Three-loop charge renormalization effects due to quartic scalar self-interactions

Thomas Curtright
The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637
(Received 2 July 1979)

Dimensionally regularized dispersion theory is used to compute the $O(\hbar^3 g^3 f^2)$ contribution to the charge renormalization function β_g, where g is a gauge field coupling and f is a quartic (pseudo) scalar self-coupling. Some motivations for and systematics of the calculation are discussed. Special attention is given to an $N=4$ globally supersymmetric gauge theory.

Higher-order corrections in perturbation theory are very interesting in two sets of circumstances. First, if experimental accuracy exceeds or threatens to improve beyond the present theoretical uncertainty in a perturbatively computable observable, one naturally wants to calculate the smaller, higher-order effects to check for discrepancies between theory and experiment. Quantum electrodynamics provides classic examples of this first situation,[1] especially in the case of anomalous magnetic moments.

Second, one may find effects occurring in low-order calculations for an unrealistic model theory which intriguingly suggest some fundamental principle is at work, e.g., a symmetry, but which do not clearly indicate what the principle is. Higher-order computations here can either show the effects to be spurious, if they are, or provide some clues for discovering the principle involved. An example of this second situation is provided by the charge renormalization function, β_g, for an $N=4$ globally supersymmetric non-Abelian gauge theory.[2] For this model, β_g vanishes in both the one- and two-loop approximation[3] yielding a conformally covariant theory with nonzero anomalous dimensions for the fields.

While this scale covariance may be sheer coincidence to $O(\hbar^2)$, it seems more natural to assume that a fundamental principle is responsible and to conjecture that β_g vanishes to all orders. Unfortunately, a rigorous argument proving such a conjecture is not yet known. Also, the full structure of the model in Ref. 2 does not really come into play when only two-loop effects are considered. For example, it is well known that quartic (pseudo) scalar self-couplings do not contribute to the gauge charge renormalization β_g until three-loop corrections are taken into account.

In this paper the problem of computing β_g to three loops in a general non-Abelian gauge theory is briefly and incompletely considered. The simplest gauge-invariant three-loop contribution to β_g is computed to illustrate what we believe is the most efficient analytic method to solve the complete problem: dimensionally regularized dispersion theory. This method is employed to determine the lowest-order contribution of a quartic (pseudo) scalar self-coupling, f, to the gauge coupling-constant renormalization. The contribution is of order $\hbar^3 g^3 f^2$.

Although the scale covariance noted in Ref. 3 is the major motivation for this analysis, the $O(\hbar^3 g^3 f^2)$ contribution to β_g will be given here for an arbitrary set of (pseudo) scalar field parameters and not just for those of the model defined in Ref. 2. Thus, the result may be of some interest for other problems, e.g., in analyzing cases of extreme ultraviolet behavior ($q^2 \gg m^2_{\rm Higgs}$) for electroweak theories with "large" Higgs self-coupling.

The rest of the paper is organized as follows. First we define our notation and give the $O(\hbar^3 g^3 f^2)$ result for β_g. Then we indicate how the dispersion-relation technique was used to obtain the result, and we briefly compare this technique with two others: Straightforward Feynman parametrization methods and hyperspherical polynomial expansions. Finally, we close with some remarks on the remaining amount of work required to completely determine β_g to three loops.

Let $\{\phi_m\}$ represent a set of real scalar and/or pseudoscalar fields with covariant derivatives

$$D_\mu \phi_m = \partial_\mu \phi_m + ig V^a_\mu (T^a)_{mn} \phi_n \,, \tag{1}$$

where V^a_μ is the vector gauge field and $(T^a)_{mn} = -(T^a)_{nm}$ is the real representation matrix for ϕ_m, possibly reducible. We assume the (pseudo) scalars interact quartically as follows:

$$\mathcal{L}_{\rm quartic} = -\tfrac{1}{24} f F_{jklm} \phi_j \phi_k \phi_l \phi_m \,. \tag{2}$$

The tensor F_{jklm} is totally symmetric and satisfies a linear constraint, with coefficients T^a, which follows from assuming $\mathcal{L}_{\rm quartic}$ is gauge invariant:

$$(T^a)_{x\underline{j}} F_{x\underline{klm}} = 0 \,. \tag{3}$$

Underlined indices are totally symmetrized and repeated indices are summed. Using $(T^a)_{mn}$ and

F_{jklm} we define group invariants $F(\phi)$, $T(\phi)$, and $C(\phi)$ as follows:

$$F(\phi)\delta_{mn}=F_{mxyz}F_{nxyz}\,,$$
$$T(\phi)\delta^{ab}=(T^a)_{xy}(T^b)_{yx}\,, \qquad (4)$$
$$C(\phi)\delta_{mn}=(T^a)_{mx}(T^a)_{xn}\,.$$

With these definitions we now present the $O(\hbar^3 g^3 f^2)$ contribution to the gauge coupling-constant renormalization function.

Using the minimal-pole-part subtraction scheme[4] so that field masses may be ignored, we find the general result

$$\beta_g(\text{quartic})=\frac{1}{24}\left(\frac{\hbar}{16\pi^2}\right)^3 g^3 f^2 T(\phi)F(\phi)\,. \qquad (5)$$

Since $T(\phi)$ and $F(\phi)$ are positive, the effect of the quartic interactions alone is to cause an increase in g^2 as the mass scale increases [cf. $M(d/dM)g^2(M) = 2g\beta_g$], an effect which could easily have been anticipated.

The general result in Eq. (5) may be restricted to the special case of the $N=4$ supersymmetric theory (SST) for purposes of comparison. For that theory, $\{\phi_m\}=\{A^a_m, B^a_m:\ m=1,2,3\}$ with A^a and B^a scalars and pesudoscalars, respectively. Both A and B are in the adjoint representation of the gauge group. Their quartic interaction is given by

$$\mathcal{L}_{\text{quartic}}\ (N=4\ \text{SST})$$
$$=-\tfrac{1}{4}g^2 f_{xab}f_{xcd}\delta_{km}\delta_{ln}$$
$$\times\,(A^a_k A^b_l A^c_m A^d_n + 2A^a_k B^b_l A^c_m B^d_n + B^a_k B^b_l B^c_m B^d_n)\,. \qquad (6)$$

The f_{abc} are totally antisymmetric, real structure constants of the (arbitrary) gauge group. Two of the pecularities of the $N=4$ SST are that the quartic coupling constant is fixed to equal g^2, the gauge coupling squared, and that the Yukawa coupling present in the model is also related to g. Consequently, it is obvious that one must also compute the three-loop corrections from gauge and Yukawa interactions before any final statement can be made about β_g to $O(\hbar^3)$. Nevertheless, we may evaluate Eq. (5) for the $N=4$ model to determine the importance of the quartic couplings alone. Rewriting Eq. (6) in the form of Eq. (2) and evaluating the corresponding invariants (we set $f=g^2$) gives

$$T(\phi)=6C(V)\ \text{and}\ F(\phi)=45C^2(V)\,, \qquad (7)$$

where the gauge group invariant $C(V)$ is defined by

$$f_{xya}f_{xyb}\equiv C(V)\delta_{ab}\,. \qquad (8)$$

Thus we have

$$\beta_g(N=4\ \text{SST, quartic})=\frac{45}{4}\left(\frac{\hbar}{16\pi^2}\right)^3 g^7 C^3(V)\,. \qquad (9)$$

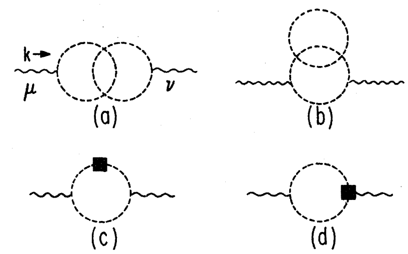

FIG. 1. Lowest-order contributions to the gauge field self-energy involving quartic (pseudo) scalar interactions.

Next we shall discuss the method by which (5) was obtained.

The number of Feynman diagrams one must consider to arrive at Eq. (5) is minimized by working in the noncovariant gauge $n^\mu V^a_\mu=0$.[5] For this gauge the Ward identities relate the vector wave-function renormalization to the charge renormalization, $Z_g=Z_V^{-1/2}$. We may thus compute the lth-loop pole part[6] of Z_g by calculating only the vector self-energy in this noncovariant gauge. The pole part $(Z_{g,l})$ then determines the lth-loop contribution to $\beta_g(\beta_{g,l})$ in the minimal subtraction scheme through the relation $\beta_{g,l}=2glZ_{g,l}$. (This relation follows from the analysis in Ref. 4.)

The three-loop contributions to the vector self-energy which involve the quartic coupling (2) are shown in Fig. 1. Diagrams (c) and (d) in Fig. 1 include two-loop counterterms (the solid boxes). The $O(\hbar^3 g^3 f^2)$ portion of (c) cancels that of (d), however, so we may neglect these graphs. The sum of the remaining two graphs is obviously gauge invariant. Diagram (b) is, of course, just an $O(f^2)$ (pseudo) scalar self-energy folded in with the familiar one-loop $O(g^2)$ (pseudo) scalar contribution to the vacuum polarization. It is easily computed for massless particles since the internal-momentum integrations are "nested" and may be performed sequentially. Diagram (a) in Fig. 1 requires some ingenuity to evaluate, however, due to the nontrivial momentum-transfer dependence occurring in the internal (pseudo) scalar-(pseudo) scalar scattering. This dependence prohibits a straightforward sequential performance of the integrations. It is for diagram (a) that we find dimensionally regularized dispersion theory (DRDT) to be quite useful.

DRDT was introduced long ago when dimensional regularization was first invented. (E.g., cf. Ref. 7. This paper also gives reference to earlier work on dispersion theory.) Nevertheless, we are not aware of any higher-loop calculations of β_g employing the method. To evaluate a Lorentz scalar diagram depending on one invariant, k^2, using DRDT, we employ the representation

$$D(k^2)=\frac{i}{2\pi}\int ds\,\frac{2i\ \text{Disc}[D(s)]}{k^2-s+i0}\,. \qquad (10)$$

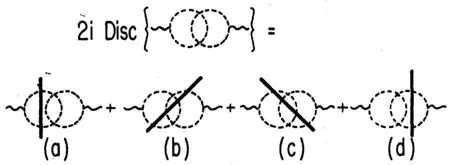

FIG. 2. All possible dispersion cuts contributing to the discontinuity of the diagram in Fig. 1(a).

The discontinuity here is obtained by "cutting" the diagram in all admissible ways, replacing cut propagators:

$$1/[(p)^2+i0] \to -2\pi i\delta(p^2)\theta(p_0)\,,$$

and evaluating the resulting weighted phase-space integrals. For diagram (a) in Fig. 1, there are four possible cuttings as shown in Fig. 2. [Cuts (a) and (d) give the same contribution in Eq. (10), as do cuts (b) and (c).] Each of these discontinuities is computed in $D=4-2\epsilon$ dimensions, and for a range of ϵ we obtain finite results even after the dispersion integral in Eq. (10) is evaluated. The final result is then continued back to $D=4$ where the pole parts arise in the usual way.

To reduce the diagrams in Fig. 1 to the Lorentz scalar form of Eq. (10), we must take either the trace D^μ_μ or the double divergence $k^\mu D_{\mu\nu} k^\nu$. Independent computation of both these scalars allows the reconstruction of the full self-energy tensor. To illustrate the use of DRDT, we explicitly consider only D^μ_μ. Note, however, that this trace is more difficult to calculate than the double divergence.

Since all particles in the diagrams are massless, simple dimensional analysis tells us the discontinuities are of the form

$$2i\,\mathrm{Disc}D^\mu_\mu(s) = i d(\epsilon)(s/M^2)^{-3\epsilon} s\,\theta(s)\,, \qquad (11)$$

where M^2 is the arbitrary renormalization-group mass scale. We may therefore perform the integration called for in Eq. (10) to obtain

$$D(k^2) = \frac{1}{2\pi} d(\epsilon)(-k^2)\left(\frac{-k^2}{M^2}\right)^{-3\epsilon}\Gamma(2-3\epsilon)\Gamma(-1+3\epsilon)\,. \qquad (12)$$

The calculation thus reduces to computing $d(\epsilon)$, the "discontinuity coefficient."

The result of the dispersion integration with the discontinuity in Eq. (11) is to produce a simple pole in ϵ as evident in the expansion

$$\Gamma(-1+3\epsilon) = -\frac{1}{3\epsilon} - 1 + \gamma_{\text{Euler}} + O(\epsilon)\,.$$

The diagrams in Figs. 1(a), (1b), may have more singular behavior as $\epsilon \to 0$ only if the phase-space integral involved in computing $\mathrm{Disc}D(k^2)$ is itself singular in the four-dimensional limit. This additional singularity can occur only if the cut diagrams contain closed loops which are not cut. From Fig. 2 we see that the (a) and (d) cuts will indeed have such an additional singularity, while the (b) and (c) cuts will be finite as $\epsilon \to 0$. Since we are only interested in the pole parts of the entire diagrams to compute β_g using the minimal subtraction scheme, it suffices to evaluate cuts (b) and (c) directly in four dimensions, i.e., determine $d(0)$. On the other hand, we need to determine both the pole part and finite part of cuts (a) and (d). [Actually, the cuts in Figs. 2(a), 2(d) are quite simple to work out, even though we do need two terms in their Laurent expansions.]

The four-dimensional cuts in Fig. 2(b), 2(c) lead to the following two weighted phase-space integrals $[\delta_+(p)=\delta(p^2)\theta(p_0)]$:

$$\int \frac{d^4p\,d^4q\,d^4r}{(2\pi)^{12}} \frac{\delta_+(p)\delta_+(k-q)\delta_+(q-r)\delta_+(r-p)}{(k-p)^2 q^2} \begin{Bmatrix} k^2 \\ p\cdot q \end{Bmatrix} = \frac{k^2\theta(k^2)\theta(k_0)}{(16\pi^2)^4} \begin{Bmatrix} \dfrac{2}{\pi}[\zeta(2)-1] & (13\text{a}) \\ \dfrac{1}{2\pi}[\zeta(2)-\frac{5}{4}] & (13\text{b}) \end{Bmatrix}\,.$$

Their evaluation involves the usual judicious choice of coordinates. Assuagingly, (13a) and (13b) appear in the (b) and (c) cuts with a relative factor of -4 so the transcendental $\zeta(2)=\pi^2/6$ cancels out.

Using (13a), (13b) we immediately determine the four-dimensional discontinuity coefficient corresponding to the cut in Fig. 2(b) for $D^\mu_\mu(k)$ [we temporarily suppress coupling constants, group invariants, and combinatoric weights, cf. Eq. (16)]:

$$d_{2(\text{b})}(0) = \frac{1}{2} i\pi\left(\frac{\hbar}{16\pi^2}\right)^3\,. \qquad (14)$$

A straightforward calculation also gives the singular discontinuity coefficient corresponding to the cut in Fig. 2(a):

$$d_{2(\text{a})}(\epsilon) = \frac{1}{2} i\pi\left(\frac{\hbar}{16\pi^2}\right)^3\left(\frac{1}{\epsilon} + \frac{17}{2} + O(\epsilon)\right)\,. \qquad (15)$$

The two internal-momentum integrations which must be evaluated to obtain Eq. (15) are both easily done due to the mass-shell constraints on the cut (pseudo) scalar lines. Combining Eqs. (12), (14), and (15) we arrive at the final expression for the singular parts of the trace of Fig. 1(a):

$$D^{\mu}_{\mu}(k)=i\left(\frac{\hbar}{16\pi^2}\right)^3 g^2 f^2 F(\phi)T(\phi)$$
$$\times k^2\left(-\frac{1}{72\epsilon^2}-\frac{19}{144\epsilon}+\frac{\ln(-k^2/M^2)}{24\epsilon}+\text{finite}\right). \tag{16}$$

The trace of Fig. 1(b) is contained in Eq. (19) below.

The double divergence $k^{\mu}D_{\mu\nu}(k)k^{\nu}$ of the diagrams in Fig. 1(a) or 1(b) may be computed either dispersively as above or with equal ease using Feynman parameters. The relevant momentum integral is

$$\int DpDqDr\frac{1}{(k-p)^2(p-q)^2(q-r)^2r^2}=i\left(\frac{\hbar}{16\pi^2}\right)^3 k^4\left(-\frac{1}{36\epsilon}+\text{finite}\right), \tag{17}$$

and allows a simple sequential evaluation of each subintegration.

We will now combine the trace and double-divergence pole-part information for both diagrams (a) and (b) in Fig. 1. As we remarked above, diagram (b) is easily computed due to the nested character of its internal-momentum integrations, so we will forego a detailed discussion of its evaluation.

The final results for the diagrams (a) and (b) in Fig. 1 are as follows:

$$D^{(a)}_{\mu\nu}(k)=i\left(\frac{\hbar}{16\pi^2}\right)^3 g^2 f^2 F(\phi)T(\phi)$$
$$\times\left[(k_\mu k_\nu-g_{\mu\nu}k^2)\left(\frac{1}{216\epsilon^2}+\frac{59/48}{27\epsilon}-\frac{\ln(-k^2/M^2)}{72\epsilon}+\text{finite}\right)+k_\mu k_\nu\left(-\frac{1}{216\epsilon}+\text{finite}\right)\right], \tag{18}$$

$$D^{(b)}_{\mu\nu}(k)=i\left(\frac{\hbar}{16\pi^2}\right)^3 g^2 f^2 F(\phi)T(\phi)$$
$$\times\left[(k_\mu k_\nu-g_{\mu\nu}k^2)\left(-\frac{1}{216\epsilon^2}-\frac{77/48}{27\epsilon}+\frac{\ln(-k^2/M^2)}{72\epsilon}+\text{finite}\right)+k_\mu k_\nu\left(\frac{1}{216\epsilon}+\text{finite}\right)\right]. \tag{19}$$

The dipole terms ($1/\epsilon^2$) cancel when (18) and (19) are added, as do the $\ln(-k^2/M^2)/\epsilon$ terms, thereby explicitly confirming that quartically coupled (pseudo) scalars do not destroy the renormalizability of a Yang-Mills theory. Also note that the sum of Eqs. (18) and (19) is conserved (true not only for the displayed pole parts, but also for the finite pieces we have neglected to show). However, each diagram separately is not conserved.

From (18) and (19) the $O(\hbar^3 g^2 f^2)$ wave-function renormalization can be deduced. The Ward identities then give the $O(\hbar^3 g^2 f^2)$ charge renormalization, as discussed earlier. From this β_g can be obtained. The result is Eq. (5).

Now we shall compare DRDT with other graph evaluation techniques. First consider the diagram in Fig. 1(a) and use the Feynman parametric method. The difficulties with this method occur because a large number (five) of parameters must be introduced and because the momentum-transfer dependence appearing in the internal (pseudo) scalar four-point function makes it very difficult to evaluate more than half the parameter integrals in closed form. One of the two basic momentum integrals encountered is

$$\int DpDqDr[(k-p)^2p^2(p+r)^2(r+q)^2q^2(k-q)^2]^{-1}.$$

The internal r integration may be performed using one Feynman parameter to obtain (dropping an overall momentum-independent factor)

$$I(\epsilon)\left(\frac{-k^2}{M^2}\right)^{-3\epsilon}=\int DpDq\frac{1}{(k-p)^2p^2[(p-q)^2]^{\epsilon}q^2(k-q)^2}. \tag{20}$$

This displays the momentum-transfer dependence mentioned previously.

The remaining two momentum integrals can be converted to a four-parameter Feynman integral. Unfortunately, only one of these parameter integrations is readily carried out in $D=4-2\epsilon$ dimensions. This produces a nontrivial hypergeometric function depending on the remaining three parameters, x, y, and z, and weighted by appropriate powers of x, $(1-x)$, etc. The remaining three-parameter integrations have stubbornly resisted our attempts at any useful simplification. In particular, we have not been able to extract the crucial $1/\epsilon$ term in the Laurent expansion of $I(\epsilon)$ using parametric methods.

The advantages of DRDT techniques here are obvious, especially for massless particles. The representation in Eq. (10) is a one-parameter integral which is always calculable for massless particles [cf. Eqs. (11) and (12)]. The only problem is to determine the phase-space integrals, which require some weighted angular averages.

Since the integrands are on-shell in the phase space, however, these angular integrations are also manageable.

This is more fully appreciated by considering one more momentum integration in Eq. (20), say q:

$$J_\epsilon(p^2/k^2, p\cdot k/k^2)\left(\frac{-k^2}{M^2}\right)^{-2\epsilon} = \int Dq \frac{1}{[(p-q)^2]^\epsilon q^2(k-q)^2}\,. \quad (21)$$

The result is a complicated dimensionless function of two invariant ratios (not a simple Gauss—but rather an Appell—hypergeometric function). This generalized hypergeometric function must then be integrated over all p weighted by the other two propagators in Eq. (20), an integration beyond both our patience and our expertise when using parametric methods. Note, however, that the integral in Eq. (21) is also encountered in the DRDT approach [cf. the cuts in Figs. 2(a), 2(d)], but there only the on-shell form of Eq. (21) enters, i.e., $J_\epsilon(0,\frac{1}{2})$, and no subsequent nontrivial integrals over p^2 and $p\cdot k$ remain. This is a significant simplification.

Next, let us compare DRDT with hyperspherical polynomial expansion methods (HM's). The latter were developed extensively long ago[8] for application in quantum electrodynamics. In fact, the specific task for which HM's were developed was precisely the three-loop calculation of the electromagnetic charge renormalization, β_e, an application which makes the method all the more attractive for the $N=4$ SST. The original $O(\hbar^3)$ calculation of β_e was not fully completed for some time,[9] but this delay was not a necessary consequence of using HM's. These methods are reasonably efficient and have been applied in other higher-order problems (e.g., cf. Refs. 10 and 11, or consult the other applications described in Ref. 1).

The essential features of HM's are as follows. After continuing momentum integrals into Euclidean space, the most difficult parts of the integrations are due to angular dependences induced by unconstrained scalar products of different four vectors. These dependences are a hindrance when they appear in propagators, e.g.,

$$1/(k-q)^2 = 1/[k^2(1-2t\cos\theta+t^2)]\,,$$

where $\cos\theta = \hat{k}\cdot\hat{q}$, $|q| = t|k|$. However, such propagators may be identified as generating functions for the hyperspherical (Gegenbauer) polynomials in four dimensions, C_n, giving a sum

$$\frac{1}{1-2t\cos\theta+t^2} = \sum_{n=0}^{\infty} t^n C_n(\cos\theta)\,.$$

Angular integrations may then be done using orthogonality relations on the three-sphere, e.g.,

$$\int d\Omega(\hat{k}) C_n(\hat{q}\cdot\hat{k}) C_m(\hat{k}\cdot\hat{p}) = 2\pi^2 \delta_{nm} C_n(\hat{q}\cdot\hat{p})/(1+n)\,,$$

and one sum is thus eliminated by δ_{mn}. Finally, all radial momentum integrations involve only simple powers of $|k|$ (for massless theories) and are calculable. The net result when products of different propagators are involved is usually a surviving sum of terms, each term being a simple radial integral. For massless theories in four dimensions, this final sum can often be explicitly done to obtain a closed-form result.

Many of the radial integrations are ultraviolet divergent, however, and must be regularized. For electrodynamics this is easily done using an invariant cutoff in Euclidean space, Λ^2. To compute β one needs the $\ln\Lambda^2$ terms, a point which was exploited in Refs. 8, 9, and 10.

The hyperspherical method, with an invariant cutoff, would probably allow one to determine β_g to $O(\hbar^3)$ for a general non-Abelian theory. A preliminary calculation indicates that the $(\Lambda\to\infty)$ divergent part of the diagram in Fig. 1(a), e.g., can be computed using HM's. The amount of analysis needed to simplify the sums encountered using HM's is comparable to that needed to work out closed forms for the discontinuities encountered using DRDT. However, we believe there is one major advantage of DRDT over HM's: Dimensional regularization.

It is desirable to use a regularization technique which manifestly preserves non-Abelian gauge invariance under radiative corrections. Dimensional regularizaion is the finest such technique for continuum theories. Unfortunately, unlike dispersion-theory methods, HM's are not readily generalizable to D dimensions in momentum space. The problem is that the angular measure in D dimensions is not compatible with both the expansion of momentum-space propagators and the orthogonality relations for the C_n's. This compatibility is achieved for arbitrary D only if one transforms from momentum to position space. Thus we may simultaneously use dimensional regularization to eliminate ultraviolet diverges and HM's to carry out angular integrations only if we evaluate Feynman diagrams in position space. The resulting formalism is quite cumbersome.

Let us make these points more explicit.[12] If the momentum-space propagator is $1/k^2$, the position-space propagator is

$$\int d^Dk\, e^{ik\cdot x}\frac{1}{k^2+i0} = \frac{i\pi^{D/2}2^{D-2}\Gamma(\frac{1}{2}D-1)}{(x^2-i0)^{D/2-1}}\,. \quad (22)$$

Only for $D=4$ does the same $1/a^2$ form occur in both momentum and position space. Continuing to

Euclidean spacetime, we may again identify inverted propagators as generators of hyperspherical polynomials on the $D-1$ sphere[13]:

$$\frac{1}{[(x-y)^2]^{D/2-1}}=\frac{1}{x^{D-2}}\frac{1}{(1-2t\cos\theta+t^2)^{D/2-1}}$$
$$=\frac{1}{x^{D-2}}\sum_{n=0}^{\infty}t^n C_n^{(D/2-1)}(\cos\theta). \quad (23)$$

$C_n^{(1)}$ is to be identified with the previous C_n. These generalized spherical polynomials obey the orthogonality relation (cf. Ref. 13, p. 488)

$$\int d\Omega(\hat{x})C_n^{(D/2-1)}(\hat{y}\cdot\hat{x})C_m^{(D/2-1)}(\hat{x}\cdot\hat{z})$$
$$=\frac{2\pi^{D/2}}{\Gamma(\frac{1}{2}D)}\frac{\delta_{mn}}{1+2n/(D-2)}C_n^{(D/2-1)}(\hat{y}\cdot\hat{z}). \quad (24)$$

The solid angle $d\Omega(\hat{x})$ is that appropriate to the $D-1$ sphere. For functions depending only on the polar angle, we have

$$d\Omega(\hat{x})\to\frac{2\pi^{(D-1)/2}}{\Gamma(\frac{1}{2}(D-1))}[1-\cos^2\theta]^{(D-3)/2}d(\cos\theta). \quad (25)$$

The power of $[1-\cos^2\theta]$ in Eq. (25) is responsible for the aforementioned incompatibility in momentum space.

Given a product of propagators in Euclidean position space,

$$1/[(y-x)^2(x-z)^2]^{D/2-1},$$

the intermediate position (x) may be integrated over by using the HM's as generalized above. Equation (24) permits the elimination of one of the two sums introduced by expanding both propagators as in Eq. (23). Each radial position integral is calculable and finite, at least for a range of D. The final remaining sum may often be expressed in closed form.

For multiloop diagrams, however, we have not found D-dimensional HM's to be very convenient. Without going into explicit examples, let us simply note the following complications. (a) Integrations over intermediate positions produce both ultraviolet and infrared pole parts,[6] in general. The two must be carefully separated. (b) The sums which are encountered are more difficult to express in closed form than those found using four-dimensional HM's due to the additional dependence on D. (c) Extracting the most singular part of a diagram is usually manageable, but nonleading singularities are discouragingly difficult. The farther the Laurent expansion in $(D-4)$ is taken, the worse this problem becomes. (d) The final results must be transformed back into momentum space, a chore which is obviously manageable but bothersome since more algebra is involved.

In summary, it appears that dispersion theory is preferable over HM's for multiloop diagrams if one wishes to have the convenience of manifestly gauge-invariant dimensional regularization. We wish to have this convenience.

In concluding this paper let us discuss the remaining analysis needed to complete the three-loop calculation of β_g. We wish to indicate some labor-saving devices, especially for the $N=4$ SST.

First, consider a general non-Abelian gauge theory with vectors and fermions (in an arbitrary representation), but no (pseudo) scalars. Much labor is saved in computing β_g in such a theory if we exploit the proven renormalizability and the Ward identities, but make no attempt to check them explicitly. We may then use the fact that the vector self-energy $\pi_{\mu\nu}$ is transverse when all diagrams are summed, so we need only compute the trace π^{μ}_{μ} to determine Z_V. Similarly, renormalizability dictates that the pole part of the vector three point function $\Gamma^{abc}_{\mu\nu\lambda}(p,q,r)$ will have the same algebraic structure as in the tree diagram, so Z_g may be determined by computing $f_{abc}p^{\lambda}g^{\mu\nu}\Gamma^{abc}_{\mu\nu\lambda}(p,-p,0)$. Thus β_g can be obtained by considering diagrams which depend on only one invariant, p^2, and for which all indices have been contracted. Further, we may conveniently set $-p^2/M^2=1$ in determining pole parts, and finally, we may similarly work in the radiatively stable, covariant Landau gauge.

The problem now breaks up into two phases: One analytic, the other purely algebraic. The first phase requires the analytic evaluation of pole parts for a complete set of three-loop integrals. As we have emphasized in this paper, DRDT seems advantageous for this phase. The second phase requires the enumeration of all diagrams (with correct combinatoric and group-theoretic weights), the algebraic reduction of the diagrams to the basic set of integrals, and lastly some careful addition. We believe this second phase may be entirely implemented on a machine. We are presently investigating this implementation.

In particular, we have written computer routines to produce the full set of vector self-energy and three-point diagrams by inserting external lines in formal algebraic expressions for vacuum bubbles. For a covariant (ghost-containing) gauge calculation, there are 17 (11) three-loop vacuum diagrams in an arbitrary vector-fermion (pure vector) theory. These lead to 114 (68) topologically distinct vector self-energies. Insertion of one more zero-momentum vector line increases the number of diagrams by a factor ≈5. The further algebraic reduction of these formal expressions to a basic set of integrals is presently under study.

Of course, to determine β_g for the $N=4$ SST, we must also include in the theory (pseudo) scalars in the adjoint representation of the gauge group. The easiest way to do this is to use dimensional reduction techniques, extended to be used in computing radiative corrections. The $N=4$ SST may be compactly formulated as a gauge theory involving only vectors and spinors in ten-dimensional spacetime.[2] The classical four-dimensional theory then materializes if the "zero-mode constraint" $k_M=0$, for $M=4,5,\ldots,9$, is imposed.

Radiative corrections may also be computed using the ten-dimensional formalism. We merely need to enforce by hand the zero-mode constraint in all internal-loop integrations. The effects of the (pseudo) scalars then arise from a $10=4+6$ cleavage of the metric tensor ($G_{MN}\rightarrow g_{\mu\nu}\oplus\delta_{mn}$) and from the use of ten-dimensional Dirac algebra ($\Gamma_M\rightarrow\gamma_\mu\oplus\gamma_m$). [Actually when integrals are evaluated using dimensional regularization, one is really making a $10-2\epsilon=(4-2\epsilon)+6$ split for the metric and Dirac matrices.] An appropriate scaling of the trace of the unit matrix in the Dirac algebra accommodates for the existence of four Majorana spinors in the four-dimensional SST.

The use of dimensional-reduction techniques to "compactify" the number of algebraic steps involved in radiative corrections for the $N=4$ SST will be described more fully elsewhere.[14] It is remarkable that these techniques allow the simultaneous calculation of β_{Yukawa} and β_{gauge} if one considers the spinor-vector-spinor vertex. This also permits a quick but nontrivial check on some of the supersymmetric Ward identities in the theory. It is also remarkable that these techniques permit an interpolation between theories with no (pseudo) scalars and the $N=4$ SST, e.g., through the dimensional reduction of Yang-Mills theory in $4+2m$ dimensions. We believe such an interpolation should be incorporated in the full three-loop calculation (using an arbitrary fermion representation of the gauge group) since it would permit the final result to be easily adapted to more realistic models such as quantum chromodynamics.

ACKNOWLEDGMENTS

Most of the analysis in this paper was done while I was at the University of California, Irvine. I wish to thank the members of the high-energy physics group at Irvine (in particular M. Bander and G. Shaw) for their discussions and especially for their hospitality. I also wish to thank the theory group at the Enrico Fermi Institute for the opportunity to write this paper in a congenial atmosphere. I was supported in part by a Robert R. McCormick Fellowship and by NSF under Contract No. PHY-78-01224.

[1]T. Kinoshita, in *Proceedings of the 19th International Conference on High Energy Physics, Tokyo, 1978*, edited by S. Homma, M. Kawaguchi, and H. Miyazawa (Physical Society of Japan, Tokyo, 1979), p. 571.

[2]F. Gliozzi, J. Scherk, and D. Olive, Nucl. Phys. B122, 253 (1977).

[3]D. R. T. Jones, Phys. Lett. 72B, 199 (1977); E. C. Poggio and H. N. Pendleton, *ibid.* 72B, 200 (1977).

[4]G. 't Hooft, Nucl. Phys. B61, 455 (1973).

[5]W. Kummer, Acta Phys. Austriaca 41, 315 (1975).

[6]The "pole part" of any quantity is the coefficient of $1/\epsilon$ in that quantity's Laurent expansion around four dimensions. We define $\epsilon=\frac{1}{2}(4-D)$ with D the dimension of space-time used in the dimensional regularization procedure.

[7]G. 't Hooft and M. Veltman, in *Particle Interactions at Very High Energies*, edited by D. Speiser, F. Halzen, and J. Weyers (Plenum, New York, 1974), Part B, p. 177.

[8]J. L. Rosner, Ann. Phys. (N.Y.) 44, 11 (1967).

[9]E. de Rafael and J. L. Rosner, Ann. Phys. (N.Y.) 82, 369 (1974).

[10]E. C. Poggio, Phys. Rev. D 8, 2431 (1973).

[11]M. J. Levine and R. Roskies, Phys. Rev. D 9, 421 (1974).

[12]The following discussion of HM's in D-dimensional spacetime is based on our unpublished notes. We are not aware of any treatment of this subject in the literature.

[13]N. J. Vilenkin, *Special Functions and the Theory of Group Representations* (American Mathematical Society, Providence, Rhode Island, 1968), Vol. 22, Chap. IX.

[14]T. Curtright (unpublished).