Gribov vacuum copies in terms of harmonic maps

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We discuss the structure of the Gribov vacuum copies for SU(2) non-Abelian gauge theories in the Landau gauge. Using the theory of the harmonic maps, we recover the known Gribov-type solutions and construct new ones which are not spherically symmetric.

I. INTRODUCTION

Recently, Misner[1] pointed out the role of the harmonic maps in different fields of particle physics and general relativity. His paper outlines the basis for a program of exploring harmonic maps as models for physical theories, such as broken-symmetry phenomena, nonlinear σ model, Yang-Mills equations, relativity, etc. The aim of this paper is to use the theory of the harmonic maps in order to find larger classes of the Gribov vacuum copies.

As is known, Gribov[2] observed that, for an SU(2) Yang-Mills gauge theory, the Coulomb transversality condition does not fix the gauge completely. It was pointed out that there are several nontrivial field configurations representing the same physical field (Gribov vacuum copies) which are connected by finite gauge transformations.[3-7] Owing to the similarity between the Landau gauge condition in Euclidean space-time and the Coulomb gauge condition, the existence of the Gribov pathologies was shown also in the Landau gauge.[8-10] On the other hand, Singer[11] established the existence of the Gribov ambiguity for general continuous gauge-fixing conditions on S^4.

In order to classify the possible vacuums in SU(2) Yang-Mills theory, the pure gauge potential is introduced in the gauge-fixing condition. The resulting nonlinear partial differential equation was solved using different particular assumptions.

Actually, the solutions of this partial differential equation define harmonic maps between the Euclidean space-time and $S^3 = \mathrm{SU}(2)$. Using this observation we find that the theory of the harmonic maps represents the natural frame for the investigation of the vacuum structure, at least in the Landau and Coulomb gauges. We apply the theory of the harmonic maps between Euclidean spheres[12,13] to obtain nontrivial pure gauge fields in the Landau gauge condition. We obtain all known solutions[8-10] and, in addition, we find new ones. We do not find all harmonic maps between R^4 and S^3, restricting ourselves only to such explicit constructions which involve ordinary differential equations.

II. CONSTRUCTION OF HARMONIC MAPS

A harmonic map is a map between Riemannian manifolds which extremizes a certain simple functional called the energy integral.

Let M,N be Riemannian manifolds with metrics $g_{ij}, h_{\alpha\beta}$ $(i,j=1,\ldots,m;\ \alpha,\beta=1,\ldots,n)$, respectively. Given a smooth map $\varphi:(M,g)\to(N,h)$, its energy is defined by the formula

$$E(\varphi)=\int_M e(\varphi)(x)dx\,, \tag{2.1}$$

with the energy density

$$e(\varphi)(x)=\tfrac{1}{2}\left|d\varphi(x)\right|^2=\tfrac{1}{2}g^{ij}h_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\,, \tag{2.2}$$

where $d\varphi(x)$ denotes the differential of φ at the point $x\in M$ and dx is the volume element of M.

The first and second fundamental forms of φ at $x\in M$ are[14]

$$\begin{aligned}(\varphi^*h)_{ij}&=\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}h_{\alpha\beta}\,,\\ (\nabla(d\varphi))^\gamma_{ij}&=\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j}-{}^M\Gamma^k_{ij}\frac{\partial\varphi^\gamma}{\partial x^k}+{}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\,,\end{aligned} \tag{2.3}$$

where φ^* is the pull-back map, ∇ is the covariant differential, and ${}^M\Gamma^k_{ij}$, ${}^N\Gamma^\gamma_{\alpha\beta}$ are the usual Christoffel symbols of M and N, respectively. Note that for a Riemannian immersion one has $\varphi^*h=g$.

The trace of the second fundamental form is called the tension field of φ, and a map with vanishing tension field is said to be harmonic. The map $\varphi: M\to N$ is harmonic if and only if it is an extremal of the energy integral $E(\varphi)$ defined by Eq. (2.1). Indeed, the Euler-Lagrange equations for E are

$$\begin{aligned}\tau(\varphi)^\gamma&=\mathrm{Tr}(\nabla(d\varphi))^\gamma_{ij}\\ &=\Delta\varphi^\gamma+{}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}g^{ij}=0\,,\end{aligned} \tag{2.4}$$

where Δ is the Laplace-Beltrami operator

$$\Delta\varphi^\gamma=g^{ij}\left(\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j}-{}^M\Gamma^k_{ij}\frac{\partial\varphi^\gamma}{\partial x^k}\right). \tag{2.5}$$

Sometimes it is useful to consider the compositions of two maps $\varphi: M \to M'$ and $\psi: M' \to N$ for which the analog of the tension from Eq. (2.4) is

$$\tau(\psi \circ \varphi) = d\psi\, \tau(\varphi) + \mathrm{Tr}\nabla d\psi(d\varphi, d\varphi). \tag{2.6}$$

In the next section we shall consider the case of a harmonic map between a manifold M and a sphere S^{n-1} which can be isometrically immersed in R^n. Applying Eq. (2.6) with $M' = S^{n-1}$, $N = R^n$, ψ being the immersion $\psi: S^{n-1} \to R^n$ and assuming that φ is harmonic $[\tau(\varphi) = 0]$, we get

$$\begin{aligned}\Delta(\psi \circ \varphi)^a &= g^{ij}\left(\frac{\partial^2 \psi^a}{\partial \varphi^\beta \partial \varphi^\gamma} - {}^{M'}\Gamma^\alpha_{\beta\gamma} \frac{\partial \psi^a}{\partial \varphi^\alpha}\right) \frac{\partial \varphi^\beta}{\partial x^i} \frac{\partial \varphi^\gamma}{\partial x^j} \\ &= (\psi \circ \varphi)^a g^{ij} h_{\beta\gamma} \frac{\partial \varphi^\beta}{\partial x^i} \frac{\partial \varphi^\gamma}{\partial x^j} \\ &= 2e(\varphi)(\psi \circ \varphi)^a \\ &= 2e(\psi \circ \varphi)(\psi \circ \varphi)^a ,\end{aligned} \tag{2.7}$$

where Δ is defined in (2.5) and g_{ij}, $h_{\alpha\beta}$ are the metric tensors of M and N, respectively. In order to obtain Eq. (2.7), we explicitly used the fact that S^{n-1} is immersed in R^n.[14,15] In fact, it is possible to prove that $\varphi: M \to S^{n-1}$ is harmonic if and only if Eq. (2.7) holds.[15] Moreover, if φ is an eigenfunction of the Laplace-Beltrami operator, we have

$$\Delta(\psi \circ \varphi) = \lambda(\psi \circ \varphi). \tag{2.8}$$

In particular, if φ is a harmonic polynomial of homogeneity k defining a map between two spheres S^n and S^m, one has

$$\lambda = k(k + n - 2). \tag{2.9}$$

Another application of Eq. (2.6) is the harmonic retraction $\varphi: R^n \backslash (0) \to S^{n-1}$ (namely, $x \to x/|x|$). A simple calculation shows that if $\psi: S^{n-1} \to N$ is harmonic, then $\psi \circ \varphi: R^n \to N$ is harmonic, too.

The main problem is to find the solutions of Eq. (2.4). In general, this is a difficult problem, but in some cases it is possible to give prescriptions to construct explicit harmonic maps. For example, an explicit construction is available for the harmonic maps between Euclidean spheres.[13] The strategy is to start with known harmonic polynomial maps between spheres with small dimensions and, by a special prescription, construct new harmonic maps between spheres of higher dimensions.

The essential harmonic polynomial maps used in the present paper are

(a) the identity,

(b) $\varphi_k: S^1 \to S^1$ realized by complex polynomials $z \to z^k$ of homogeneity $|k|$,

(c) the Hopf polynomial map $h: S^3 \to S^2$.

Suppose that $\varphi_l: S^{p-1} \to S^{q-1}$ and $\varphi_k: S^{r-1} \to S^{s-1}$ are homogeneous harmonic polynomials of degrees l and k, respectively. The join of these two maps is the map $\varphi_l * \varphi_k: S^{p+r-1} \to S^{q+s-1}$, defined as follows. The coordinates of point $z \in S^{p+r-1}$ are parametrized in the form

$$z = \frac{x}{|x|} \sin\alpha + \frac{y}{|y|} \cos\alpha , \tag{2.10}$$

with $x \in R^p \backslash (0)$ $(x/|x| \in S^{p-1})$, $y \in R^q \backslash (0)$ $(y/|y| \in S^{q-1})$, and $0 \leq \alpha \leq \pi/2$. Then we set

$$(\varphi_l * \varphi_k)(x, y) = \left(\sin\alpha(t)\varphi_l\left(\frac{x}{|x|}\right), \cos\alpha(t)\varphi_k\left(\frac{y}{|y|}\right)\right), \tag{2.11}$$

where $(x, y) \in R^p \times R^r \backslash (0)$ and

$$t = \ln\frac{|x|}{|y|} \in (-\infty, \infty).$$

A very interesting theorem is due to Smith,[13] who was able to construct a harmonic representative (in homotopy) of $\varphi_l * \varphi_k$. Suppose that φ_l and φ_k are defined as above and the following conditions are satisfied:

$$k > \theta(r - 2), \quad l > \theta(p - 2), \quad \theta = (\sqrt{2} - 1)/2. \tag{2.12}$$

Then, the join $\varphi_l * \varphi_k$ can be deformed into a harmonic map $S^{p+r-1} \to S^{q+s-1}$. This can be realized by constraining the function $\alpha(t)$ to satisfy the differential equation

$$\begin{aligned}\ddot{\alpha}(t) + (e^t + e^{-t})^{-1}\{[(p-2)e^{-t} - (r-2)e^t]\dot{\alpha}(t) \\ + (\lambda_2 e^t - \lambda_1 e^{-t}) \sin\alpha(t) \cos\alpha(t)\} = 0 .\end{aligned} \tag{2.13}$$

This equation can easily be obtained by imposing the map (2.11) to satisfy the condition of harmonicity in the form (2.7). In addition, we must keep in mind that φ_l and φ_k satisfy equations of the type (2.8) with the eigenvalues given by (2.9), namely, $\lambda_1 = l(l + p - 2)$ and $\lambda_2 = k(k + r - 2)$. Smith proved that Eq. (2.13), subject to (2.12), has a solution α with $0 < \alpha < \pi/2$ and which is asymptotic to 0 and $\pi/2$ at $-\infty$ and ∞, respectively.

We note that Eq. (2.13) can be interpreted as that of a pendulum with variable gravity, changing sign with position.

III. GRIBOV VACUUM COPIES IN THE LANDAU GAUGE

We deal with SU(2) Yang-Mills theory in four-dimensional Euclidean space-time. Our aim is to discuss the structure of the vacuum states in the Landau gauge, i.e., to find which potentials $A_\mu(x)$ satisfying the condition

$$\partial_\mu A_\mu(x) = 0 \quad (\mu = 1, 2, 3, 4) \tag{3.1}$$

generate a vanishing field strength tensor $F_{\mu\nu} = 0$, for which one can write the potential

$$A_\mu = U^{-1}\partial_\mu U, \qquad (3.2)$$

where $U(x)$ is an SU(2) gauge group element which can be parametrized by a unit four-vector $N_a(x)$ $(a=1,2,3,4)$ lying on the unit sphere S^3

$$U = N_4 + i\sigma_i N_i \quad (i=1,2,3). \qquad (3.3)$$

Then the Landau condition (3.1) becomes

$$\Delta N_a = (\partial_\mu N_b \partial_\mu N_b) N_a, \qquad (3.4)$$

where Δ is the Laplacian in R^4 defined with the convention $\Delta f = -\mathrm{div}\, df$, so that it has positive eigenvalues. We are looking for a solution of Eq. (3.4) which leads to potential $A_\mu(x)$ satisfying the so-called weak boundary condition[5]

$$A_\mu(x)_{|x| \to \infty} O\left(\frac{1}{|x|}\right). \qquad (3.5)$$

We note that Eq. (3.4) is the condition that the map

$$N: R^4 \to S^3 \subset R^4 \qquad (3.6)$$

is harmonic. Indeed, Eq. (3.4) is essentially Eq. (2.7) with the "energy" density

$$e(N) = \tfrac{1}{2}\partial_\mu N_a \partial_\mu N_a. \qquad (3.7)$$

Another proof that U is a harmonic map can be done in an invariant way. As is known, the vector potential A_μ represents a connection for a principal bundle P with the gauge group SU(2) over the base space R^4. The gauge-fixing condition can be written as[16]

$$d^*A = 0, \qquad (3.8)$$

where d^* is the adjoint operator of the covariant differential d. For a vanishing field tensor

$$F = dA = 0, \qquad (3.9)$$

we can write A_μ in the form (3.2) for which Eq. (3.8) is equivalent with Eq. (3.1). But Eqs. (3.8) and (3.9) say that the one-form (3.2) is harmonic, which implies that the map U is harmonic.[12] We conclude that the problem of obtaining the Gribov vacuum copies in the Landau gauge is equivalent to finding the harmonic representatives of the map (3.6).

The first example of harmonic map between R^4 and S^3 will be the composition of a harmonic retraction $R^4\backslash(0) \to S^3$,

$$(x_1 x_2, x_3, x_4) \to \left(\frac{x_1}{|x|}, \frac{x_2}{|x|}, \frac{x_3}{|x|}, \frac{x_4}{|x|}\right), \quad x = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}, \qquad (3.10)$$

followed by a harmonic map $S^3 \to S^3$. As we stated in Sec. II, this composition is a harmonic map between R^4 and S^3, so that we have to construct a family of harmonic maps between spheres of dimension three. For this purpose we shall use the join of the harmonic polynomials $\varphi_l: S^1 \to S^1$, $\varphi_k: S^1 \to S^1$, where

$$\varphi_l(x_1/(x_1^2+x_2^2)^{1/2}, x_2/(x_1^2+x_2^2)^{1/2})$$

is a harmonic polynomial of degree l and

$$\varphi_k(x_3/(x_3^2+x_4^2)^{1/2}, x_4/(x_3^2+x_4^2)^{1/2})$$

is a harmonic polynomial of degree k. These polynomial maps are of the type z^k, where $z = x+iy$ is a point on the circle S^1.

Using the prescription presented in Sec. II, we construct the join of φ_l and φ_k,

$$(\varphi_l * \varphi_k)\left(\frac{x_1}{|x|}, \frac{x_2}{|x|}, \frac{x_3}{|x|}, \frac{x_4}{|x|}\right) = (\sin\alpha(t)\varphi_l(x_1/(x_1^2+x_2^2)^{1/2}, x_2/(x_1^2+x_2^2)^{1/2}), \cos\alpha(t)\varphi_k(x_3/(x_3^2+x_4^2)^{1/2}, x_4/(x_3^2+x_4^2)^{1/2})), \qquad (3.11)$$

where

$$t = \tfrac{1}{2}\ln\frac{x_1^2+x_2^2}{x_3^2+x_4^2} \in (-\infty, \infty). \qquad (3.12)$$

The condition that this map is harmonic implies

$$\ddot{\alpha}(t) + (e^t + e^{-t})^{-1}(\lambda_2 e^t - \lambda_1 e^{-t})\sin\alpha(t)\cos\alpha(t) = 0, \qquad (3.13)$$

which is just Eq. (2.13) for $p = r = 2$ and $\lambda_1 = l^2$, $\lambda_2 = k^2$. Taking into account the asymptotic behavior of the solution of Eq. (3.13), we note that the corresponding potential $A_\mu(x)$ satisfies the weak boundary condition (3.5).

We remark that in (3.11) we have a family of equations depending on the integer numbers k and l. The final sphere S^3 spanned by the vectors N_a is parametrized as

$$N_a = \begin{bmatrix} \sin\alpha\sin\beta \\ \sin\alpha\cos\beta \\ \cos\alpha\sin\gamma \\ \cos\alpha\cos\gamma \end{bmatrix}, \qquad (3.14)$$

where α is given by Eq. (3.13) and

$$\beta = l \arctan\frac{x_2}{x_1}, \quad \gamma = k \arctan\frac{x_4}{x_3}. \tag{3.15}$$

The map (3.11) can be characterized by an integer m [$\pi_3(S^3)=Z$]:

$$m = -\frac{1}{2\pi^2}\int_{(\varphi_l * \varphi_k)(S^3)} dS = kl, \tag{3.16}$$

where dS is the surface element on the S^3 sphere and $(\varphi_l * \varphi_k)(S^3)$ is the image of the initial S^3 sphere on the final S^3 sphere by the join $\varphi_l * \varphi_k$. This quantity gives the number of times the group manifold S^3 is covered by the image of the join map $\varphi_l * \varphi_k$.

Another construction of a harmonic map between R^4 and S^3 will be done with the aid of the Hopf polynomial map $h: S^3 \to S^2$. For this purpose we shall consider again the map $R^4 \to S^3$ given by (3.10). The Hopf polynomial map is defined as[12]

$$h\left(\frac{x_1}{|x|}, \frac{x_2}{|x|}, \frac{x_3}{|x|}, \frac{x_4}{|x|}\right) = \left(\frac{x_1^2+x_2^2-x_3^2-x_4^2}{|x|^2}, 2\frac{x_1x_3-x_2x_4}{|x|^2}, 2\frac{x_1x_4+x_2x_3}{|x|^2}\right), \tag{3.17}$$

for which the topological invariant is called the Hopf invariant and equals one. At this point we shall modify the preceding construction in order to suspend this harmonic map to one from R^4 to S^3:

$$N(x_1,x_2,x_3,x_4) = \left(\sin\alpha(t) h\left(\frac{x}{|x|}\right), \cos\alpha(t)\right), \tag{3.18}$$

with $t = \ln|x|^2$.

Now using the same technique we obtain the condition that N is a harmonic map in the form

$$\ddot{\alpha}(t) + \dot{\alpha}(t) - \sin 2\alpha(t) = 0, \tag{3.19}$$

which coincides with the Gribov equation in the Coulomb gauge. Since we know only one polynomial Hopf map (3.17), we find just one equation but not a family as in (3.13). We note that both the above constructions give new vacuum copies which are not spherically symmetric.

Finally, we shall write the spherical symmetric solution of the Gribov ambiguity in the Landau gauge.[8] We shall start with a harmonic polynomial map

$$\varphi(x_i/(x_1^2+x_2^2+x_3^2)^{1/2}) \quad (i=1,2,3)$$

between S^2 and S^2. Now using a similar trick as in the above, we shall construct a map from R^4 to S^3:

$$N(x_1,x_2,x_3,x_4) = (\sin\alpha(t)\varphi(x_i/(x_1^2+x_2^2+x_3^2)^{1/2}), \cos\alpha(t)), \tag{3.20}$$

where

$$t = \ln(x_1^2+x_2^2+x_3^2)^{1/2}/|x_4|.$$

The condition that N is harmonic implies the following equation for α:

$$\ddot{\alpha}(t) + \dot{\alpha}(t) - \tfrac{1}{2}\lambda(1+e^{2t})^{-1}\sin 2\alpha(t) = 0, \tag{3.21}$$

where $\lambda = l(l+1)$, l being the degree of the polynomial map φ. Unfortunately, harmonic polynomials between two-dimensional spheres are known only for $l=0$ (the constant map) and $l=1$ (the identity).[17] We shall look only at the nontrivial case $l=1$. After a change of variables

$$t = \ln\tau - \ln(e^\tau - e^{-\tau}),$$

Eq. (3.21) becomes

$$\frac{d^2\alpha(\tau)}{d\tau^2} - \tanh\tau\frac{d\alpha(\tau)}{d\tau} - \tfrac{1}{2}\sin 2\alpha(\tau) = 0, \tag{3.22}$$

which was obtained by Itabashi.[8] Again, we have only one equation (3.22) due to the fact that we start with a single polynomial map $S^2 \to S^2$ (the identity). For a discussion of Eq. (3.22) we refer the reader to Itabashi's paper.[8]

We remark that all the above examples use as starting point known polynomial maps between spheres of lower dimensions. But only in the first construction were we able to find a family of equations of the Gribov type since the initial polynomial map $\varphi_k: S^1 \to S^1$ can be explicitly realized for any integer k. In this case we are able to find vacuum copies characterized by any integer number ($m = kl$).

Finally, let us briefly discuss the Gribov ambiguity in the Coulomb gauge. In this case, Eq. (3.4) becomes

$$\Delta N_a = (\partial_i N_b \partial_i N_b) N_a, \quad i = 1,2,3 \tag{3.23}$$

and Δ is now the Laplacian in R^3. Hence the problem is reduced to the construction of the harmonic maps between R^3 and S^3. We observe that we cannot use the join of maps between spheres as we did in the Landau case because we finally arrive at spheres of dimension higher than necessary. Of course we can suspend the identity map $S^2 \to S^2$ to $R^3 \to S^3$, but this is essentially the solution known in the literature.[2-7] This is due to the fact that beteen two-dimensional spheres there are no polynomial harmonic maps except the constant and the identity maps.[17]

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