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Equation of state at ultrahigh densities and the speed of sound

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We consider a system of baryons interacting through the exchange of massive neutral vector mesons at densities of $\rho \geq 10^{15}$ g/cm^3, where they form a lattice. For densities $10^{16} \geq \rho \geq 10^{15}$ g/cm^3 we show that $\epsilon > 3P$ and $v_s < 1$ even though the interaction energy dominates the kinetic energy. We discuss the practicality of this model and its implications for the neutron-star mass limit. We also consider its relation to the work of Zel'dovich and of Bludman and Ruderman.

I. INTRODUCTION

One of the unsettled problems of theoretical physics is to find an equation of state at ultrahigh densities. Even though the structure of matter is pretty much understood for densities up to $\rho \sim 3 \times 10^{14}$ g/cm^3, there seem to be no reliable calculations for densities above this value.[1] A knowledge of the equation of state for $10^{15} \lesssim \rho \lesssim 10^{16}$ g/cm^3 is necessary in order to find a mass limit for neutron stars.[2] This mass limit is critical in the identification of some observed compact objects as black holes, one of the novel predictions of general relativity. Even though the exact equation of state is going to be rather complicated, one still can make arguments about its asymptotic form and the limitations placed on this form by the theory of relativity.

For some time it was generally accepted that the equation of state for massive particles at high densities approaches $3P = \epsilon$, the equation of state for massless particles, and satisfies $3P < \epsilon$ for low densities. The reason for this is that at high densities the rest mass in energy density ϵ will be negligible compared to the kinetic energy, and hence all the particles will behave as massless particles. This is the case for a noninteracting Fermi gas and remains the case even when electromagnetic interactions are considered.[3]

In general let ϵ have a power-law dependence on the number density n, so that

$$\epsilon = an^{\gamma}, \quad P = (\gamma - 1)\epsilon$$

and

$$v_s^2 = \frac{dP}{d\epsilon} = (\gamma - 1) \quad (c = 1).$$

Thus $3P = \epsilon$ corresponds to $\gamma = \frac{4}{3}$ and $dP/d\epsilon = \frac{1}{3}$, which is the speed of low-frequency sound waves. Of course, from causality we expect $\gamma \leqslant 2$ for $v_s^2 \leqslant 1$.

Zel'dovich[4] was the first to show that for a gas of particles interacting through the exchange of massive neutral vector mesons, the inequality $\epsilon > 3P$ could be violated. He found that, in the continuum limit (infinite densities), P/ϵ and the speed of sound approach unity but never exceed it. The interaction potential he considered was

$$\phi = g^2 \frac{e^{-\mu r}}{r},$$

where g is the baryonic charge and μ^{-1} is the range of the interaction.

The total energy density of the system can be given as

$$\epsilon = n\left(m + \frac{\epsilon_{\text{Fermi}}}{n} + \frac{g^2}{2a^3} \sum_{i \neq j} \frac{e^{-\mu r_{ij}}}{r_{ij}} a^3\right). \tag{1.1}$$

In the continuum limit $a^3 \to dv$, where a is the interparticle spacing, and the summation becomes an integral giving

$$\epsilon = nm + \epsilon_{\text{Fermi}} + \frac{2\pi g^2 n^2}{\mu^2}, \quad \epsilon_{\text{Fermi}} = a_0 n^{4/3}, \tag{1.2}$$

with

$$P = -\epsilon + n\frac{d\epsilon}{dn} = \tfrac{1}{3} a_0 n^{4/3} + \frac{2\pi g^2 n^2}{\mu^2}. \tag{1.3}$$

Hence, for infinite densities the interaction energy dominates both the rest energy and the kinetic energy allowing $3P > \epsilon$. Zel'dovich also mentioned that the continuum limit would be applicable at densities $\rho \gtrsim 100\rho_{\text{nuclear}}$.

Later Bludman and Ruderman[5] modified Zel'dovich's results to first order by including corrections due to finite interparticle distance. For high densities they found an interaction energy given by

$$V_L = [1 - \alpha(\mu a)^2] V_c, \quad \mu a \ll 1 \tag{1.4}$$

where

$$V_L = \tfrac{1}{2} g^2 \sum_{i \neq j} \frac{e^{-\mu |r_i - r_j|}}{|r_i - r_j|} ,$$
$$V_c = \frac{Ng^2}{2} \int \frac{e^{-\mu r}}{r} d^3 r \, \frac{N}{V} . \tag{1.5}$$

For a simple cubic lattice $\alpha = \frac{1}{12}$ and the pressure and the energy density expressions become

$$\epsilon = n\{m + \tfrac{1}{2} n v(0) [1 - \alpha(\mu a)^2]\} , \tag{1.6}$$

$$P = \tfrac{1}{2} n^2 v(0) [1 - \tfrac{1}{3} \alpha(\mu a)^2] , \tag{1.7}$$

$$\frac{dP}{d\epsilon} = \frac{n d^2\epsilon/dn^2}{d\epsilon/dn} = \frac{1 - \frac{2}{9}\alpha(\mu a)^2}{1 - \frac{2}{3}\alpha(\mu a)^2 + (m/4\pi g^2)\, a(\mu a)^2} , \tag{1.8}$$

where $V_c = (N^2/V)\, v(0)$. From Eqs. (1.6)–(1.8) they concluded that even though for $n \to \infty$ Zel'dovich's results are obtained; for finite n it is possible for the speed of sound to be greater than unity.

However, the densities which one expects in neutron-star mass limit calculations are not high enough to consider nucleons to be crushed into a continuum[2] ($\leqslant 10^{16}$ g/cm^3). In this paper we calculate the lattice energy for a three-dimensional simple cubic lattice for densities $\leqslant 10^{16}$ g/cm^3 and show that $\epsilon \geqslant 3P$. For densities higher than 10^{16} g/cm^3 we argue that we have a phase transition from neutron matter to quark matter before we reach the density regime where the results of Zel'dovich and of Bludman and Ruderman would be valid. Hence, we conclude with the suggestion that $\epsilon \geqslant 3P$ might very well be true even in the presence of repulsive interactions with interaction energy dominating kinetic energy. We also discuss the implications of this result for the neutron-star mass limit and its practicality.

II. THE INTERACTION OF THE CHARGES AND THE EQUATION OF STATE

First we consider a linear chain of particles interacting through the following two-body repulsive interaction with interparticle separation a:

$$\phi_{ij} = g^2 \frac{e^{-\mu r_{ij}}}{r_{ij}} . \tag{2.1}$$

The potential energy of the lattice is

$$v_L = \tfrac{1}{2} \sum_{i \neq j} \phi_{ij} . \tag{2.2}$$

v_L is also equal to Nv_1, where v_1 is the potential energy of one particle,

$$v_1 = g^2 \sum_{n=1}^{N} \frac{e^{-\mu n a}}{na} . \tag{2.3}$$

The series in (2.3) can be summed by using the well-known formula

$$\sum_{n=0}^{N} X^n = \frac{1 - X^N}{1 - X} . \tag{2.4}$$

This is true for all X, but for $N \to \infty$ it converges only for $X < 1$, hence the series in (2.3) in the thermodynamic limit, where $N \to \infty$, $L \to \infty$ while $N/L \to$ constant, becomes

$$\sum_{n=1}^{\infty} \frac{X^n}{n} = \ln \frac{1}{1 - X} , \tag{2.5}$$

where $X = e^{-\mu a}$. Hence the lattice energy for a linear chain can be given as

$$v_L = \frac{ng^2}{a} \ln \frac{1}{1 - X} , \tag{2.6}$$

where $X = e^{-\mu a}$. This result can be generalized to a three-dimensional simple cubic lattice, which is actually an infinite number of linear chains with different interparticle spacing and different orientations.

The distance of the ith nucleon to the origin r_i and its orientation θ_i and ϕ_i can be given as

$$r_i = pa [1 + (m/p)^2 + (n/p)^2]^{1/2} , \tag{2.7}$$

$$\tan\phi_i = [(m/p)^2 + (n/p)^2]^{1/2} , \tag{2.8}$$

$$\cos\theta_i = \frac{(m/p)}{(n/p)} , \tag{2.9}$$

where m, n, and p are integers. For a given chain, θ and ϕ are fixed. This fixes the ratios (m/p) and (n/p); hence for the ith chain, from Eq. (2.7) the interparticle distance a_i is

$$a_i = a [1 + (m/p)^2 + (n/p)^2]^{1/2} . \tag{2.10}$$

This gives the following lattice energy for a three-dimensional system:

$$v_L = \sum_i \frac{ng^2}{a_i} \ln \frac{1}{1 - X_i} , \tag{2.11}$$

where $X_i = e^{-\mu a_i}$ and the summation is over all possible a_i. Now we can give the energy density as

$$\epsilon = nm + (4\pi/3)^{1/3} g^2 n^{4/3} \sum_i \frac{1}{[1 + (m/p)^2 + (n/p)^2]^{1/2}} \times \ln \frac{1}{1 - X_i} + a_1 n^{4/3} , \tag{2.12}$$

where $a_1 n^{4/3}$ is the Fermi energy and we define a_0 and $f_i(\theta, \phi)$ as

$$a_0 = (4\pi/3)^{1/3} g^2 ,$$
$$f_i(\theta, \phi) = \frac{1}{[1 + (m/p)^2 + (n/p)^2]^{1/2}} . \tag{2.13}$$

In this notation we obtain for the pressure

$$P = \tfrac{1}{3} n^{4/3}\left[a_0\left(\sum_i f_i(\theta,\phi)\ln\frac{1}{1-X_i}\right)+a_1\right] + n^{4/3} a_0 n \sum_i \frac{1}{1-X_i}\left(\frac{dX_i}{dn}\right) f_i . \tag{2.14}$$

Since we are interested in the asymptotic form of the equation of state at high densities, we will consider the limit $\mu a \ll 1$, which gives

$$\epsilon = nm + \left(\frac{4\pi}{3}\right)^{1/3} g^2 n^{4/3}\left(\alpha \ln\frac{1}{\mu a}+\beta\right)+a_1 n^{4/3} \tag{2.15}$$

and

$$P = \tfrac{1}{3} n^{4/3}\left[a_0\left(\alpha\ln\frac{1}{\mu a}+\beta\right)+a_1\right]+n^{4/3}\left(\frac{a_0}{3}\right)\alpha , \tag{2.16}$$

where

$$\alpha = \sum_i f_i(\theta,\phi), \quad \beta=\sum_i f_i(\theta,\phi)\ln f_i(\theta,\phi). \tag{2.17}$$

Evaluating $3P/\epsilon$ and $dP/d\epsilon$ we have

$$\frac{3P}{\epsilon} = 1 + \frac{(4\pi/3)^{1/3} m}{\left[a_0\left(\alpha\ln\frac{1}{\mu a}+\beta\right)+a_1\right]}(\alpha g^2/m - a), \tag{2.18}$$

$$\frac{dP}{d\epsilon} = \frac{1}{3} + \frac{(4\pi/3)^{1/3}\frac{1}{3}m}{\left[\frac{4}{3}a_0\left(\alpha\ln\frac{1}{\mu a}+\beta+\frac{a_1}{a_0}\right)+a_0\frac{1}{3}\alpha\right]} \times \left(\tfrac{4}{3}\alpha g^2/m - a\right). \tag{2.19}$$

For electromagnetic interactions $\mu=0$, and $3P/\epsilon$ and $3dP/d\epsilon$ become unity in the high-density limit. Note that $\mu a_i \geqslant \mu a$, hence for the high-density limit we have to consider contributions from chains, where $\mu a_i \ll 1$, $\mu a_i \sim 1$, and $\mu a_i \gg 1$. Contribution from chains with $\mu a_i \gg 1$ goes to zero logarithmically while the contribution from chains with $\mu a_i \sim 1$ is proportional to n, hence does not contribute to pressure and could be absorbed into the mass term, leaving the expression (2.15) for the total energy of the system.

Even though in the limit of infinite densities the expressions $3P/\epsilon$ and $3dp/d\epsilon$ go to unity, we would like to point out that in this limit these equations are no longer valid since the contributions from chains with $\mu a_i \gg 1$ can no longer be neglected. The proper way to achieve this limit is from Eq. (2.10),

$$v_L = n^2 g^2 \sum_i f_i(\theta,\phi)\ln\frac{1}{1-e^{-\mu a_i}}a^2 . \tag{2.20}$$

In the continuum limit $a^2 \to d\Omega$, the summation becomes an integral over all angles and the n^2 dependence of the total energy becomes apparent. Our results are valid for densities roughly ten times nuclear densities which is not high enough to crush nucleons to form a continuum.

For low densities, $\mu a \gg 1$, we only have to consider the nearest-neighbor interaction, and for a simple cubic lattice this can be given as

$$\epsilon = n\left(m + \tfrac{1}{2}\times 6 g^2 \frac{e^{-\mu a}}{a}\right). \tag{2.21}$$

If we take $n = 1/a^3$,

$$\epsilon = nm + 3g^2 n^{4/3} e^{-\mu n^{-1/3}}, \tag{2.22}$$

$$P = 3g^2 e^{-\mu n^{-1/3}}\left(\tfrac{1}{3}n^{4/3}+\tfrac{1}{3}\mu n\right), \tag{2.23}$$

with

$$\frac{3P}{\epsilon} = \frac{3g^2\mu}{m} e^{-\mu a}, \quad \text{for } \mu a \gg 1 . \tag{2.24}$$

Note that $\eta g^2/m$ is the classical baryon radius, where η is a geometric factor depending on the assumed baryonic charge distribution and μ^{-1} is the range of interaction. Since $\mu^{-1} > \eta(g^2/m)$ and also $e^{-\mu a} < 1$, in the low-density limit we get $3P < \epsilon$ as expected. For the case where the $\mu^{-1} < \eta(g^2/m)$, the above model is no longer correct. Note that at low densities kinetic energy which goes as $n^{4/3}$ dominates interaction energy. As the density increases, interaction energy takes over kinetic energy with $\epsilon \gtrsim 3P$ still being true.

III. CONCLUSIONS AND THE PRACTICALITY OF THIS MODEL

Since for densities up to $\sim 3\times 10^{16}$ g/cm^3 (nuclear densities) the structure of matter is well understood and no causality violation is found, we were concerned with the asymptotic structure of the equation of state. We examined the equation of state for densities roughly ten times larger than the nuclear densities. At these densities it has been shown that neutron matter prefers to form a lattice,[1] hence minimizing its energy. We found the lattice energy for a cubic lattice exactly within the relevant density range, aside from a numerical factor that depends on the geometry of the crystal. The energy we found for particles interacting through the exchange of neutral vector mesons is

$$v_L = n^{4/3}(A_0 \ln n + B_0). \tag{3.1}$$

Here A_0 and B_0 are constants which could be found by inspection from (2.15) using $n = 3/4\pi a^3$. As expected, for large densities our lattice energy is less than the energy found from the continuum model

$$v_c = \frac{2\pi g^2 n^2}{\mu^2} . \tag{3.2}$$

In the relevant density range we found $\epsilon > 3P$ if

$$\alpha \frac{g^2}{m} - a \tag{3.3}$$

is negative. It is well known that the classical radius of baryons can be given as $\eta g^2/m$, where η is a numerical factor depending on the assumed baryonic charge distribution. In Eq. (3.3) we also have a quantity that depends on the geometry of the crystal. Hence, α/η can be arranged to be around unity, and Eq. (3.3) being negative implies an interparticle distance to be larger than the classical radius of baryons.

Actually for $(\alpha g^2/m - a) > 0$, baryons can no longer be treated as point particles occupying lattice sites and the above model is no longer valid. Hence within the range of applicability our model gives $\epsilon \geq 3P$. We also reconsidered[5] the low-density limit and found that $\epsilon > 3P$ as expected.[6]

It is still true that for densities higher than ~10^{16} g/cm^3 we will be approaching the continuum limit where the equations of Zel'dovich and of Bludman and Ruderman are valid. However, we expect a phase transition between neutron matter to quark matter long before this happens and from asymptotic freedom, the quark equation of state always satisfies $\epsilon < 3P$.[7,8] It is our opinion that the equation of state may very well satisfy $\epsilon \leq 3P$ even in the presence of interactions with interaction energy dominating kinetic energy.

The density range for which Eq. (3.1) is valid is sufficient for limiting-mass calculations for neutron stars. If we use the equation of state $P = \alpha\rho$ for densities above ρ_m below which a plausible nuclear model is attached, the "core mass" of such a star has mass[7]

$$M = \left(\frac{\alpha}{\alpha^2 + 6\alpha + 1}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{\rho_m^{1/2}} . \tag{3.4}$$

Hence, if we take $P = \epsilon$ as our limiting equation of state, the mass limit obtained will be 1.26 times the value obtained by $P = \frac{1}{3}\epsilon$. In this paper we argued in favor of the latter. Zero-point energy and quantum corrections are considered by Zel'dovich[4] and by Bludman and Ruderman[9] and do not affect our conclusions.

Canuto and Lodenquai[10] have studied the behavior of matter at densities higher than 10^{16} g/cm^3. This is roughly the same region in which we think our Eq. (3.1) is no longer valid. They use p-p collisions in the GeV region and argue in favor of $P = \epsilon$. However, their results are hampered by a lack of satisfactory knowledge of the viscosity of hadronic matter.

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[6]If we look at the present calculations of equations of state [Fig. 1 in Canuto (Ref. 2)], we see that, owing to the attractive nature of the nuclear interactions at low densities, we have $\epsilon > 3P$. These equations of state cross the $\epsilon = 3P$ line around $\rho \approx 8 \times 10^{14}$ g/cm^3 which is already close to the point where these calculations cannot be considered as reliable. We have argued that even in the presence of repulsive interactions $\epsilon \leq 3P$ would still be true.

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