Anharmonic oscillator: A new approach

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We construct a novel perturbation series for the anharmonic oscillator. This has elegant, qualitative properties. It also converges, unlike conventional perturbation theory, as we prove by a generalization of Lipatov's technique.

I. INTRODUCTION

Since the early days of quantum electrodynamics it has been realized that the standard perturbation series in α is probably divergent.[1] Recently Lipatov has given a powerful impetus to the study of this question.[2] He showed how in many theories the Nth term of the perturbation series could be investigated elegantly by studying the instanton solutions of an artificial functional integral. By considering zero space dimensions this technique may also be used to estimate the terms in the conventional g^n expansion of the energy levels of the anharmonic oscillator,

$$H=\tfrac{1}{2}(p^2+m^2x^2)+\tfrac{1}{4}gx^4\,. \tag{1.1}$$

These results confirm the old results of Bender and Wu[3] that the Nth term behaves like $(N!)A^NN^B C[1+O(1/N)]g^N$ where A,B,C are computable g-independent numbers. However, the technique shows that genuine field theories in nonzero space dimensions have essentially identical large-N behavior with A,B,C altered. Moreover, almost all of the present-day approximation techniques have similarly been shown to be divergent. This includes the ϵ expansion and the $1/N$ expansion.[1,4]

In this paper we shall give a finite, convergent expansion scheme for the energy levels of the anharmonic oscillator. This avoids all of the problems of the conventional perturbation series but is just as easy to calculate. We will prove in Sec. III that is is a convergent scheme both by using Lipatov's functional technique and by a more direct and laborious technique.

The conventional approach to these divergences has been to use Borel summation.[5] The Lipatov analysis shows that in some cases the formal Borel sum constructed from the perturbation series exists. Only in simple cases such as the anharmonic oscillator has it been proved that this must equal the correct answer.[6] Indeed there exist some spectacular counterexamples to the notion that a convergent or Borel-summable formal series must give the correct answer.[7] This Borel summation technique fails for renormalizable theories. Thus any other possibilities merit attention.

We also strees that our expansion clearly includes the known qualitative structure of the exact eigenvalues of (1.1). Thus it is known that[8]

$$E\simeq g^{1/3}\left[A+B\frac{m^2}{g^{2/3}}+c\left(\frac{m^2}{g^{2/3}}\right)^2+\cdots\right] \tag{1.2}$$

is convergent for large g. The existence of the $g^{1/3}$ factor, necessary on dimensional grounds, is hard to see from a power-series expansion in g.

The nth energy level is known for large n to behave like $n^{4/3}$. For negative m^2 the potential in H has a double minimum and leads to the well-known "symmetry breaking" shift. These results show up clearly in our answers.

In conclusion we stress that we have a convergent, easily computable technique for deriving the energy levels of a classical quantum-mechanical problem which contains many features of field theories. Our hope is that this may give a new insight into how field theories can be solved.

II. GENERAL PROPERTIES

We first consider conventional perturbation theory for the anharmonic oscillator

$$\begin{aligned}H_0&=\tfrac{1}{2}(p^2+m^2x^2)=m(a^\dagger a+\tfrac{1}{2})\,,\\ H_I&=\frac{g}{4}x^4=\frac{g}{16m^2}(a^\dagger+a)^2\,.\end{aligned} \tag{2.1}$$

Here we have $a^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle$, $a|n\rangle=\sqrt{n}\,|n-1\rangle$ as usual for the eigenstates of H_0. Thus $a^\dagger$ raises the eigenvalues of H_0 by one unit of m. If we consider standard Rayleigh-Schrödinger or Brillouin-Wigner perturbation theory then each extra order in g implies an extra factor of the form $\langle n|H_I|r\rangle/(E-E_r^0)$ where for a given n we are only allowed to have $r=n$, $n\pm2$, $n\pm4$ because of (2.1). Thus at order g^{2N} the states n,r can be of order $4N$. Thus $E_r^0\sim m4N$ while $\langle n|H_I|r\rangle\simeq(g/16m^2)(4N)^2$. This means that the terms in the perturbation series which reach high excited states imply the existence of $N!$ factors for large N. This arises because H_I is fourth order in

$a^\dagger, a$ while H_0 is second order.

If we were to use a Feynman diagram technique then we must think of $|n\rangle$ as being n independent particles each of energy m over a vacuum of energy $1/2m$. Then H_I creates or annihilates up to four particles. The overall $N!$ now comes not from individual terms in the series but from the very large number of diagrams.[9] Thus when H_I acts on an n-particle state, only one of these particles turns into three and the n factor comes from the choice of which one H_I acts on.

Our method is to rewrite $H = H_0 + H_I$ by using

$$H_0 = \frac{g}{4}\left(x^2 + \frac{1}{\Omega^2}p^2\right)^2,$$
$$H_I = (p^2 + m^2x^2)/2 + \frac{g}{4}\left[x^4 - \left(x^2 + \frac{1}{\Omega^2}p^2\right)^2\right]. \tag{2.2}$$

H_0 is now a squared harmonic oscillator with frequency Ω. Ω is at this stage an arbitrary parameter. We now use $a^\dagger, a$ operators of frequency Ω. For the moment the harmonic $(p^2 + m^2x^2)$ term is put into H_I. Later we will move it into H_0. Thus the eigenvalues of H_0 are

$$E_n^0 = \frac{g}{\Omega^2}(n + \tfrac{1}{2})^2. \tag{2.3}$$

H_I is still fourth order in $a^\dagger, a$. The extra factor in going from Nth-order to $(N+1)$th-order perturbation theory is now of order 1, not N. This eliminates the terms that give rise to the $N!$. However, all graphs in a given order must now be counted, and it is not at all clear how the sum behaves. This is the subject of the last section.

The parameter Ω is at this stage entirely free except that it must not be zero or infinity. In both these cases the spectrum of H_0 becomes continuous rather than discrete and the standard perturbation theory is not well defined. Order by order the perturbation theory defined by (2.3) depends on Ω. This must vanish in the final summed answer. We will show how to estimate Ω later on. Clearly this fake dependence imposes strong renormalization-group-type constraints in our series, but we have been unable to use this.

In terms of $a^\dagger, a$ we now have in normal-ordered form

$$H_I = -\frac{\Omega}{4}(a^{\dagger 2} + a^2 - 2a^\dagger a - 1) + \frac{m^2}{4\Omega}(a^{\dagger 2} + a^2 + 2a^\dagger a + 1)$$
$$+ \frac{g}{16\Omega^2}(a^{\dagger 4} + a^4 + 4a^{\dagger 3}a + 4a^\dagger a^3 - 10a^{\dagger 2}a^2 - 20a^\dagger a + 6a^{\dagger 2} + 6a^2 - 1). \tag{2.4}$$

We stress that $a^\dagger, a$ have dimensionless, Ω-independent matrix elements. Clearly calculating low orders orders of perturbation theory is trivial, if tedious, since H_I is at most fourth order in $a^\dagger, a$.

Before writing out the first few terms we change Ω for a new variable $Z = \Omega^3/g$ which is dimensionless. Then

$$E_K^0 = g^{1/3}Z^{-2/3}(K + \tfrac{1}{2})^2,$$
$$H_I = -\frac{g^{1/3}Z^{1/3}}{4}(a^{\dagger 2} + a^2 - 2a^\dagger a - 1) + \frac{m^2}{4g^{1/3}Z^{1/3}}(a^{\dagger 2} + a^2 + 2a^\dagger a + 1)$$
$$+ \frac{g^{1/3}}{16Z^{2/3}}(a^{\dagger 4} + a^4 + 4a^{\dagger 3}a + 4a^\dagger a^3 - 10a^{\dagger 2}a^2 - 20a^\dagger a + 6a^{\dagger 2} + 6a^2 - 1). \tag{2.5}$$

In each order of perturbation theory we explicitly remove $g^{1/3}Z^{-2/3}$.

For simplicity we set $m^2 = 0$, for the moment. Then for the ground-state energy in Rayleigh-Schrödinger perturbation theory we obtain

$$E = \frac{g^{1/3}}{Z^{2/3}}\left[\{(\tfrac{1}{2})^2\} + \{\tfrac{1}{4}Z - \tfrac{1}{16}\} - \left\{\frac{(Z - \frac{3}{2})^2}{48} + \frac{3}{640}\right\}\right.$$
$$+ \left\{\tfrac{1}{18}(\tfrac{3}{8} - \tfrac{1}{4}Z)^2(\tfrac{5}{4}Z - \tfrac{61}{16}) + \tfrac{1}{40}(\tfrac{7}{8} - \tfrac{1}{4}Z)(\tfrac{3}{8} - \tfrac{1}{4}Z) + \frac{3}{16\times 800}(\tfrac{9}{4}Z - \tfrac{201}{16}) - \frac{(\frac{3}{8} - \frac{1}{4}Z)^2(\frac{1}{4}Z - \frac{1}{16})}{18}\right.$$
$$\left.\left. - \frac{3}{32\times 400}(\tfrac{1}{4}Z - \tfrac{1}{16})\right\}\right], \tag{2.6}$$

and where the curly brackets enclose different orders in H_I. For the coefficient A of Eq. (1.2) a choice of $Z = 1$ gives 0.4247 as opposed to the correct 0.420 805. Further numerical studies may be found in Ref. 10.

The question which must now be raised is: How do we know what value to choose for Ω or equivalently Z? We shall prove in the next sec-

tion that the nth term $\sim\exp(-n^{3/7}Z^{4/7}\,\text{const})$ as $n\to\infty$. Thus the larger the choice of Z the faster the convergence of the nth term to zero. However, large Z implies $E_0\to 0$ and so we start further away from the correct answer. For small Z the convergence is slower and we start even further from the correct answer.

One way is to choose Z so that the sum of the corrections to the zeroth-order term E_n^0 vanishes. Referring to (2.6) this gives $Z=0.25$, 0.345, or 0.412 keeping one, two, or three orders. Notice that our expansion is not an expansion in any parameter of the theory.

Returning to (2.6) and keeping the m^2 term we see that the numerators of the Rayleigh-Schrödinger series are polynomials in $(m^2/g^{2/3})$ while the denominators are constants after we cancel the $g^{1/3}$ factors and remove the sole overall $g^{1/3}$ factor. Thus our perturbation series automatically falls into the form of Eq. (1.2) and we obtain perturbation series for the separate A, B, C terms. In other words, the qualitative structure of the large-g limit is built into our expansion. The terms $(m^2/g^{2/3})^N$ first appear in Nth-order perturbation theory.

Now let us turn to the calculation of the Kth energy level for large K. This is known to be $\sim K^{4/3}$ lying between the K of the harmonic oscillator and the K^2 of our unperturbed Hamiltonian H_0. We use the strategy employed above of solving for the Z which makes the corrections to E_K^0 zero. This gives $Z\sim N$ and remembering $E_K^0=g^{1/3}Z^{-2/3}(K+\frac{1}{2})^2$ we see immediately that $E_K\sim K^{4/3}$.

We have checked the above calculation of $Z\sim K$ up to third order in H_I. The calculation involves surprising cancellations whose origin and significance are obscure to us. Thus, although each term in say third order $\sim K^4$, the sum of all Rayleigh-Schrödinger graphs $\sim K^2$.

It is convenient for symmetry purposes to introduce $r=K+\frac{1}{2}$. Then

$$\begin{aligned}&E_K^0=g^{1/3}Z^{-2/3}r^2,\\&\langle K|H_I|K\rangle=g^{1/3}Z^{-2/3}\\&\qquad\times\left[\frac{Zr}{2}-\tfrac{1}{16}(10(r+\tfrac{1}{2})(r-\tfrac{1}{2})+1)\right].\end{aligned}\tag{2.7}$$

Setting the second term equal to zero gives $Z\sim 5/4r\sim 5/4K$ as expected. Now turn to second order. The energy denominators are all of the form

$$r^2-(r\pm 2)^2,\quad r^2-(r\pm 4)^2,$$

since we are not allowed to return to the state K. Moreover, $\langle K|H_I|K\pm 2\rangle$ are equal for large K. Thus in second order we have

$$\sum_{\pm}\frac{\langle K|H_I|K\pm 2\rangle\langle K\pm 2|H_I|K\rangle}{E_K^0-E_{K\pm 2}^0}+\sum_{\pm}\frac{\langle K|H_I|K\pm 4\rangle\langle K\pm 4|H_I|K\rangle}{E_K^0-E_{K\pm 4}^0}.$$

Each of these terms $\sim K^3$ but of course, due to the oddness of the denominators and evenness of the numerators, these cancel. We are left with terms Z^2, ZK, K^2. Again we obtain $Z\sim K$.

Now turn to third order in H_I. Assuming $Z\sim r$ and keeping up to $1/r$ corrections we may easily show that

$$\begin{aligned}&\langle K+2|H_I|K\rangle=-\tfrac{1}{4}(Zr-r^2)+(-\tfrac{1}{4}Z+\tfrac{1}{2}r),\\&\langle K-2|H_I|K\rangle=-\tfrac{1}{4}(Zr-r^2)-(-\tfrac{1}{4}Z+\tfrac{1}{2}r),\\&\langle K+4|H_I|K\rangle=-\tfrac{1}{16}r^2-\tfrac{1}{4}r,\\&\langle K-4|H_I|K\rangle=-\tfrac{1}{16}r^2+\tfrac{1}{4}r,\\&\langle K+4|H_I|K+2\rangle=-\tfrac{1}{4}(Zr-r^2)-(\tfrac{3}{4}Z-\tfrac{3}{2}r),\\&\langle K-4|H_I|K-2\rangle=-\tfrac{1}{4}(Zr-r^2)+(\tfrac{3}{4}Z-\tfrac{3}{2}r),\\&\langle K|H_I|K\rangle=\tfrac{1}{2}r(Z-\tfrac{5}{4}r)+O(\text{const}),\\&\langle K+2|H_I|K-2\rangle=-\tfrac{1}{16}r^2+O(\text{const}).\end{aligned}\tag{2.8}$$

Thus in leading order in K or Z we have four graphs where we jump up or down four excited states and then come down or up in two jumps of two. These give

$$4\frac{[-\frac{1}{4}(Zr-r^2)]^2(-\frac{1}{16}r^2)}{(-4r)(-8r)}.$$

Then there are two graphs were we jump up or down two states, stay in that state, and then jump back to K:

$$2\frac{[-\frac{1}{4}(Zr-r^2)]^2\frac{1}{2}r(Z-\frac{5}{4}r)}{(-4r)(-4r)};$$

the same graphs but with jumps of four:

$$2\frac{(-\frac{1}{16}r^2)^2\frac{1}{2}r(Z-\frac{5}{4}r)}{(-8r)(-8r)};$$

two graphs where we jump up two, down four, and up two:

$$2\frac{[-\frac{1}{4}(Zr-r^2)]^2(-\frac{1}{16}r^2)}{(-4r)(-4r)}.\tag{2.9}$$

Then in Rayleigh-Schrödinger theory we must subtract the four terms

$$\frac{\langle K|H_I|K+a\rangle\langle K+a|H_I|K\rangle}{E_K-E_{K+a}}\frac{\langle K|H_I|K\rangle}{E_K-E_{K+a}},$$

where $a=\pm 2,\pm 4$. These are again equal in pairs in leading order giving

$$-2\frac{[-\frac{1}{4}(Zr-r^2)]^2\,\frac{1}{2}r(Z-\frac{5}{4}r)}{(-4r)^2}$$

$$-2\frac{(-\frac{1}{16}r^2)^2\,\frac{1}{2}r(Z-\frac{5}{4}r)}{(-8r)^2}$$

for $a=\pm 2$ and ± 4, respectively. Thus the terms add because the energy denominators have the opposite signs for a plus or minus but they occur in pairs of the same sign except for (2.7). The above terms add up to zero.

Now turning to the correction $1/r$ down on the dominant term, we see that the corrections to the energy denominators and matrix elements are all odd under $+\longleftrightarrow -$, i.e., $r+2\to r-2$. Thus any given correction term is canceled by a similar term with $r+2\longleftrightarrow r-2$. Notice the even $\langle K|H_I|K\rangle$, $\langle K+2|H_I|K-2\rangle$ terms have no $1/r$ corrections.

The final answer, thus, is of order r^2, Z^2, Zr just like the previous terms. This is a truly remarkable result. It also shows clearly that if we try to calculate excited states we must use different Ω for the fastest convergence.

The above split of $H=H_0+H_I$ thus turns out to naturally fit the problem of investigating $g\to\infty$ so that $m^2/g^{2/3}\to 0$. The limit $g\to 0$ is much more complicated. Clearly for g positive or negative we expect violently different spectra. Thus $E(g)$ is expected to be extremely delicate near $g=0$. We now write

$$H_0'=\tfrac{1}{4}g\left(x^2+\frac{1}{m^2}p^2\right)^2+\tfrac{1}{2}(p^2+m^2x^2)\,,$$
$$H_I'=\tfrac{1}{4}g\left[x^4-\left(x^2+\frac{1}{m^2}p^2\right)^2\right]. \tag{2.10}$$

The harmonic term is now included in the "free" Hamiltonian. This means H_I' is proportional to g, unlike H_I. Thus nth order in H_I' is at least nth order in g since H_0' is linear in g. To make H_0' soluble we have assumed $\Omega^2=m^2$. Clearly this only works if $m^2>0$. Then the perturbation series defined by this split has the remarkable property that it is convergent for all $g\neq 0,\infty$ and, moreover, the first N terms of conventional perturbation theory are contained in the first N terms of the H_0', H_I' series plus higher-order corrections. This can be checked by expanding the energy denominators. Notice that the terms proportional to g in the denominator are the terms of order N^2 which give us convergence. Hence the expansion of these denominators is dangerous as $N\to\infty$. The H_I' series has thus picked up parts of the conventional expansion in g and inserted them into the energy denominators in a truly remarkable manner.

In our original H_I split with $\Omega^3/g=Z=1$, say, the energy denominators took the form $(\frac{1}{2})^2-(N+\frac{1}{2})^2, N\geq 1$. Thus there are no g singularities apart from the explicit $g^{1/3}$ in a given order of perturbation theory. Our expansion, however, contains powers of $m^2/g^{2/3}$ which hardly allows us to investigate $g\to 0$.

Now $E_n^{0\prime}=(g/m^2)(n+\frac{1}{2})^2+m(n+\frac{1}{2})$ and energy denominators $E_0^{0\prime}-E_n^{0\prime}=(g/m^2)[\frac{1}{4}-(n+\frac{1}{2})^2]-mn$ clearly have zeros as functions of g which tend to zero for large n at fixed m. These clearly lie at negative g. In Nth-order perturbation theory these poles appear to the power $(n-1)$ and the behavior of the sum is tricky. Thus, although we cannot compute the behavior as $g\to 0$, our series contains clear warning signs that $g\to 0$ is dangerous.

For $m^2<0$ when the potential has two minima we cannot set $\Omega^2=m^2$. We then are forced into setting $\Omega^2=\lambda^2=-m^2$. When we shift $y=x\pm(\lambda^2/g)^{1/2}$ then

$$H=\tfrac{1}{2}p^2+\lambda^2y^2+\tfrac{1}{4}gy^4\pm y^3(g\lambda^2)^{1/2}+\lambda^4/4g\,.$$

Now remove the $\lambda^4/4g$ and write

$$H_0''=\frac{g}{4}\left(y^2+\frac{p^2}{2\lambda^2}\right)^2+\left(\frac{p^2}{2}+\lambda^2y^2\right).$$

Then H_0'' is soluble and H_I'' is proportional to g. The y^3 term is proportional to $\sqrt{g}$ and must appear an even number of times. Thus apart from the $\lambda^4/4g$ term we obtain a series in g^n as above with denominators as rational functions of g.

We expect that the above technique also works for potentials of the form $g(x^2)^R/2^R$ for R an integer ≥ 2. We again write

$$H_0=g\left(x^2+\frac{1}{\Omega^2}p^2\right)^R/2^R$$
$$=\frac{g}{\Omega^R}(a^\dagger a+\tfrac{1}{2})^R\,.$$

We also expect that if we have two coupled oscillators

$$H=(p_1^2+m_1^2x_1^2)/2+(p_2^2+m_2^2x_2^2)/2$$
$$+\lambda(x_1-x_2)^2+g_1x_1^4/4+g_2x_2^4/4$$

and we write

$$H_0=\tfrac{1}{4}g_1\left(x_1^2+\frac{1}{\Omega_1^2}p_1^2\right)^2+\tfrac{1}{4}g_2\left(x_2^2+\frac{1}{\Omega_2^2}p_2^2\right)^2,$$
$$H_I=H-H_0\,,$$

then this perturbation expansion converges. The coupling is all provided by the $\lambda(X_1-X_2)^2$ term which is of lower order in creation and annihilation operators. The convergence properties are hence similar to our technique applied to two independent anharmonic oscillators.

We can clearly take the sum of M such anharmonic terms. This Hamiltonian is then the lattice

analog of ϕ^4 theory where $(\Delta\phi)^2 \to [(x_i - x_{i-1})/a]^2$ where i labels the lattice points and a is the lattice size. This then gives, for example, a model for anharmonic effects on the phonon spectrum in a crystal lattice. Unlike the conventional perturbation theory in g_i, we construct a convergent expansion.

III. CONVERGENCE PROOFS

In this section we prove the convergence of our perturbation series for both Rayleigh–Schrödinger (RS) and Brillouin-Wigner (BW) perturbation theory. We do this by estimating E_N, the Nth-order contribution to the ground-state energy.

The leading large-N asymptotic contributions to E_N are obtained by entirely different methods for RS and BW theory and agree. Such agreement is in accordance with our previous numerical results that showed a diminishing difference between ground-state BW and RS energies even in low orders.

Let us concentrate first on the convergence of the BW series. Suppose we wish to calculate the energy levels of a system described by the Hamiltonian

$$H = (p^2 + x^2)/2 + V(x), \tag{3.1}$$

where $V(x)$ is a polynomial in x^2 and $V(x) \simeq g(x^2/2)^L$ for $x \to \infty$, where L is an integer. We then define

$$\begin{aligned} H_0 &= (x^2 + p^2)/2 + g(p^2 + x^2)^L/2^L, \\ H_I &= V(x) - g(p^2 + x^2)^L/2^L, \end{aligned} \tag{3.2}$$

and prove that the corresponding BW series is convergent for arbitrary E in the denominators. For the sake of simplicity we take $L = 2$ and $V(x) = g(x^2/2)^2$ and only quote the results for general L and V. Here we have chosen $\Omega = m = 1$. Different choices of Ω, m do not alter our conclusion.

Then

$$E_N = \langle 0 | H_I M^{N-1} | 0 \rangle, \tag{3.3}$$

where

$$M = (E - H_0)^{-1} P H_I. \tag{3.4}$$

P is the projection operator for the space orthogonal to $|0\rangle$, the ground state of H_0, i.e., of the harmonic oscillator. We remark that in BW perturbation theory E is the exact energy level and at Nth order we should in principle solve an Nth-order polynomial equation for E. Our proof works for arbitrary E.

We may omit the projection operator by restricting the space of state to the one spanned by the even excited states of the harmonic oscillator if we write

$$E_N = \sum_{l,m=1}^{2} \langle 0 | H_I | l \rangle \langle l | M^{N-2} | m \rangle \frac{1}{E - E_m} \langle m | H_I | 0 \rangle, \tag{3.5}$$

where

$$E_m = (2m + \tfrac{1}{2}) + g(2m + \tfrac{1}{2})^2. \tag{3.6}$$

We are now using l, m, which are half the previous values. Since H_I is a fourth-order polynomial of $a^\dagger, a$ it only generates finite steps in the space of eigenstates of H_0. Thus (3.5) may be calculated by replacing M by a finite matrix. Thus labeling the rows and columns by the even excited states of the harmonic oscillator we omit all $l, m > N$.

The new matrix has N eigenvalues $\lambda^{(i)}$ and eigenvectors $a_k^{(i)}$, $i, k = 1, \ldots, N$. The eigenvalues $\lambda^{(i)}$ are real since the symmetrized form of M is Hermitian. Then (3.6) becomes

$$E_N = \sum_{l,m=1}^{2} \sum_{r=1}^{N} \langle 0 | H_I | l \rangle a_l^{(r)*} \lambda^{(r)N-2} a_m^{(r)} \times \frac{1}{E - E_m} \langle m | H_I | 0 \rangle. \tag{3.7}$$

The eigenvalue equation

$$\sum_{K=1}^{N} \langle m | H_I | k \rangle a_k + \lambda(E_n - E) a_m = 0 \tag{3.8}$$

then leads to a 5-term difference equation for a_m. We investigate (3.8) as $N \to \infty$.

If the initial conditions of the linear homogeneous difference equation (3.8) are not fixed, there are four linearly independent solutions. Defining $a_k/a_{k-1} = (-1)^k C_k$ each of the linearly independent solutions is such that $(C_k - 1)/(C_{k-1} - 1) \to 1$ as $k \to \infty$ as shown later.

Setting $C_{l+r} = C_l = C$ for $|r| \leq 2$, one obtains the following leading terms and the most important correction term:

$$\frac{g(2l + \tfrac{1}{2})^2 (C-1)^4}{16} - g(2l + \tfrac{1}{2})^2 C^2 + \lambda[g(2l + \tfrac{1}{2})^2 + (2l + \tfrac{1}{2})]C^2 + O(l(C-1)^3) = 0. \tag{3.9}$$

The deviation of C_{l+r} from C_l generates a term similar to the correction term of Eq. (3.9).

This equation can be solved to give

$$\sqrt{C} - \frac{1}{\sqrt{C}} = 2(1 - \lambda - \lambda/2gl)^{1/4} e^{i\phi_s} + O(1/\sqrt{l}), \tag{3.10}$$

where $\phi_s = (s-1)\pi$, $s = 1, \ldots, 4$. If $|1 - \lambda| \ll 1$ then $|C_l - 1| \ll 1$ for $l \gg 1$, and we can write the left-hand side as $C_l - 1$. The solutions of (3.9) are generated by

$$a_n(s)=\prod_t^n C_t a_t = a_t \exp\left(\sum_t^n \ln C_t\right)$$
$$= a_t \exp\left\{2\sum_t^n\left[\left(1-\lambda-\frac{\lambda}{2gl}\right)^{1/4} e^{i\varphi_s}+O\left(\frac{1}{\sqrt{l}}\right)\right]\right\}$$
$$= a_t \exp\left[2\int dx\left(1-\lambda-\frac{\lambda}{2gx}\right)^{1/4} e^{i\varphi_s}+O(\sqrt{n})\right], \tag{3.11}$$

where t is a large independent number, $t \ll N$.

Now our recurrence relation leaves a_1 and a_2 independent and all subsequent a_i are linear functions of a_1 and a_2. Thus a_t is determined by the recursion relations independently of N. This means that a_n is a linear function of a_1 and a_2 and the four $a_n(s)$ functions

$$a_n = \sum_{s=1}^{4}\sum_{m=1}^{2} \mathrm{Re}(\rho_s^m a_n(s)) a_m , \tag{3.12}$$

where ρ_s^m are complex numbers independent of N, a_1, a_2. They can be calculated exactly from the recursion relations for small l. The real parts in (3.12) are taken because a_n is real but the $a_n(s)$ are complex.

The eigenvalue λ is to be determined from the conditions $a_{N+1}=a_{N+2}=0$. Substitute into (3.12) and we obtain two homogeneous linear equations for the two initial values. The determinant of that equation provides the characteristic equation for λ.

Suppose first that $\lambda/2gN > 1-\lambda$. Then the dominant solutions are $a_n(1), a_n(1)^*$:

$$a_n(1)=a_t \exp\left[2\int^n dx\left(\frac{\lambda}{2gx}-1+\lambda\right)^{1/4} e^{i\phi_s}+O(\sqrt{n})\right]. \tag{3.13}$$

Substituting (3.12) one obtains

$$a_n = \exp\left[2\int dx\left(\frac{\lambda}{2gx}-1+\lambda\right)^{1/4}\cos\frac{\pi}{4}\right] \times \sum_{m=1}^{2} a_m \rho^m \cos(\phi_m+\eta_n), \tag{3.14}$$

where ϕ_m is the phase of $\rho^m=\rho_1^m$ and

$$\eta_n = \sin\frac{\pi}{4}\int^n dx\left(\frac{\lambda}{2gx}-1+\lambda\right)^{1/4}. \tag{3.15}$$

Substitute this into (3.14) and solving the determinantal equation gives

$$\sqrt{2}\left(\frac{\lambda}{2g(N+2)}-1+\lambda\right)^{1/4}=M\pi, \tag{3.16}$$

or since $N\to\infty$,

$$\lambda = 1+\frac{1}{2g(N+2)}, \tag{3.17}$$

$$\lambda = 1+\left(\frac{M\pi}{2}\right)^4. \tag{3.18}$$

The solutions (3.17) will be discussed later with the solutions $(1-\lambda > \lambda/2gN)$. The solutions (3.18) give a contribution $e^{-\alpha N}$, $\alpha>0$, to E_N much smaller than those of the next paragraph.

Suppose that $1>1-\lambda>\lambda/2gN$. For $n<n_0=\lambda/2(1-\lambda)g$, a_n is still given by (3.14). For $n>n_0$,

$$a_n = e^{2\eta_{n_c}}\{[\alpha_1 e^{2g_n}+\gamma_1 e^{-2g_n}+\mathrm{Re}\beta_1 e^{2ig_n}]a_1+[\alpha_2 e^{2g_n}+\gamma_2 e^{-2g_n}+\mathrm{Re}\beta_2 e^{2ig_n}]a_2\}, \tag{3.19}$$

where

$$g_n=\int_{n_c}^{n} dx(1-\lambda-\lambda/2gx)^{1/4}.$$

The α, β, γ coefficients are the analogs of the ρ's of (3.12). Neglecting the low order γ_1, γ_2 we find

$$\sin(2g_N-\psi)=0, \tag{3.20}$$

where ψ is a fixed, N-independent constant. Thus the eigenvalues λ are such that g_N is large in general, i.e., $2g_N \sim M\pi$ with M large. Thus we obtain

$$\lambda \simeq 1-(M\pi/2N)^{1/4}, \quad M=1,2,\ldots. \tag{3.21}$$

Using (3.19) and inserting α_1, α_2 we find

$$a_n = e^{2\eta_{n_c}}\{[e^{2(g_n-g_{N+1})}\mathrm{Re}(\beta_1 e^{2ig_{N+1}})+\mathrm{Re}(\beta_1 e^{2ig_N})]a_1+[e^{2(g_n-g_{N+1})}\mathrm{Re}(\beta_2 e^{2ig_{N+1}})+\mathrm{Re}(\beta_2 e^{2ig_n})]a_2\}. \tag{3.22}$$

Thus e^{2g_n} is only important when $N-n=0(1)$. Thus

$$a_n = e^{2\eta_n}\alpha_n, \tag{3.23}$$

where α is bounded.

Now, disregarding slowly varying functions of n, low-order entries of the normalized vector $\tilde{a}_k$, $\tilde{a}_{1,2}$ are

$$\tilde{a}_1^2 \sim \frac{1}{(\sum a_n^2)} \sim \frac{1}{a_n^2} \sim e^{-4\eta_{n_c}}.$$

Using $4\eta_{n_c}=4\pi(1-\lambda)^{-3/4}/2g$ we obtain

$$E_N \simeq \sum_\lambda \lambda^N a_0^2$$

$$\sim \sum_\lambda \lambda^N \exp[-\pi(1-\lambda)^{-3/4}\lambda/2g]. \quad (3.24)$$

Using the fact that the spectrum of M is dense near $\lambda \sim 1$, the sum of (3.24) can be converted to an integral and evaluated by the saddle-point method. The optimum value of λ is obtained from

$$N/\lambda = \pi(1-\lambda)^{-7/4}3/8g.$$

Consequently $\lambda_{max} \simeq 1 - (8Ng/3\pi)^{-4/7}$ and substituting into (3.24) we obtain

$$E_N \simeq \exp[-N^{3/7}g^{-4/7}(3\pi/8)^{4/7}7/3 + O(n^{2/7})]. \quad (3.25)$$

The behavior of E_N for the general problem of $V(x)$ Eq. (3.1) gives

$$E_N \simeq \exp[-N^{(L+1)/(2L^2-L+1)}g^{-2L(2L^2-L+1)} \times d_L(1+O(N^{-1/2L}))], \quad (3.26)$$

where

$$d_L = \frac{(2L^2-L+1)}{L+1}\rho^{2L(L-1)/(2L^2-L+1)},$$

$$\rho = \frac{L+1}{L(L-1)^2} \times \cos\left[\frac{\pi}{2L}\frac{\Gamma(1/L+1-1/2L)\Gamma(1+1/2L)}{\Gamma(1/L-1+1)}\right].$$

Thus the BW series defined by Eq. (3.3) is convergent for every $g>0$ and integer $L \geq 2$.

Let us now turn to using Lipatov's[2] method to investigate the RS series for our split. This method has been used by Lipatov and Brezin, LeGuillou, and Zinn-Justin to investigate large order of conventional perturbation theory. We again restrict ourselves to $L=2$. The generalization for $L>2$ is agian possible and gives results identical to (3.26).

Let us define

$$H(\lambda) = H_0 + \lambda H',$$
$$Z(\lambda) = \mathrm{Tr}(e^{-\beta H(\lambda)}), \quad (3.27)$$

where $H_0 = R + gR^2$, $H' = -gR^2 + gx^4/4$, $R = (p^2+x^2)/2$. Then

$$Z(\lambda) \sim \exp(e^{-\beta E(\lambda)}), \quad \beta \to \infty$$

where $E(\lambda)$ is the ground-state energy of $H(\lambda)$. Now define

$$E_N = \frac{1}{N!}\frac{\partial^N E(\lambda)}{\partial \lambda^N}\bigg|_{\lambda=0}. \quad (3.28)$$

Then we prove that if $\sum_N E_N < \infty$ and $(E_{N-k}-E_N)/E_N \to 0$ for $N \to \infty$, k fixed, then

$$Z_N \sim -TE_N e^{-TE(1)}, \quad N \to \infty. \quad (3.29)$$

The proof of the theorem is fairly straightforward; it follows from the expansion of $Z(\lambda)$ in powers of λ. The theorem implies that the leading asymptotic term of $\ln E_N$ is given by $\ln E_N / \ln Z_N \to 1$. We calculation Z_N by using the Hamiltonian functional integral form of Z:

$$Z \sim \int Dp\,Dx \exp\left\{\int_{-\beta/2}^{\beta/2} d\tau\left[ip(\tau)\dot{x}(\tau) - \frac{(p^2(\tau)+x^2(\tau))}{2} - g\left(\frac{p^2(\tau)+x^2(\tau)}{2}\right)^2(1-\lambda) + \frac{g\lambda x^4(\tau)}{4}\right]\right\}. \quad (3.30)$$

We use the boundary conditions $p(-\beta/2) = p(\beta/2)$ in the functional integral.

Before proceeding to the calculation we remark that this functional integral has an ordering problem. The operator $(p^2+x^2)^2$ does not simply translate to a function $(p^2(\tau)+x^2(\tau))^2$ but terms corresponding to the commutation $[p^2, x^2] \sim 2i(px+xp)$ also appear. It is easy to see that such terms do not change the leading asymptotic contribution to $\ln E_N$, the only quantity we are going to calculate. Therefore for the moment we are going to ignore the corresponding contributions.

Thus

$$Z_N = \frac{1}{N!}\int Dp\,Dx \times \exp\left(\int d\tau\left[ip\dot{x} - \frac{p^2+x^2}{2} - g\frac{(p^2+x^2)^2}{4}\right] + N\ln\left\{g\int d\tau\left[\frac{(p^2+x^2)^2}{4} - \frac{x^4}{4}\right]\right\}\right)$$
$$= \frac{1}{N!}\int Dp\,Dx\, e^{-A}. \quad (3.31)$$

We have dropped the arguments of the functions $p(\tau)$ and $x(\tau)$. The functional integral (3.31) is dominated by classical paths as $N \to \infty$. The canonical equations for the classical paths are

$$\frac{\delta(-A)}{\delta p} = i\dot{x} - p - gp(x^2+p^2) + \frac{N}{C}p(x^2+p^2) = 0,$$
$$\frac{\delta(-A)}{\delta x} = -i\dot{p} - x - gx(p^2+x^2) + \frac{N}{C}xp^2 = 0, \quad (3.32)$$

where

$$C = -\tfrac{1}{4}\int_{-\beta/2}^{\beta/2} d\tau[(x^2+p^2)^2 - x^4]. \quad (3.33)$$

The first integral of the canonical equations is obtained from

$$\frac{\delta(-A)}{\delta p}\dot{p} + \frac{\delta(-A)}{\delta x}\dot{x} = 0, \quad (3.34)$$

which integrated over τ gives

$$\frac{p^2+x^2}{2}+g\frac{(p^2+x^2)^2}{4}+\frac{N}{C}\left[\frac{x^4}{4}-\frac{(x^2+p^2)^2}{4}\right]=E=\text{const}. \quad (3.35)$$

The second integral is obtained from

$$\frac{\delta(-A)}{\delta x}p-\frac{\delta(-A)}{\delta p}x=0. \quad (3.36)$$

This gives

$$i\frac{d}{d\tau}\left(\frac{x^2+p^2}{2}\right)+\frac{N}{C}px^3=0. \quad (3.37)$$

It is advantageous to introduce the polar coordinates

$$p=\sqrt{2R}\cos\theta,\qquad x=\sqrt{2R}\sin\theta. \quad (3.38)$$

The boundary condition on the polar coordinates is $\theta(-\beta/2)=\pm\theta(\beta/2)$. Then Eqs. (3.35) and (3.37) can be restated:

$$R+gR^2-\frac{N}{C}R^2(1-\sin^4\theta)=E, \quad (3.39)$$

$$i\frac{d}{d\tau}\frac{1}{R}=4\sin^3\theta\cos\theta. \quad (3.40)$$

The instantonlike classical paths dominate the functional integral. These are the solutions that satisfy $R(\tau)\to 0$ for $\tau\to\pm\infty$. This implies $E=0$. If $E=0$, then R can be calculated from (3.39) and inserted into (3.40) to give

$$\dot\theta=i \quad (3.41)$$

or

$$\theta=i(\tau-\tau_0).$$

The condition $p(\beta/2)=p(-\beta/2)$ requires $\tau_0\sim 0$. The constant C is to be determined from the self-consistency Eq. (3.33). Substituting $Cg/N=1+\Delta$ and using (3.41) and (3.39) we find

$$Ng/(1+\Delta)=\int_{-\beta/2}^{\beta/2}d\tau\frac{[1-\sinh^4(\tau-\tau_0)]}{[\Delta+\sinh^4(\tau-\tau_0)]^2}. \quad (3.42)$$

The right-hand side is large only if $\Delta\to 0$. This gives

$$Ng\approx\int_{-\infty}^{\infty}d\tau\frac{1}{[\Delta+(\tau-\tau_0)^4]^2}. \quad (3.43)$$

There are two solutions of (3.43) for Δ. A $\Delta>0$ solution is

$$\Delta=\left(\frac{8Ng}{3\sqrt{2}\pi}\right)^{-4/7}. \quad (3.44)$$

Alternatively for $\Delta<0$ and choosing $\tau_0=\pm i\epsilon$ or using principal values on the integral

$$\Delta=-\left(\frac{8Ng}{3\pi}\right)^{-4/7}, \quad (3.45)$$

all prescriptions giving the same.

Now calculate the classical action corresponding to (3.45). The other solution will be shown later to be unstable. We can write

$$-A_0=\int_{-\infty}^{\infty}d\tau(iR\dot\theta-R-gR^2)+N\ln gC. \quad (3.46)$$

Now we use (3.39) and (3.41) and the above Δ to obtain

$$-A_0=-\tfrac{7}{3}N^{3/7}g^{-4/7}\left(\frac{3\pi}{8}\right)^{4/7}+N\ln(N/l).$$

Thus

$$\ln E_N=\tfrac{7}{3}N^{3/7}g^{-4/7}(3\pi/8)^{4/7}+O(N^{2/7}). \quad (3.47)$$

This agrees perfectly with our previous BW result.

At this point the effect of extra ordering terms is easily computed:

$$i\int d\tau\, x(\tau)p(\tau)\simeq 2i\int d\tau\, R\sin\theta\cos\theta\simeq O(N^{2/7}).$$

This proves our earlier statement that ordering problems do not affect the leading large-N behavior. The small oscillations around the classical solution do not alter the leading term of (3.47).

Let us now return to the $\Delta>0$ solution. We shall show that the corresponding classical solution is not stable; (3.44) does not correspond to a minimum of the action.

The action A can be expanded in a functional series around the classical solutions $p(\tau)$ and $x(\tau)$ as follows:

$$-A=-A_0+\tfrac{1}{2}\int d\tau_1 d\tau_2\left[\pi(\tau_2)\frac{\delta^2(-A)}{\delta p(\tau_2)\delta p(\tau_1)}\pi(\tau_1)+2\pi(\tau_1)\frac{\delta^2(-A)}{\delta p(\tau_1)\delta x(\tau_2)}\phi(\tau_2)+\phi(\tau_1)\frac{\delta^2(-A)}{\delta x(\tau_1)\delta x(\tau_2)}\phi(\tau_2)\right], \quad (3.48)$$

where

$$\frac{\delta^2(-A)}{\delta p(\tau_1)\delta p(\tau_2)}=-\left\{1+[3p^2(\tau_1)+x^2(\tau_2)]\left(g-\frac{N}{C}\right)\right\}\delta(\tau_1-\tau_2)-\frac{N}{C^2}p(\tau_1)[p^2(\tau_1)+x^2(\tau_1)]p(\tau_2)[p^2(\tau_2)+x^2(\tau_2)], \quad (3.49)$$

and similar equations for the other derivatives.

We show that at least one of the eigenvalues of the Schrödinger equation corresponding to the eigenmodes of the oscillations around the classical solution is negative. We use the variational approach by choosing a particular pair (ϕ, π) and calculate the quadratic part of the right-hand side of (3.48). Our choice is

$$\begin{bmatrix} \phi \\ \pi \end{bmatrix} = \begin{bmatrix} 0 \\ Rp \end{bmatrix} \left[-\int_{-\infty}^{\infty} R^2 p^2 d\tau \right]^{-1/2} . \quad (3.50)$$

Then we have the lowest eigenvalue,

$$\lambda_0 < \frac{N}{C^2} \int R^2 p^2 d\tau = -\frac{21}{32} (Ng)^{4/7} \left(\frac{8}{3\sqrt{2}\pi} \right)^{11/7} . \quad (3.51)$$

Thus a large negative eigenvalue exists. The $\Delta > 0$ solution is unstable and (3.47) gives the dominant term.

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