^2 + \left(\frac{K\alpha^2}{\eta^2} - E + \xi^{\pm}\right)^2\right]^{-1/2} .$$

To calculate the interaction between blocks we must evaluate T_{x_1} in the basis of $|\xi^+\rangle$ $|\xi^0\rangle$ and $|\xi^-\rangle$ of one block and T_{x_2} in the corresponding basis of the other. The result for the new block (see Fig. 6) is

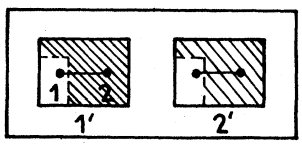

FIG. 6. Block spin. The shaded area indicates that only the energy of site 2 and the link are kept.

$$H_b' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon' & 0 \\ 0 & 0 & -K' \end{pmatrix}_{2'} - \Delta' \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \alpha' \\ 0 & \alpha' & 0 \end{pmatrix}_{1'} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \alpha' \\ 0 & \alpha' & 0 \end{pmatrix}_{2'}, \tag{29}$$

where

$$\begin{aligned} \epsilon' &= \xi^+ - \xi^0 \\ &= -\tfrac{1}{2}[2(K-\epsilon)+E] \\ &\quad + \tfrac{1}{2}\{[2(K-\epsilon)+E]^2 + 4\beta^2\}^{1/2}, \\ K' &= \xi^+ - \xi^- = \{[2(K-\epsilon)+E]^2 + 4\beta^2\}^{1/2}, \\ \Delta' &= -2\Delta^2 (N^+)^2 \beta^2 N^2 \eta^2 E(K+E)(K+E\eta^2), \\ \alpha' &= N^-/N^+ = [\epsilon'/(K'-\epsilon')]^{1/2}. \end{aligned} \tag{30}$$

Equation 30 is the renormalization-group recursion formula, which gives ϵ', K', Δ' in terms of ϵ, K, Δ. The parameter α' is not independent and is related to the others according to the same law obeyed by α.

One sees from Eqs. (24) and (29) that the form of the Hamiltonian is preserved by the renormalization-group transformations. Therefore, Eq. (30) is the basis for a numerical computation which allows to draw the phase-transition diagram of Fig. 7. Since the relevant parameters are Δ/K and ϵ/K, it turns out that starting from any point in region I one gets a fixed point $\epsilon = \Delta = 0$ which corresponds to an effective configuration where the first and second excited levels become degenerate. In the same way, starting from region II the iterative procedure leads to a point with ϵ/K between 0.5 and 1 and $\Delta = 0$; this is equivalent to the spin-$\frac{1}{2}$ Ising disordered phase. Finally, starting from any point of region III one ends up to a point with $\epsilon/K = 1$ and $\Delta \neq 0$, which is the ordered phase for the spin-$\frac{1}{2}$ Ising model.

The dashed curve corresponds to a second-order phase transition analogous to the one of the spin-$\frac{1}{2}$ case. The solid curve represents a phase transition of first order since a small modification of the initial conditions changes the output from an excited degenerate doublet to a ground-state degenerate doublet. The dashed-dotted curve does not describe a phase transition in the traditional sense of the word since the ground state is nondegenerate and the corresponding order parameter is zero on both sides of this line. But on the other hand, crossing this line a discontinuous expectation value for the first excited state appears so that, defining in this way an order parameter, we still have a first-order transition. The three

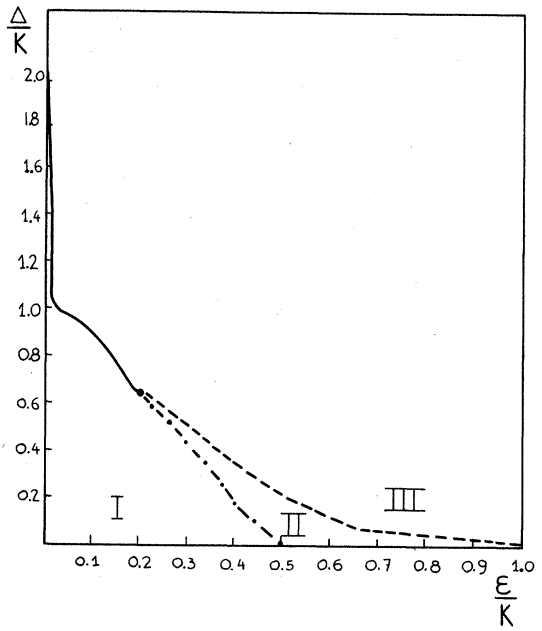

FIG. 7. Phase diagram obtained with block-spin method. The dashed curve denotes a second-order transition. The solid curve corresponds to a first-order transition from a degenerate ground state to a degenerate excited one. The dashed-dotted curve shows a discontinuous transition from nondegenerate states to a degenerate excited state. The tricritical point is represented by a large dot.

curves join at the tricritical point

$$\epsilon/K \simeq 0.2, \quad \Delta/K \simeq 0.65. \tag{31}$$

From this analysis it is clear that for the spin-1 Ising model (by which we mean $\epsilon/K = 0.5$) only the second-order transition is available and the critical point $(\epsilon/\Delta)_c = 2.5$ is only slightly different from that of the spin-$\frac{1}{2}$ case. This value cannot be strictly compared with the exact critical ratio 2 for the spin-$\frac{1}{2}$ case because our treatment of three-level systems introduces a small error. In fact, one can check that for $\epsilon/K \gtrsim 0.8$ the method of Ref. 1 modifies ϵ' less than 4% (as is reasonable since the effect of the third level must be small), whereas with the prescription of Ref. 6 the change is $\lesssim 18\%$. As a result, for $\epsilon/K \approx 1$ instead of the exact critical ratio for the spin-$\frac{1}{2}$ case $\epsilon/\Delta_{1/2} = 2$ our transition occurs at $(K-\epsilon)/\Delta\alpha^2 \approx 1.6$.

Even though our method is therefore approximate but better than the mean-field treatment, we expect that the above-described phase-transition pattern is general. In particular, we see from Fig. 7 that the first-order transition, with the sense we have given to it, may appear only if the potential $\mathfrak{U}(\phi_j)$ shows lateral wells less deep than the

central one.

Regarding the tricritical point, we may evaluate again the concentration of He^3. This is related to the expectation value of

$$S_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the lowest eigenstate of Eq. (16), or alternatively to the ground-state expectation value of this matrix transformed to the basis in which Eq. (6) is written. This corresponds to calculate

$$1-x = \left\langle \xi^- \left| \begin{pmatrix} 1/(\alpha^2+1) & 0 & \alpha/(\alpha^2+1) \\ 0 & 1 & 0 \\ \alpha/(\alpha^2+1) & 0 & \alpha^2/(\alpha^2+1) \end{pmatrix} \right| \xi^- \right\rangle .$$

Since the tricritical point is a fixed one, it is allowed to compute this expectation value in the first step. Because of the asymmetry in sites of our treatment, the result is different if one computes the expectation value of the matrix for site 2 or 1. Using Eq. (28), the two expressions are

$$\begin{aligned} 1-x_2 = (N^-)^2 \Big\{ & \beta^2 N^2 [\Delta^2 (K+E)^2 + 2\Delta^2 \alpha^2 E(K+E) \\ & + \Delta^2 E^2 \alpha^4 + E^2 (K+E)^2] \\ & + \left(\frac{K\alpha^2}{\eta^2} - E_b + \xi^- \right)^2 \frac{\alpha^2}{\alpha^2+1} \Big\}, \\ 1-x_1 = (N^-)^2 \beta^2 N^2 & [\Delta^2 \eta^2 (K+E)^2 + \Delta^2 \eta^2 E^2 \alpha^2 + E^2 (K+E)^2] , \end{aligned} \tag{32}$$

which, for the tricritical values in Eq. (31), give $x_2 = 0.69$ and $x_1 = 0.75$. One may observe that these results are similar to that coming from the mean-field theory even though the tricritical parameters Eq. (22) and (31) are quite different and the influence of the dimensionality of the renormalization-group treatment cannot be anticipated.

VI. CONCLUSIONS

We have applied the nonperturbative methods of the mean-field approximation and renormalization group to the ϕ^6 field theory finding a second-order phase transition similar to that of the ϕ^4 theory. In particular, with the three-level approximation, we have seen that the critical point is slightly affected by the presence of the third level even if it is close to the second one. Therefore, one gains confidence in the spin-$\frac{1}{2}$ approximation of models such as Reggeon field theory where the third level is not too far from the second. One the other hand, we have seen that the richness of the phase diagram increases since in the ϕ^6 theory a critical line of first order appears joining the second-order curve at a tricritical point.

The mean-field treatment, as in the case of spin-$\frac{1}{2}$, gives a large error in the position of the second-order critical line. The use of the block-spin method where a block is formed by one site and one link has the virtue of preserving the form of the Hamiltonian, giving for all range of parameters the same set of three lowest block levels. The second-order critical parameters are more accurate than in the mean-field approximation though not as good as in the spin-$\frac{1}{2}$ case, probably because of the lack of self-duality,[4] a property which was at the basis of the method of Ref. 6. It is interesting to remark that the ground-state expectation value $\langle S_z^2 \rangle$ seems to be rather independent of the methods used giving, both with the mean-field and the renormalization-group technique, good agreement with the experimental value of the He^3 concentration in the He^3-He^4 mixture.

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