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σ models of nonlinear evolution equations

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A systematic method is developed of constructing the σ model associated with any given nonlinear evolution equation solvable by the inverse scattering method. The σ model is obtained from the adjoint representation of the group associated with the Lax representation of the evolution equation. Bäcklund transformations for the σ model and for the evolution equation are realized as gauge transformations. The complete integrability of the σ model follows from Pohlmeyer's R transformation which is systematically constructed in each case. The examples of the sine-Gordon, nonlinear Schrödinger, Korteweg–de Vries , and modified Korteweg–de Vries equations are discussed in detail.

I. INTRODUCTION

Among the attempts to understand the origin of the Lax representation of completely integrable nonlinear evolution equations,[1,2] the group-theoretical and differential-geometric interpretation of solitons, which began with the work of Wahlquist and Estabrook,[3] has attracted considerable attention.[4-6]

The differential-geometric character is brought about by casting such problems in the framework of fiber bundles with a structure group such as SU(2) or SL(2,R), where the Lax representation corresponds to a flat connection form on the bundle, describing the parallel transport of fibers. The flatness of the connection is equivalent to the nonlinear evolution equation.

Recent work on completely integrable σ models, on the other hand, has resulted in an interesting interconnection between σ models and nonlinear evolution equations. Namely, every σ model has an underlying nonlinear evolution equation associated with it. Using a process of "reduction", Pohlmeyer first showed how the relativistic O(n)-invariant σ model is intimately related to the sine-Gordon equation,[7] or its generalizations.[5,7,8] This reduction procedure was further clarified by Neveu and Papanicolaou,[9] who showed how to actually reconstruct the σ-model solutions from the Jost eigenfunctions of the Lax representation of the sine-Gordon equation. Similarly, the continuous Heisenberg spin chain, the complete integrability of which was first shown by Takhtadzhyan,[10] has been found by Jevicki and Papanicolaou,[11] Lakshmanan,[12] and Zakharov and Takhtadzhyan,[13] to be associated with the nonlinear Schrödinger equation.[14]

In this paper, we present a systematic method of constructing the σ model associated with any given nonlinear evolution equation solvable by the inverse scattering method. Instead of using a direct reduction procedure that assumes the knowledge of the σ model, we begin with a given nonlinear evolution equation and its Lax representation, which is then recast in the adjoint representation (moving trihedral) of the underlying structure group. The σ model is obtained from the adjoint representation by defining the σ field using Neveu and Papanicolaou's prescription[9] and finding its equation of motion from the moving trihedral equations. The two principal features of the resulting σ model, namely, the existence of Pohlmeyer's R transformations which imply the complete integrability of the σ model, and Bäcklund transformations, become simple consequences of the group-theoretical character of the problem. Bäcklund transformations as well as space-time transformations such as scale, Galilean, or Lorentz transformations are realized as gauge transformations, which, in the group space, correspond to group action from the left. R transformations, on the other hand, correspond to group action from the right. During the preparation of our work we became aware of the work of Zakharov and Takhtadzhyan in Ref. 13 where a similar method of constructing the R transformation was presented for the case of the Heisenberg spin chain. In Sec. II, we present the general formalism, and apply it in Secs. III, IV, and V to the examples of the sine-Gordon equation, nonlinear Schrödinger equation, and the Korteweg–de Vries (KdV) equations.

II. GENERAL FORMALISM

In this section we present a general method of constructing the σ model associated with any given nonlinear evolution equation. We follow closely the methods of Neveu and Papanicolaou in Ref. 9. We begin by briefly reviewing the geometric and group-theoretical aspects of nonlinear evolution equations, with particular emphasis on three well-known group representations in which to cast the Lax representation of the evolution equation. Namely, (i) the group space representation, in which the solution of the Lax-Zakharov-Shabat eigenvalue problem is most conveniently discussed, (ii) the coset space representation, which is most appropriate for the discussion of Bäcklund transformations, and (iii) the adjoint representation, which forms the starting point of our discussion of σ models. For a more precise and detailed treatment of the differential-geometric approach, the reader is referred to the original references (3-5) or the more recent ones.[6] For the moment, we need not specify the evolution equation, but shall assume that it admits a Lax representation of the Ablowitz-Kaup-Newell-Segur (AKNS)[2] type

$$\psi_x = A(u,\gamma)\psi, \quad \psi_t = B(u,\gamma)\psi, \tag{2.1}$$

where $u = u(x,t)$ is the solution of the evolution equation, γ is the eigenvalue parameter assumed to be real, and $\psi = \binom{\psi_1}{\psi_2}$ is the corresponding eigenstate. The 2×2 matrices A and B, which are functionals of u and γ, must satisfy the integrability condition

$$B_x - A_t = [A, B], \tag{2.2}$$

which should be equivalent to the nonlinear evolution equation for u.

The group-theoretical aspects of such problems arise from the fact that, in most cases of interest, the matrices A and B are infinitesimal elements of a group G which, depending on the evolution equation, is the group SU(2), SU(1,1), or SL(2,R). The examples discussed in the remaining sections cover all these cases.

To fix notations, let τ_i, $i=1,2,3$, denote the generators of G defined in terms of the Pauli matrices as

$$\tau_i = \sigma_i, \quad i=1,2,3 \tag{2.3}$$

for the SU(2) case,

$$\tau_1 = -i\sigma_1, \quad \tau_2 = -i\sigma_2, \quad \tau_3 = \sigma_3 \tag{2.4}$$

for the SU(1,1) case, and

$$\tau_1 = -i\sigma_1, \quad \tau_2 = \sigma_2, \quad \tau_3 = -i\sigma_3 \tag{2.5}$$

for SL(2,R). The τ matrices satisfy

$$\tau_i\tau_j = g_{ij} + ic_{ijk}\tau_k, \tag{2.6}$$

where g_{ij} and c_{ijk} are the Killing metric and structure constants of G. Inner and cross products between three-vectors, transforming by the adjoint representation of G, are defined in terms of g_{ij} and c_{ijk}, respectively.

Equation (2.1) can now be written, in an alternative representation, in the group space of G by using two linearly independent Jost solutions ψ to form a Jost matrix $g(x,t)$ which also satisfies (2.1). For example, it is easily shown that if ψ is a solution of (2.1) then so is $\hat{\psi} = -i\sigma_2 J\psi^*$, where J is the metric matrix of G defined to be $J=1$, σ_3, $i\sigma_2$, for SU(2), SU(1,1), SL(2,R), respectively. Then $g(x,t)$ may be defined as $g=[\psi,\hat{\psi}]$ for SU(2) and SU(1,1) and $g=[\sqrt{2}\,\mathrm{Re}\psi, \sqrt{2}\,\mathrm{Im}\psi]$ for SL(2,R). Since the matrices A and B are infinitesimal elements of G, they satisfy the Hermiticity property $A^\dagger J + JA = 0$, and similarly for B, from which it follows that the quantity $\psi^\dagger J\psi$ is constant in x and t, and thus may be set equal to $\psi^\dagger J\psi = 1$ for SU(2) and SU(1,1) and $\psi^\dagger J\psi = i$ for SL(2,R). This normalization is necessary in order to make g an element of G, satisfying $g^\dagger Jg = J$. The group space version of (2.1) is then

$$g_x = Ag, \quad g_t = Bg. \tag{2.7}$$

It can be rewritten in a form that brings out the fiber bundle structure of the problem as

$$d\,gg^{-1} = A\;dx + B\;dt, \tag{2.8}$$

where the left-hand side can be recognized as the canonical right-invariant Maurer-Cartan form of G, and the right-hand side as a connection form defined on the fiber bundle with base the space-time manifold and fiber the group G. The integrability conditions (2.2) imply that this connection is flat. Moreover, Eq. (2.8) may be interpreted as the parallel transport of fibers, as defined by this connection form.

In this representation, gauge transformations by $S(x,t)\in G$ correspond to group multiplication from the left,

$$\hat{g}(x,t) = S(x,t)g(x,t), \tag{2.9}$$

and Eqs. (2.7) transform to

$$\hat{g}_x = \hat{A}\hat{g}, \quad \hat{g}_t = \hat{B}\hat{g}, \tag{2.10}$$

where

$$\begin{aligned}\hat{A} &= SAS^{-1} + S_xS^{-1},\\ \hat{B} &= SBS^{-1} + S_tS^{-1}.\end{aligned} \tag{2.11}$$

Bäcklund transformations are realized as gauge transformations, in the sense that if the solution $\hat{u}$ of the evolution equation is obtained from a solution u by a Bäcklund transformation, then the Lax representation of $\hat{u}$ must be connected to that of u by a gauge transformation S such that $\hat{A} = A(\hat{u},\gamma)$ and $\hat{B} = B(\hat{u},\gamma)$. The explicit form of S was first constructed by Neveu and Papanicolaou[9] for the sine-Gordon equation, and we have done the same for the other examples that we discuss here. It must be emphasized that this view of Bäcklund transformations is different from that usually presented.[1-3,6] As stressed by Neveu and Papanicolaou,[9] the eigenvalue parameter γ must be clearly distinguished from the Bäcklund parameter appearing in the Bäcklund transformation, whereas in the usual approaches, the two parameters are identified.

A more general transformation is a space-time transformation $(x,t)\to(x',t')$ accompanied by a gauge transformation

$$g'(x',t')=S(x,t)g(x,t). \tag{2.12}$$

Galilean or Lorentz transformations of the evolution equation are realized in this manner. From the geometric point of view, Eq. (2.12) corresponds to a general coordinate transformation of the bundle onto itself. The property that a space-time symmetry requires that the space-time transformation be compensated by a simultaneous gauge transformation is very common in theories with magnetic monopoles.[15] This property will prove important in our discussion of the σ models.

Next we discuss two other useful representations of the inverse scattering equations. The first is realized in the coset space $G/\mathrm{U}(1)$ on which the group G acts nonlinearly by fractional transformations, and the second is the adjoint representation. Defining the coset space variable $z(x,t) = \psi_1/\psi_2$, we find that it moves according to

$$\begin{aligned} z_x &= a_{12}+(a_{11}-a_{22})z-a_{21}z^2, \\ z_t &= b_{12}+(b_{11}-b_{22})z-b_{21}z^2, \end{aligned} \tag{2.13}$$

where a_{ij} and b_{ij} are the entries of A and B. These are, of course, the Ricatti equations that are usually associated with the inverse scattering representations.[1,2] Equation (2.13) corresponds to the infinitesimal action of G on the coset space. Under a gauge transformation (2.9), z undergoes a linear fractional transformation

$$\hat{z}=\frac{s_{11}z+s_{12}}{s_{21}z+s_{22}}, \tag{2.14}$$

where s_{ij} are the entries of S.

The adjoint representation may be introduced by defining an orthonormal trihedral set of unit vectors $\vec{e}_i$, $i=1,2,3$, in terms of the Jost matrix

$$E_i=\vec{e}_i\cdot\vec{\tau}=g^{-1}\tau_i g, \quad i=1,2,3 \tag{2.15}$$

where the inner product is computed with the group metric g_{ij}. We find it more convenient to work, from now on, with the matrices E_i rather than the three-vectors $\vec{e}_i$. The trihedral moves according to

$$E_{ix}=A_{ij}E_j, \quad E_{it}=B_{ij}E_j, \tag{2.16}$$

where A_{ij} and B_{ij} are the three-dimensional (adjoint) representations of the matrices A and B. Using (2.6), the orthonormality and Hermiticity properties of the trihedral are expressed by

$$\begin{aligned} E_iE_j &= g_{ij}+ic_{ijk}E_k, \\ E_i^\dagger &= JE_iJ^{-1}. \end{aligned} \tag{2.17}$$

Under the gauge transformation (2.9) the trihedral transforms as

$$\hat{E}_i=\hat{g}^{-1}\tau_i\hat{g}=S_{ij}E_j, \tag{2.18}$$

where S_{ij} is the three-dimensional representation of S, and $\hat{E}_i$ moves according to (2.16) but with the transformed matrices $\hat{A}$ and $\hat{B}$, given in (2.11).

The moving trihedral (2.15) can also be subjected to any constant G rotation, under which (2.16) and (2.17) are left invariant. Such transformations correspond to group action from the right,

$$\hat{g}=gR, \tag{2.19}$$

where R is a constant element of G. Then E_i undergo a rotation by R

$$\hat{E}_i=R^{-1}E_iR, \quad i=1,2,3. \tag{2.20}$$

This transformation clearly leaves the matrices A and B in (2.16) invariant. From the bundle point of view, this reflects the particular convention that the group fiber is defined in terms of its right-invariant 1-forms, and therefore the group is defined to act on the bundle space by left translations, that is, gauge transformations. As we shall see shortly, Pohlmeyer's R transformation corresponds to (2.19) but R is allowed to have space-time dependence.

The moving trihedral E_i may be used next to define a σ model. Following Neveu and Papanicolaou's prescription[9] we define the σ field $\vec{q}(x,t)$ to be one of the unit vectors $\vec{e}_i$, most conveniently, the vector $\vec{e}_3$. That is, let

$$Q=\vec{q}\cdot\vec{\tau}=g^{-1}\tau_3 g. \tag{2.21}$$

Using (2.17) we find it satisfies the constraint

$$Q^2=\vec{q}\cdot\vec{q}=g_{33}, \tag{2.22}$$

where $g_{33}=1$ for SU(2) and SU(1,1), and $g_{33}=-1$ for SL(2,R).

The equation of motion satisfied by Q is obtained from (2.16) by differentiating it enough times with respect to x and t and eliminating the other two unit vectors E_1 and E_2 in favor of Q and its x and t derivatives. The exact form of the equation of motion depends, of course, on the evolution equation. In the process of deriving the equation for Q the various quantities, invariant under G, are identified in terms of the solution u and γ. The eigenvalue parameter γ which is usually related to the energy-momentum densities of Q can be changed in two ways: first, by performing a space-time transformation which is implemented as a gauge transformation of the form (2.12), and second, by Pohlmeyer's R transformation which can be done in the same frame of reference. In fact, the R transformation will be defined precisely by the requirement that it change the eigenvalue parameter from an initial value γ to a new value $\hat{\gamma}$, while leaving the solution u of the evolution equation unchanged. It should be distinguished from a Bäcklund transformation which does the opposite: It changes u but not γ. Another requirement for the R transformation is that it rotate Q as in (2.20),

$$\hat{Q}=R^{-1}QR\,, \tag{2.23}$$

and therefore, must be implemented by (2.19), where $\hat{g}$ will satisfy the Lax representation (2.7) with the new eigenvalue $\hat{\gamma}$:

$$\hat{g}_x=A(u,\hat{\gamma})\hat{g}\,,\quad \hat{g}_t=B(u,\hat{\gamma})\hat{g}\,. \tag{2.24}$$

To find the equations satisfied by R, we rewrite (2.19) as

$$R=g^{-1}\hat{g}\,, \tag{2.25}$$

which shows that R is obtainable from $\hat{g}$ by a special gauge transformation $S=g^{-1}$. Therefore, R will satisfy

$$R_x=\mathcal{A}R\,,\quad R_t=\mathcal{B}R\,, \tag{2.26}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the gauge transforms of $A(u,\hat{\gamma})$. That is,

$$\begin{aligned}\mathcal{A}&=g^{-1}A(u,\hat{\gamma})g-g^{-1}g_x\,,\\ \mathcal{B}&=g^{-1}B(u,\hat{\gamma})g-g^{-1}g_t\,.\end{aligned} \tag{2.27}$$

Using (2.7) we obtain

$$\begin{aligned}\mathcal{A}&=g^{-1}[A(u,\hat{\gamma})-A(u,\gamma)]g\,,\\ \mathcal{B}&=g^{-1}[B(u,\hat{\gamma})-B(u,\gamma)]g\,.\end{aligned} \tag{2.28}$$

Next we express $\mathcal{A}$ and $\mathcal{B}$ in terms of the initial moving trihedral E_i belonging to the eigenvalue γ. Expanding $A(u,\hat{\gamma})-A(u,\gamma)=ia^i\tau_i$ and $B(u,\hat{\gamma})-B(u,\gamma)=ib^i\tau_i$ and using the definition (2.15) we find from (2.28)

$$\begin{aligned}\mathcal{A}&=ia^iE_i\,,\\ \mathcal{B}&=ib^iE_i\,.\end{aligned} \tag{2.29}$$

Finally, eliminating E_i in favor of Q and its x and t derivatives we may express $\mathcal{A}$ and $\mathcal{B}$ only in terms of Q. The integrability condition of (2.26),

$$\mathcal{B}_x-\mathcal{A}_t=[\mathcal{A},\mathcal{B}]\,, \tag{2.30}$$

follows from that of (2.24) which is in turn equivalent to the evolution equation for u. The R transformation has a dual purpose[11]: First, it serves to establish the complete integrability of the resulting σ model, with Eqs. (2.26) as the corresponding Lax representation, and therefore, the compatibility conditions (2.30) must also follow directly from the σ-model equation of motion. Second, Eq. (2.23) defines a family of rotated σ fields $\hat{Q}$, which even though they may not necessarily satisfy the original σ-model equation of motion, they can be made to do so by subjecting $\hat{Q}$ to a further space-time transformation. These remarks are considered in more detail in the specific examples that follow.

III. THE SINE-GORDON EQUATION

In this section we illustrate our method by reproducing the well-known results of the O(3)-invariant σ model.[7,9] First we treat the sine-Gordon equation as an SU(2) problem, and then, as an SU(1,1) problem. The latter results into an O(2,1)-invariant σ model which has been discussed recently in connection with the axially symmetric solutions of Einstein's equations.[16] The sine-Gordon equation

$$\phi_{xt}=s\,in\phi \tag{3.1}$$

admits a Lax representation of the form (2.1), or (2.7) with matrices A and B defined as

$$\begin{aligned}A(\phi,\gamma)&=-\tfrac{1}{4}i\phi_x\sigma_3-\tfrac{1}{2}i\gamma(\sigma_1\cos\phi-\sigma_2\sin\phi)\,,\\ B(\phi,\gamma)&=\tfrac{1}{4}i\phi_t\sigma_3+\tfrac{1}{2}i\gamma^{-1}(\sigma_1\cos\phi+\sigma_2\sin\phi)\,,\end{aligned} \tag{3.2}$$

which are infinitesimal elements of SU(2). The corresponding moving trihedral and σ field Q are defined by (2.15) and (2.21). Defining the combination $E=E_1+iE_2$ and its conjugate $E^\dagger=E_1-iE_2$, the trihedral equations (2.16) take the form

$$\begin{aligned}Q_x&=-\tfrac{1}{2}i\gamma(Ee^{i\phi/2}-E^\dagger e^{-i\phi/2})\,,\\ E_x&=\tfrac{1}{2}i\phi_xE-i\gamma e^{-i\phi/2}Q\,,\\ Q_t&=\tfrac{1}{2}i\gamma^{-1}(Ee^{-i\phi/2}-E^\dagger e^{i\phi/2})\,,\\ E_t&=-\tfrac{1}{2}i\phi_tE+i\gamma^{-1}e^{i\phi/2}Q\,.\end{aligned} \tag{3.3}$$

Using the orthogonality properties (2.17) we obtain the O(3) invariants

$$Q_x^{\,2}=\vec{q}_x^{\,2}=\gamma^2\,,\quad Q_t^{\,2}=\vec{q}_t^{\,2}=\gamma^{-2}\,,\quad \vec{q}_x\cdot\vec{q}_t=-\cos\phi\,. \tag{3.4}$$

Differentiating the first of (3.3) with respect to t and using (3.4) we find the equation of motion for Q,

$$\vec{q}_{xt}+(\vec{q}_x\cdot\vec{q}_t)\vec{q}=0\,. \tag{3.5}$$

The well-known Bäcklund transformations of (3.1),[1]

$$\begin{aligned}\phi_x-\hat{\phi}_x&=2a\sin\left(\frac{\phi+\hat{\phi}}{2}\right),\\ \phi_t+\hat{\phi}_t&=2a^{-1}\sin\left(\frac{\phi-\hat{\phi}}{2}\right),\end{aligned} \tag{3.6}$$

are realized as gauge transformations of the form (2.9) with a gauge matrix S[9]:

$$S(\hat{\phi},\phi;a)=e^{-i\sigma_3\phi/4}e^{-i\sigma_1\theta}e^{i\sigma_3\hat{\phi}/4},\quad \tan\theta=a/\gamma. \tag{3.7}$$

As emphasized by Neveu and Papanicolaou,[9] the eigenvalue parameter γ must be clearly distinguished from the Bäcklund parameter a. In the usual way of deriving the Bäcklund transformation (3.6) from the inverse scattering representation, the eigenvalue and Bäcklund parameters are identified and the inverse scattering equations are then transformed into the Bäcklund equations (3.6).[1,2] The connection between the two approaches can be best seen in the coset space representation (2.13). Under the gauge transformation (3.7) the coset variable z transforms as in (2.14). Inverting (2.14) we write

$$z=\frac{\cos\theta e^{i(\phi-\hat{\phi})/4}\hat{z}+i\sin\theta e^{-i(\phi+\hat{\phi})/4}}{i\sin\theta e^{i(\phi+\hat{\phi})/4}\hat{z}+\cos\theta e^{i(\hat{\phi}-\phi)/4}}. \tag{3.8}$$

It is evident from (3.7) that in the limit $\gamma\to ia$, S develops a singularity. However, (3.8) remains finite and gives

$$z\big|_{\gamma=ia}=e^{-i\hat{\phi}/2}, \tag{3.9}$$

which is the usual identification (up to a gauge transformation). In fact, inserting this expression for z into (2.13) with a_{ij} and b_{ij} evaluated at $\gamma=ia$, (2.13) turns into the Bäcklund equations (3.6).

Under the gauge transformation (3.7), the moving trihedral and in particular the σ field Q transform by (2.18). Thus, the Bäcklund transformation for Q becomes

$$\hat{Q}=\cos(2\theta)Q-\tfrac{1}{2}i\sin(2\theta)\,(Ee^{-i\hat{\phi}/2}-E^{\dagger}e^{i\hat{\phi}/2}). \tag{3.10}$$

Lorentz transformations of the σ model are also realized as gauge transformations of the form (2.12), but in a trivial manner. Under the Lorentz transformation $(x',t')=(\rho^{-1}x,\rho t)$ the field ϕ transforms as a scalar $\phi'(x',t')=\phi(x,t)$, the eigenvalue parameter γ changes to $\gamma'=\rho\gamma$, and the Lax representation (2.7) undergoes a gauge transformation of the type (2.12) with S equal to the identity matrix, such that the transformed matrices A and B become $A'=A(\phi',\gamma')$ and $B'=B(\phi',\gamma')$. The σ field Q transforms according to (2.18) which gives in this case $Q'(x',t')=Q(x,t)$, and the invariants (3.4) are changed to the corresponding primed quantities.

Next we construct Pohlmeyer's R transformation. Considering Q as the initial σ-model solution, the R transformation will generate a new solution $\hat{Q}$ belonging to a new eigenvalue, say $\hat{\gamma}$, and satisfying (3.3)–(3.5) with $\hat{\gamma}$, but with the same ϕ. That is,

$$\hat{Q}_x=-\tfrac{1}{2}i\hat{\gamma}(\hat{E}e^{i\phi/2}-\hat{E}^{\dagger}e^{-i\phi/2}),\quad \hat{E}_x=\tfrac{1}{2}i\,\phi_x\hat{E}-i\hat{\gamma}e^{-i\phi/2}\hat{Q}, \tag{3.11}$$

$$\hat{Q}_t=\tfrac{1}{2}i\hat{\gamma}^{-1}(\hat{E}e^{-i\phi/2}-\hat{E}^{\dagger}e^{i\phi/2}),\quad \hat{E}_t=-\tfrac{1}{2}i\,\phi_t\hat{E}+i\hat{\gamma}^{-1}e^{i\phi/2}\hat{Q}. \tag{3.12}$$

The generators of the R transformation (2.26) can be computed easily from (2.29). We illustrate here some of the details. According to (2.28) we compute

$$A(\phi,\hat{\gamma})-A(\phi,\gamma)=-\tfrac{1}{4}i(\hat{\gamma}-\gamma)\left[(\sigma_1+i\sigma_2)\,e^{i\phi/2}+(\sigma_1-i\sigma_2)e^{-i\phi/2}\right], \tag{3.13}$$

$$B(\phi,\hat{\gamma})-B(\phi,\gamma)=\tfrac{1}{4}i\,(\hat{\gamma}^{-1}-\gamma^{-1})\left[(\sigma_1+i\sigma_2)e^{-i\phi/2}+(\sigma_1-i\sigma_2)e^{i\phi/2}\right],$$

and therefore (2.29) becomes

$$\mathcal{A}=-\tfrac{1}{4}i(\hat{\gamma}-\gamma)\left[Ee^{i\phi/2}+E^{\dagger}e^{-i\phi/2}\right],\quad \mathcal{B}=\tfrac{1}{4}i(\hat{\gamma}^{-1}-\gamma^{-1})\left[Ee^{-i\phi/2}+E^{\dagger}e^{i\phi/2}\right]. \tag{3.14}$$

The right-hand side may be written now in terms of Q alone. Using the trihedral equations (3.3) and the orthogonality properties (2.17) we find

$$QQ_x=-\tfrac{1}{2}i\gamma(Ee^{i\phi/2}+E^{\dagger}e^{-i\phi/2}),\quad QQ_t=\tfrac{1}{2}i\gamma^{-1}(Ee^{-i\phi/2}+E^{\dagger}e^{i\phi/2}). \tag{3.15}$$

Therefore, (3.14) become

$$\mathcal{A}=\frac{1}{2}\left(\frac{\hat{\gamma}}{\gamma}-1\right)QQ_x,\quad \mathcal{B}=\frac{1}{2}\left(\frac{\gamma}{\hat{\gamma}}-1\right)QQ_t, \tag{3.16}$$

which are the usual equations.[7] Their integrability conditions (2.30) follow directly from the σ-model equation for Q. Under the R transformation, Q and $\hat{Q}$ are related by

$$\hat{Q}=R^{-1}QR. \tag{3.17}$$

Next we consider briefly the SU(1,1) version of this formalism. It can be obtained if throughout the above the eigenvalue parameter γ is replaced by $\gamma\to i\gamma$. Then the matrices A and B become elements of SU(1,1), but their compatibility condition (2.2) is still equivalent to the sine-Gordon equation. The corresponding Jost matrix $g(x,t)$ defined in Sec. II will be an element of SU(1,1). The moving trihedral and the σ field will be defined by (2.15) and (2.21) with the choice (2.4) for the τ matrices. The Killing metric is now $g_{ij}=(-1,-1,1)$ and the σ field will satisfy the constraint (2.22) and Eqs. (3.4) and (3.5) with inner products computed with the metric g_{ij}.

We close this section by showing an interesting application of the gauge property of Bäcklund transformations. That is, we may derive the well-known algebraic combination formula among four solutions of the sine-Gordon equation that are connected by successive Bäcklund transformations[1]

$$\tan\left(\frac{\phi_{ab}-\phi}{4}\right)=\frac{a+b}{a-b}\tan\left(\frac{\phi_a-\phi_b}{4}\right), \quad (3.18)$$

as the condition of commutativity of Bäcklund transformations. In (3.18) the solutions ϕ_a and ϕ_b are obtained from an initial solution ϕ by the Bäcklund transformation (3.6) with parameters a and b, respectively, and ϕ_{ab} is obtained from ϕ_a or ϕ_b by a further Bäcklund transformation with parameter b or a, respectively. The commutativity of the corresponding gauge transformations (3.7)

$$S(\phi_{ab},\phi_a;b)S(\phi_a,\phi;a)=S(\phi_{ab},\phi_b;a)S(\phi_b,\phi;b) \quad (3.19)$$

is easily seen to be equivalent to (3.18).

IV. THE NONLINEAR SCHRÖDINGER EQUATION

In this section we consider the example of the nonlinear Schrödinger equation

$$iu_t+u_{xx}+2|u|^2u=0, \quad (4.1)$$

which is known to be associated with the classical continuous Heisenberg spin chain.[10-13] Equation (4.1) admits a Lax representation of the form (2.1) or (2.7) with matrices A and B,[14]

$$A(u,\gamma)=\begin{pmatrix} i\gamma & u \\ -u^* & -i\gamma\end{pmatrix},$$
$$B(u,\gamma)=\begin{pmatrix} i|u|^2-2i\gamma^2 & iu_x-2\gamma u \\ iu_x^*+2\gamma u^* & 2i\gamma^2-i|u|^2\end{pmatrix}, \quad (4.2)$$

which are infinitesimal elements of SU(2). The consistency condition (2.2) is equivalent to (4.1). The corresponding moving trihedral E_i and σ field Q are again defined by (2.15) and (2.21). Equations (2.16) take the form

$$\begin{aligned} Q_x&=uE+u^*E^\dagger,\\ E_x&=-2i\gamma E-2u^*Q,\\ Q_t&=(iu_x-2\gamma u)E-(iu_x^*+2\gamma u^*)E^\dagger,\\ E_t&=2(2i\gamma^2-i|u|^2)E+2(iu_x^*+2\gamma u^*)Q, \end{aligned} \quad (4.3)$$

where $E=E_1+iE_2$. The O(3) invariants are

$$\begin{aligned} Q_x{}^2&=\vec{q}_x{}^2=4|u|^2,\\ \vec{q}\cdot(\vec{q}_x\times\vec{q}_{xx})&=2i(uu_x^*-u^*u_x)-8\gamma|u|^2. \end{aligned} \quad (4.4)$$

The equation of motion for the σ field Q is found from (4.3) to be

$$Q_t+4\gamma Q_x+\frac{1}{2i}[Q,Q_{xx}]=0, \quad (4.5)$$

or equivalently

$$\vec{q}_t+4\gamma\vec{q}_x+\vec{q}\times\vec{q}_{xx}=0. \quad (4.6)$$

For $\gamma=0$, (4.6) is the Heisenberg chain. The eigenvalue γ can be changed to zero, or any other value, in two ways: first, by a Galilean transformation, and second, by Pohlmeyer's R transformation.

Galilean transformations $(x',t')=(x-vt,t)$ are implemented as gauge transformations. It is easily seen that the gauge transformation (2.12), where

$$S(x,t)=e^{i\sigma_3\theta(x,t)/2},\quad \theta(x,t)=\tfrac{1}{4}v^2t-\tfrac{1}{2}vx, \quad (4.7)$$

takes the Lax representation defined with the matrices (4.2) into

$$g'_{x'}=A'g',\quad g'_{t'}=B'g', \quad (4.8)$$

where $A'=A(u',\gamma')$ and $B'=B(u',\gamma')$, where u' is the Galilean transformed u,

$$u'(x',t')=e^{i\theta(x,t)}u(x,t), \quad (4.9)$$

satisfying (4.1) in the primed coordinate frame, and γ' is

$$\gamma'=\gamma-\tfrac{1}{4}v. \quad (4.10)$$

Thus, clearly, by choosing $v=4\gamma$, the eigenvalue γ may be set equal to zero. The σ field Q transforms according to (2.18), which gives in this case $Q'(x',t')=Q(x,t)$, and the equation of motion

$$Q'_{t'}+4\gamma'Q'_{x'}+\frac{1}{2i}[Q',Q'_{x'x'}]=0. \quad (4.11)$$

The other components of the trihedral transform as $E'(x',t')=e^{-i\theta(x,t)}E(x,t)$. The field Q transforms as a scalar in both the sine-Gordon and the present cases. This is due to the fact that the gauge matrix S, implementing the space-time transformation, commutes with the generator τ_3 in the definition (2.21) of Q. A more interesting situation appears in the KdV case in Sec. V.

Next we construct the R transformation. We begin with the σ-model solution Q belonging to the eigenvalue γ, and transform it by R to the new solution $\hat{Q}$ belonging to $\hat{\gamma}$. Looked at from an appropriate Galilean frame, we could have started with $\gamma=0$. The trihedral equations for $\hat{Q}$ are

$$\begin{aligned} \hat{Q}_x&=u\hat{E}+u^*\hat{E}^\dagger,\\ \hat{E}_x&=-2i\hat{\gamma}\hat{E}-2u^*\hat{Q},\\ \hat{Q}_t&=(iu_x-2\hat{\gamma}u)\hat{E}-(iu_x^*+2\hat{\gamma}u^*)\hat{E}^\dagger,\\ \hat{E}_t&=2(2i\hat{\gamma}^2-i|u|^2)\hat{E}+2(iu_x^*+2\hat{\gamma}u^*)\hat{Q}, \end{aligned} \quad (4.12)$$

and its equation of motion

$$\hat{Q}_t+4\hat{\gamma}\hat{Q}_x+\frac{1}{2i}[\hat{Q},\hat{Q}_{xx}]=0. \quad (4.13)$$

The R transformation is designed to rotate Q into $\hat{Q}$,

$$\hat{Q} = R^{-1}QR . \tag{4.14}$$

The generators $\mathcal{A}$ and $\mathcal{B}$ of R, are computed using the procedure explained in (2.28) and (2.29). We find

$$A(u,\hat{\gamma}) - A(u,\gamma) = i(\hat{\gamma}-\gamma)\sigma_3 ,$$
$$B(u,\hat{\gamma}) - B(u,\gamma) = -2i(\hat{\gamma}^2-\gamma^2)\sigma_3 - (\hat{\gamma}-\gamma)\,[u(\sigma_1+i\sigma_2) - u^*(\sigma_1 - i\sigma_2)] , \tag{4.15}$$

and (2.29) becomes

$$\mathcal{A} = i(\hat{\gamma}-\gamma)Q ,$$
$$\mathcal{B} = -2i(\hat{\gamma}^2-\gamma^2)Q - (\hat{\gamma}-\gamma)(uE - u^*E^\dagger) . \tag{4.16}$$

Using (4.3) we rewrite (4.16) only in terms of Q, as

$$\mathcal{A} = i(\hat{\gamma}-\gamma)Q ,$$
$$\mathcal{B} = -2i(\hat{\gamma}^2-\gamma^2)Q - \tfrac{1}{2}(\hat{\gamma}-\gamma)\,[Q,Q_x] , \tag{4.17}$$

which for $\gamma = 0$ are the expressions given by Takhtadzhyan.[10] Similar methods for constructing R have been given by Zakharov and Takhtadzhyan.[13] The integrability conditions (2.30) for $\mathcal{A}$ and $\mathcal{B}$ are easily shown to also follow from the σ-model equations of motion (4.5) of Q. If the field $\hat{Q}$ is further subjected to a Galilean transformation with velocity $v = 4(\hat{\gamma}-\gamma)$, the parameter $\hat{\gamma}$ may be set back to its initial value γ, thus obtaining a σ field that satisfies the initial equation (4.5) in the new frame.[11]

Finally, we discuss Bäcklund transformations in the present model. Equation (4.1) admits a Bäcklund transformation of the form[17]

$$u_x - \hat{u}_x = (u+\hat{u})\,(4\eta^2 - |u-\hat{u}|^2)^{1/2} ,$$
$$u_t - \hat{u}_t = i(u_x + \hat{u}_x)\,(4\eta^2 - |u-\hat{u}|^2)^{1/2} + i(u-\hat{u})\,(|u|^2 + |\hat{u}|^2) , \tag{4.18}$$

where η is the Bäcklund parameter, and $\hat{u}(x,t)$ satisfies (4.1) if $u(x,t)$ does. In Ref. 17, a more general form for the Bäcklund transformation is given which, however, may be reduced to (4.18) by an appropriate Galilean transformation. Much like the sine-Gordon case, Bäcklund transformations are realized as gauge transformations of the form (2.9) with

$$S = \begin{pmatrix} \cos\theta + i\sin\theta a & -i\sin\theta b \\ -i\sin\theta b^* & \cos\theta - i\sin\theta a \end{pmatrix}, \quad \tan\theta = \eta/\gamma \tag{4.19}$$

$$b = \frac{u-\hat{u}}{2\eta}, \quad a = (1 - |b|^2)^{1/2} ,$$

in the sense that the transformed matrices (2.11) become $\hat{A} = A(\hat{u},\gamma)$ and $\hat{B} = B(\hat{u},\gamma)$ provided (4.18) holds. Using (2.18) we find the corresponding transformation for σ field Q,

$$\hat{Q} = (1 - 2\sin^2\theta\,|b|^2)Q - b\sin\theta(i\cos\theta + a\sin\theta)E + b^*\sin\theta(i\cos\theta - a\sin\theta)E^\dagger . \tag{4.20}$$

The relationship to the usual approach of obtaining Bäcklund transformations is best seen, again, in the coset representation (2.13). Under (4.19) the coset variable z transforms by (2.14), the inverse of which is in this case

$$z = \frac{(\cos\theta - ia\sin\theta)\hat{z} + i\sin\theta b}{i\sin\theta b^*\hat{z} + \cos\theta + ia\sin\theta} . \tag{4.21}$$

In the limit $\gamma \to i\eta$, (4.21) gives

$$z\big|_{\gamma=i\eta} = \frac{2\eta - (4\eta^2 - |u-\hat{u}|^2)^{1/2}}{(u-\hat{u})^*} , \tag{4.22}$$

which is the usual identification.[17]

V. KdV EQUATIONS

In this section we consider two equivalent Lax representations for the KdV equation. One is of the AKNS type,[2] and the other is that of Wahlquist and Estabrook.[3] They are connected to each other by a gauge transformation, and they lead to two different σ models. The analysis of these two cases leads us to the conclusion that the right choice for the σ field may not always be given by Eq. (2.21), but is dictated primarily by the form of the R transformation and the requirement that the compatibility of the R Eqs. (2.26) should be equivalent to the σ-model equations of motion, as well as to the underlying evolution equation.

In addition, we consider the SL(2,R) version of the modified KdV equation, and show how Miura's transformation is realized as a gauge transformation between the Lax representations of the KdV and modified KdV equations.

The first Lax representation that we consider is defined by the infinitesimal SL(2,R) matrices A and B,[2]

$$A(u,\gamma) = -i\gamma\tau_3 + \tfrac{1}{2}iu(\tau_1+\tau_2) - i(\tau_1 - \tau_2) ,$$
$$B(u,\gamma) = ia\tau_3 + \tfrac{1}{2}ib(\tau_1+\tau_2) + \tfrac{1}{2}ic(\tau_1-\tau_2) , \tag{5.1}$$

with the choice (2.5) for the τ matrices, where

$$a = 4\gamma^3 + 4\gamma u - 2u_x ,$$
$$b = 2\gamma u_x - 4\gamma^2 u - u_{xx} - 4u^2 , \tag{5.2}$$
$$c = 8u + 8\gamma^2 .$$

The integrability conditions (2.2) are equivalent to the KdV equation

$$u_t + u_{xxx} + 12uu_x = 0 . \tag{5.3}$$

The moving trihedral and the σ field Q are defined by (2.15) and (2.21). The trihedral Eqs. (2.16) become

$$\begin{aligned} &Q_x = 2E + uF , \quad Q_t = bF - cE , \\ &E_x = -2\gamma E - 2uQ , \quad E_t = 2aE - 2bQ , \\ &F_x = 2\gamma F - 4Q , \quad F_t = 2cQ - 2aF , \end{aligned} \tag{5.4}$$

where $E = E_1 - E_2$ and $F = E_1 + E_2$. The equation of motion for the σ field is easily found from (5.4) to be

$$Q_t + Q_{xxx} - \tfrac{3}{2}(Q_x^2 Q)_x = 3i\gamma [Q, Q_{xx}] - 12\gamma^2 Q_x . \tag{5.5}$$

The SL(2,R) invariants that follow from (5.4) and (2.17) are

$$Q_x^2 = \vec{q}_x^{\,2} = -8u , \quad \vec{q}\cdot(\vec{q}_x \times \vec{q}_{xx}) = u_x + 8\gamma u . \tag{5.6}$$

Galilean transformations of the KdV equation (5.3) are realized as gauge transformations of the form (2.12). Under the transformation

$$\begin{aligned} &x' = x - vt , \\ &t' = t , \\ &u'(x', t') = u(x, t) - \tfrac{1}{12} v , \end{aligned} \tag{5.7}$$

the KdV equation remains invariant. The Lax representation (5.1) gets transformed according to (2.12) with the SL(2,R) gauge matrix S,

$$S = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad \alpha = [-\gamma \pm (\gamma^2 - \tfrac{1}{6}v)^{1/2}]/2 \tag{5.8}$$

into $A' = A(u', \gamma')$ and $B' = B(u', \gamma')$, where $\gamma' = \pm(\gamma^2 - \frac{1}{6}v)^{1/2}$. The trihedral (5.4), and in particular the σ field, transforms as

$$\begin{aligned} &Q'(x', t') = Q(x, t) + \alpha F(x, t) , \\ &E'(x', t') = E(x, t) - 2\alpha Q(x, t) - \alpha^2 F(x, t) , \\ &F'(x', t') = F(x, t) , \end{aligned} \tag{5.9}$$

and Q' satisfies (5.4) and (5.5) in the primed variables. The eigenvalue γ, therefore, can be changed to zero or any other value by an appropriate Galilean transformation.

Next we construct the R transformation. As in previous sections, we begin with the initial σ field Q belonging to the eigenvalue γ, and transform it to $\hat{Q}$ belonging to $\hat{\gamma}$. The trihedral equations for $\hat{Q}$ are

$$\begin{aligned} &\hat{Q}_x = 2\hat{E} + u\hat{F} , \quad \hat{Q}_t = \hat{b}\hat{F} - \hat{c}\hat{E} , \\ &\hat{E}_x = -2\hat{\gamma}\hat{E} - 2u\hat{Q} , \quad \hat{E}_t = 2\hat{a}\hat{E} - 2\hat{b}\hat{Q} , \\ &\hat{F}_x = 2\hat{\gamma}\hat{F} - 4\hat{Q} , \quad \hat{F}_t = 2\hat{c}\hat{Q} - 2\hat{a}\hat{F} , \end{aligned} \tag{5.10}$$

and the equation of motion

$$\hat{Q}_t + \hat{Q}_{xxx} - \tfrac{3}{2}(\hat{Q}_x^2 \hat{Q})_x = 3i\hat{\gamma}[\hat{Q}, \hat{Q}_{xx}] - 12\hat{\gamma}^2 \hat{Q}_x . \tag{5.11}$$

The generators $\mathcal{A}$ and $\mathcal{B}$ of the R transformation are constructed according to (2.28) and (2.29). Using (5.4) we may express them in terms of Q only, as

$$\begin{aligned} &\mathcal{A} = -i(\hat{\gamma} - \gamma)Q , \\ &\mathcal{B} = 4i(\hat{\gamma}^3 - \gamma^3)Q + i(\hat{\gamma} - \gamma)(Q_{xx} - \tfrac{3}{2}Q_x^2 Q) \\ &\qquad + (\hat{\gamma} - \gamma)(\hat{\gamma} + 2\gamma)[Q, Q_x] . \end{aligned} \tag{5.12}$$

The integrability condition (2.30) is equivalent to the equation of motion (5.5) for Q.

The second Lax representation that we discuss has generator matrices

$$\begin{aligned} &A(u, \gamma) = \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix}, \\ &B(u, \gamma) = \begin{pmatrix} \alpha_x & -\beta \\ \alpha\beta + \alpha_{xx} & -\alpha_x \end{pmatrix}, \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} &\alpha = 2u - \gamma^2 , \\ &\beta = 4u + 4\gamma^2 . \end{aligned} \tag{5.14}$$

Their compatibility condition (2.2) is equivalent to the KdV equation. The two Lax representations are related to each other by a γ-dependent SL(2,R) gauge transformation

$$S_c = \frac{1}{\sqrt{2}} \begin{pmatrix} -\gamma & 1 \\ -2 & 0 \end{pmatrix}, \tag{5.15}$$

which transforms (5.13) into (5.1). The corresponding moving trihedrals transform into each other by (2.18). The trihedral of the representation (5.13) moves according to

$$\begin{aligned} &Q_x = \alpha E + F , \quad Q_t = -(\alpha\beta + \alpha_{xx})E - \beta F , \\ &E_x = -2Q , \quad E_t = 2\beta Q + 2\alpha_x E , \\ &F_x = -2\alpha Q , \quad F_t = 2(\alpha\beta + \alpha_{xx})Q - 2\alpha_x F . \end{aligned} \tag{5.16}$$

Galilean transformations are implemented with a gauge matrix which can be computed from (5.8) and the connecting matrix (5.15). It turns out to be the unit matrix, and therefore under (5.7) the trihedral (5.16) transforms as

$$\begin{aligned} &Q'(x', t') = Q(x, t) , \\ &E'(x', t') = E(x, t) , \\ &F'(x', t') = F(x, t) , \end{aligned} \tag{5.17}$$

and satisfies (5.16) in the primed variables.

At first glance it might be thought that the proper choice for the σ field should be the component Q of the trihedral. However, the derivation of the R transformation in Eqs. (2.28) and (2.29) suggests that the field $E = E_1 - E_2$ be chosen as the σ field. E is a "lightlike" vector satisfying $E^2=0$. To see this, we may start with an initial value for the eigenvalue parameter γ. Without loss of generality we may choose the initial value $\gamma=0$. The trihedral equations become

$$\begin{aligned} &Q_x=F+2uE\,,\quad Q_t=-2(4u^2+u_{xx})E-4uF\,,\\ &E_x=-2Q\,,\quad E_t=8uQ+4u_xE\,,\\ &F_x=-4uQ\,,\quad F_t=4(4u^2+u_{xx})Q-4u_xF\,. \end{aligned}\tag{5.18}$$

Using these and (2.28) and (2.29) we compute the generator matrices $\mathcal{A}$ and $\mathcal{B}$ of the R transformation as

$$\begin{aligned} &\mathcal{A}=\tfrac{1}{2}i\hat{\gamma}^2E\,,\\ &\mathcal{B}=2i\hat{\gamma}^2(u-\hat{\gamma}^2)E-2i\hat{\gamma}^2F\,. \end{aligned}\tag{5.19}$$

Comparing these with the expressions in the previous sections, we choose E as the σ field. The equation of motion for E follows from (5.18)

$$\begin{aligned} &E_t+Q_{xx}+\tfrac{3}{2}i[Q,Q_{xx}]=0\,,\\ &E_x=-2Q\,. \end{aligned}\tag{5.20}$$

The R transformation rotates E into $\hat{E}$,

$$\hat{E}=R^{-1}ER\,,\quad \hat{Q}=R^{-1}QR\,,\tag{5.21}$$

which satisfies

$$\begin{aligned} &\hat{E}_t+\hat{Q}_{xx}+\tfrac{3}{2}i[\hat{Q},\hat{Q}_{xx}]=-6\hat{\gamma}^2\hat{E}_x\,,\\ &\hat{E}_x=-2\hat{Q}\,. \end{aligned}\tag{5.22}$$

The extra term $-6\hat{\gamma}^2\hat{E}_x$ may in turn be removed by a Galilean transformation, of the form (5.17) with velocity $v=6\hat{\gamma}^2$, which resets the eigenvalue parameter $\hat{\gamma}$ to its zero value. The matrices $\mathcal{A}$ and $\mathcal{B}$ can also be expressed in terms of E only:

$$\begin{aligned} &\mathcal{A}=\tfrac{1}{2}i\hat{\gamma}^2E\,,\\ &\mathcal{B}=-2i\hat{\gamma}^4E-\tfrac{1}{2}i\hat{\gamma}^2(Q_x+\tfrac{3}{2}i[Q,Q_x])\,. \end{aligned}\tag{5.23}$$

Their compatibility condition (2.30) is equivalent to the equation of motion (5.20), provided one uses the orthonormality property $QE=iE$ which follows from (2.17).

The KdV equation admits Bäcklund transformations, first constructed by Wahlquist and Estabrook,[18] of the form

$$\begin{aligned} &w_x+\hat{w}_x=-\eta^2+(w-\hat{w})^2\,,\\ &w_t+\hat{w}_t=4(u^2+u\hat{u}+\hat{u}^2)+2(u_x-\hat{u}_x)(w-\hat{w})\,, \end{aligned}\tag{5.24}$$

where w and $\hat{w}$ are pseudopotentials related to the two solutions u and $\hat{u}$ of (5.3) by

$$u=-w_x\,,\quad \hat{u}=-\hat{w}_x\,,\tag{5.25}$$

and η is the Bäcklund parameter. Bäcklund transformations are realized as gauge transformations (2.9) with SL(2,R) gauge matrix

$$\begin{aligned} S&=S(\hat{w},w;\eta)\\ &=\frac{1}{(\eta^2-\gamma^2)^{1/2}}\begin{pmatrix} w-\hat{w} & -1\\ \eta^2-\gamma^2-(w-\hat{w})^2 & w-\hat{w}\end{pmatrix}. \end{aligned}\tag{5.26}$$

The transformed Lax representation has generator matrices $\hat{A}=A(\hat{u},\gamma)$ and $\hat{B}=B(\hat{u},\gamma)$. The gauge matrix that corresponds to the first Lax representation (5.1) can be obtained from the connecting matrix S_c as $S_cSS_c^{-1}$. The moving trihedrals, and in particular the σ fields of either representation, transform according to (2.18). The relationship to the usual way of deriving Bäcklund transformations can be found using the coset representation (2.14) which reads

$$z=\frac{(w-\hat{w})\hat{z}+1}{[\gamma^2-\eta^2+(w-\hat{w})^2]\hat{z}+w-\hat{w}}\,,\tag{5.27}$$

and in the limit $\gamma\to\eta$ gives

$$z\big|_{\gamma=\eta}=\frac{1}{w-\hat{w}}\,,\tag{5.28}$$

which is the usual identification. Using (5.26) we may rederive the algebraic combination formula among four solutions of the KdV equation connected by successive Bäcklund transformations. Requiring the commutativity of the matrices (5.26), that is,

$$S(w_{12},w_1;\eta_2)S(w_1,w;\eta_1)=S(w_{12},w_2;\eta_1)S(w_2,w;\eta_2),$$

we find[18]

$$w_{12}-w=\frac{\eta_1{}^2-\eta_2{}^2}{w_1-w_2}.\tag{5.29}$$

Finally we discuss the SL(2,R) version of the modified KdV equation

$$v_t+v_{xxx}-6v^2v_x=0\,.\tag{5.30}$$

Its Lax representation is[2]

$$A(v,\gamma)=\begin{pmatrix}-\gamma & v\\ v & \gamma\end{pmatrix},\quad B(v,\gamma)=\begin{pmatrix}a & b\\ c & -a\end{pmatrix},\tag{5.31}$$

where

$$\begin{aligned} &a=4\gamma^3-2\gamma v^2\,,\\ &b=2\gamma v_x-4\gamma^2v-v_{xx}+2v^3\,,\\ &c=-2\gamma v_x-4\gamma^2v-v_{xx}+2v^3\,. \end{aligned}\tag{5.32}$$

The corresponding trihedral moves according to

$$\begin{aligned} &Q_x=vF-vE\,,\quad Q_t=bF-cE\,,\\ &E_x=-2\gamma E-2vQ\,,\quad E_t=2aE-2bQ\,,\\ &F_x=2\gamma F+2vQ\,,\quad F_t=2cQ-2aF\,, \end{aligned}\tag{5.33}$$

where again $E=E_1-E_2$ and $F=E_1+E_2$. The σ field satisfies the equation of motion

$$Q_t + Q_{xxx} - \tfrac{3}{2}(Q_x{}^2 Q)_x = 3i\gamma [Q, Q_{xx}] - 12\gamma^2 Q_x, \qquad (5.34)$$

which is the same as for the KdV case, Eq. (5.5), but with SL(2,R) invariants

$$Q_x{}^2 = \vec{q}_x{}^2 = 4v^2, \quad \vec{q}\cdot(\vec{q}_x \times \vec{q}_{xx}) = -4v^2\gamma. \qquad (5.35)$$

The R transformation can be easily constructed.

The KdV and modified KdV equations are related by Miura's transformation[19]

$$u = -\tfrac{1}{2}(v_x + v^2), \qquad (5.36)$$

which is a sort of Bäcklund transformation, and therefore can be realized as a gauge transformation between the Lax representations. In fact, it is easily shown that the SL(2,R) gauge matrix

$$S = \gamma^{-1/2}\begin{pmatrix} \frac{1}{2}v - \gamma & \frac{1}{2}v \\ -1 & -1 \end{pmatrix} \qquad (5.37)$$

takes (5.31) into (5.1), provided u and v are related by (5.36).

VI. CONCLUSION

The above method of constructing σ models from underlying evolution equations is quite general. The basic requirement is that the Lax representation of the evolution equation be associated with a structure group G. The σ model and its R transformation are constructed from the adjoint representation of the group. For example, we have used this method to derive an SU(n)-invariant Heisenberg spin chain[20] starting from a U($n-1$)-invariant nonlinear Schrödinger equation. It can be applied also to discrete evolution equations as long as they are associated with a structure group G, as for example, in the discrete sine-Gordon equation case, which results in a discrete version of the O(3)-invariant σ model.[2] It may also be applicable to evolution equations in more than one space dimension. We have not discussed the infinite number of nonlocal conserved charges that characterize σ models, since they can be derived systematically from the R transformation as in Refs. 22 and 11. The method has a number of shortcomings. First, to use it one must know the Lax representation of the evolution equation. Second, if one considers the σ model as the basic theory, rather than the evolution equation, the method is incomplete, in the sense that it generates σ-model solutions, where the group invariants are not necessarily independent of each other, as was the case for the KdV equation. It may very well be possible that if the σ model is "reduced" directly, it might result in a larger underlying evolution equation than the one used to derive it.

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