Nonsingular particle solutions in variations on Einstein's nonsymmetric unified field theory

Robert Huerta and Barry R. Parker
Department of Physics, Idaho State University, Pocatello, Idaho 83209
(Received 18 June 1979)

The Einstein-Kurşunoğlu nonsymmetric field theory has a distributed particle solution for the spherically symmetric time-independent case where $g_{23} \neq 0$. The mass of the particle is derived from the field energy when the integration constants are chosen to avoid singularities. Linearization of the equations yields the Klein-Gordon equation for the charge density. The field functions are similar to a solution previously found for Bonnor's modification of the nonsymmetric theory.

I. INTRODUCTION

After twenty years of neglect, Einstein's nonsymmetric theory has become the basis for a number of attempts to unify the fields of nature.[1] These attempts have generally been centered around two modifications of Einstein's original theory. The first was published in 1952 by Kurşunoğlu,[2] the other in 1954 by Bonnor.[3] They differ from Einstein's theory by the addition of a term involving g_{ij}. This form was considered and rejected by Einstein earlier.[4] Most of the recent papers in the area have been concerned with the unification of the electromagnetic and gravitational fields; however, Borchsenius[5] has attempted to combine gravitational and electromagnetic interactions with the strong interaction by extending Bonnor's form of Einstein's nonsymmetric theory to include the Yang-Mills field. Moffat[6] has also attempted to bring other fields of nature into the formalism by using Bonnor's theory in conjunction with an extension of f-g theory.[7] Finally, Kurşunoğlu has reinterpreted and generalized his earlier nonsymmetric theory to include further fields.[8,9]

The closed-form solutions of Kurşunoğlu and of Pant[10] both suffer from field singularities at the origin which lead to an infinite rest mass. This infinite rest mass is made finite by the artifice of adding an infinite-negative-mass singularity to provide a long-range mass which is finite. The resulting uncoupling of the rest mass and total field energy was presented by Bonnor[11] as an objection to Kurşunoğlu's[2] contention that the rest mass of the particle was derived from the total field energy in the Kurşunoğlu theory. When Pant[10] presented the closed-form solution for the magnetic monopole in Bonnor's theory he failed to find the corresponding electric (g_{23}) charge solution. He offered a proof that no such solution existed if the charge could be confined to a finite sphere. Boal and Moffat[12] obtained the same solution and offered a reinterpretation by interchanging the electric and magnetic fields. Boal and Moffat[13] also offered a proof that there could be no solution to the electric (g_{23}) charge problem if $g_{11} g_{44} = -1$. The present authors found both assumptions to be disposable and found the equations could be reduced to an integro-differential equation. The charge structure of the spherically symmetric, time-independent fields which satisfy Bonnor's equations was investigated by numerical methods.[14]

The integro-differential equation method can easily be generalized to a class of theories which include Bonnor's and Kurşunoğlu's. In this paper a solution for Kurşunoğlu's theory will be obtained by numerical integration of the corresponding integro-differential equation. It will be shown that in the case of small charge the solution goes over to a solution of the Klein-Gordon equation found by Kurşunoğlu as a result of linearizing his unified field theory.[2] Kurşunoğlu's evaluation of the total field energy and claim that the total mass is derived from the field energy is also supported by the numerical integrations.

II. EINSTEIN-KURŞUNOĞLU FIELD EQUATIONS

Kurşunoğlu derives his equations from a variational principle

$$\delta \int [\mathfrak{g}^{\alpha\beta} R_{\alpha\beta} - \kappa^2(\sqrt{-b} - \sqrt{-g})]\, d^4x = 0\,, \qquad (1)$$

which yields the following set of equations:

$$R_{(\mu\nu)} = \tfrac{1}{2}\kappa^2 (b_{\mu\nu} - a_{\mu\nu})\,, \qquad (2)$$

$$R_{[\mu\nu],\lambda} + R_{[\nu\lambda],\mu} + R_{[\lambda\mu],\nu} = \tfrac{1}{2}\kappa^2 (g_{[\mu\nu],\lambda} + g_{[\nu\lambda],\mu} + g_{[\lambda\mu],\nu})\,, \qquad (3)$$

$$\mathfrak{g}^{\beta\gamma}{}_{;\alpha} = \mathfrak{g}^{\beta\gamma}{}_{,\alpha} + \mathfrak{g}^{\beta\sigma}\Gamma^{\gamma}_{\alpha\sigma} + \mathfrak{g}^{\sigma\gamma}\Gamma^{\beta}_{\sigma\alpha} - \mathfrak{g}^{\beta\gamma}\Gamma^{\rho}_{\alpha\rho} = 0\,, \qquad (4)$$

when the torsion vector vanishes, i.e.,

$$\Gamma^{\lambda}_{\mu\lambda} - \Gamma^{\lambda}_{\lambda\mu} = 0\,. \qquad (5)$$

Here $\mathfrak{g}^{\alpha\beta}$ is defined as the contravariant metric tensor density. $R_{\mu\nu}$ is the Ricci tensor formed with the nonsymmetric affine connections $\Gamma^{\gamma}_{\mu\nu}$ according to

$$R_{\mu\nu} \equiv \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\mu\rho,\nu} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\sigma\nu}\,. \qquad (6)$$

The symmetric and antisymmetric parts are written as

$$2R_{(\mu\nu)} \equiv R_{\mu\nu} + R_{\nu\mu}\,, \tag{7}$$

$$2R_{[\mu\nu]} \equiv R_{\mu\nu} - R_{\nu\mu}\,. \tag{8}$$

The comma stands for partial differentiation and the semicolon for covariant differentiation as defined in Eq. (4). The covariant metric tensor is defined by

$$g_{\mu\nu}\,\mathfrak{g}^{\nu\lambda} = \sqrt{-g}\,\delta_\mu^\lambda\,, \tag{9}$$

where

$$g = \det(g_{\mu\nu})\,, \quad b^{\mu\nu} = \mathfrak{g}^{(\mu\nu)}/(-\det \mathfrak{g}^{(\mu\nu)})^{1/2}\,, \tag{10}$$

$$b = \det(g^{(\mu\nu)})\,, \quad a_{\mu\nu} = g_{(\mu\nu)}\,, \quad b_{\mu\nu}\,b^{\epsilon\nu} = \delta_\mu^\epsilon\,. \tag{11}$$

The sixty-four equations (4) are sufficient to define the Γ's in terms of the g's. The system is simplified by requiring spherical symmetry and time independence. When we further restrict the problem by seeking only the electric field, we have a metric of the form

$$g_{\mu\nu} = \begin{bmatrix} \dfrac{-e^{-u}}{v^2} & 0 & 0 & 0 \\ 0 & -e^\rho \sin\phi & e^\rho \cos\phi \sin\theta & 0 \\ 0 & -e^\rho \cos\phi \sin\theta & -e^\rho \sin\phi \sin^2\theta & 0 \\ 0 & 0 & 0 & e^u \end{bmatrix}. \tag{12}$$

The equations (4) are solved for the Γ's and then the Γ's are used to form six nonvanishing equations from (2) and (3):

$$\begin{aligned} R_{11} &= \rho'' + \rho' v'/v + \tfrac{1}{2}(\phi'^2 + \rho'^2) \\ &\quad + \tfrac{1}{2}(u'' + u'^2 + u' v'/v + u'\rho') \\ &= \tfrac{1}{2}\kappa^2 \frac{e^{-u}}{v^2}(1 - \sin\phi)\,, \end{aligned} \tag{13}$$

$$\begin{aligned} R_{22} &= -1 + \tfrac{1}{2}v[v e^{u+\rho}(\rho' \sin\phi - \phi' \cos\phi)]' \\ &\quad - \tfrac{1}{2}\phi' v^2 e^{u+\rho}(\phi' \sin\phi + \rho' \cos\phi) \\ &= \tfrac{1}{2}\kappa^2 e^\rho(\sin\phi - 1)\,, \end{aligned} \tag{14}$$

$$R_{33} = R_{22}\sin^2\theta\,, \tag{15}$$

$$\begin{aligned} R_{44} &= v^2 e^{2u}[-\tfrac{1}{2}(u'' + u'^2 + u' v'/v + u'\rho')]' \\ &= -\tfrac{1}{2}\kappa^2 e^u(1 - \sin\phi)\,, \end{aligned} \tag{16}$$

$$\begin{aligned} \csc\theta\, R_{23} &= -\tfrac{1}{2}v[v e^{u+\rho}(\phi' \sin\phi + \rho' \cos\phi)]' \\ &\quad + \tfrac{1}{2}v^2 e^{u+\rho}\phi'[\phi' \cos\phi - \rho' \sin\phi] \\ &= \tfrac{1}{2}\kappa^2(l^2 - e^\rho \cos\phi)\,, \end{aligned} \tag{17}$$

$$R_{23} = -R_{32}\,. \tag{18}$$

Since Eq. (14) is proportional to Eq. (15) and Eqs. (17) and (18) also are proportional, we have only four equations to satisfy. Kurşunoğlu[9] has shown that one of the four remaining equations is redundant (due to the Bianchi identity).

III. SOLUTIONS

A. Analytic Results

A linear combination of Eqs. (13) and (16) can be conveniently manipulated to give

$$\rho'' + \rho' v'/v + \tfrac{1}{2}(\rho'^2 + \phi'^2) = 0\,. \tag{19}$$

We may formally integrate to obtain

$$\ln(\rho' v) = -\tfrac{1}{2}\int_0^r \left(\rho' + \frac{\phi'^2}{\rho'}\right) dr\,. \tag{20}$$

Solving for v gives

$$v = \frac{1}{\rho'}\exp\left[-\tfrac{1}{2}\int_0^r \left(\rho' + \frac{\phi'^2}{\rho'}\right) dr\right] \equiv \frac{1}{\rho'}e^{-\gamma}\,. \tag{21}$$

By forming the linear combinations $R_{22}\cos\phi + R_{23}\sin\phi/\sin\theta$ and $R_{22}\sin\phi - R_{23}\cos\phi/\sin\theta$ we obtain

$$(v e^{u+\rho}\phi')' = 1/v[\kappa^2(e^\rho \cos\phi - l^2 \sin\phi) - 2\cos\phi] \tag{22}$$

and

$$(v e^{u+\rho}\rho')' = 1/v\{\kappa^2[e^\rho(1 - \sin\phi) - l^2\cos\phi] + 2\sin\phi\}\,. \tag{23}$$

These two equations are formally integrated and a ratio is formed to eliminate the variable u:

$$\frac{d\phi}{d\rho} = \frac{\int_0^r 1/v[\kappa^2(e^{\rho_1}\cos\phi_1 - l^2\sin\phi_1) - 2\cos\phi_1]dr_1}{\int_0^r 1/v\{\kappa^2[e^{\rho_1}(1 - \sin\phi_1) - l^2\cos\phi_1] + 2\sin\phi_1\}dr_1}\,. \tag{24}$$

The integration constants are set equal to zero to eliminate singularities. When Eq. (21) is used to eliminate v we have an ordinary integro-differential equation in ϕ as a function of ρ,

$$\frac{d\phi}{d\rho} = \frac{\int_{\rho_0}^{\rho} e^\gamma[\kappa^2(e^{\rho_1}\cos\phi_1 - l^2\sin\phi_1) - 2\cos\phi_1]d\rho_1}{\int_{\rho_0}^{\rho} e^\gamma\{\kappa^2[e^{\rho_1}(1 - \sin\phi_1) - l^2\cos\phi_1] + 2\sin\phi_1\}d\rho_1}\,, \tag{25}$$

where

$$\gamma = \tfrac{1}{2}\int_{\rho_0}^{\rho_1}\left[1+\left(\frac{d\phi_2}{d\rho_2}\right)^2\right]d\rho_2 . \tag{26}$$

Notice only ϕ, ρ, and its dummy replacements ρ_1, ρ_2 remain. ρ_0 is a constant.

The complete elimination of r is consistent with the Bianchi identity which allows an arbitrary choice for r. We choose

$$e^{\rho} \equiv r^2, \quad p^2 = -\kappa^2 \tag{27}$$

and

$$\phi \equiv i\psi + \tfrac{1}{2}\pi . \tag{28}$$

The integro-differential equation becomes

$$i\frac{d\psi}{dr} = \frac{2}{r}\,\frac{\int_0^r e^{\delta}[p^2(r_1^2 i \sinh\psi_1 + l^2\cosh\psi_1) + 2i\sinh\psi_1]dr_1}{\int_0^r e^{\delta}\{-p^2[r_1^2(1-\cosh\psi_1) + l^2 i\sinh\psi_1] + 2\cosh\psi_1\}dr_1}, \tag{29}$$

where

$$\delta \equiv \tfrac{1}{4}\int_0^{r_1}\left(\frac{d\psi_2}{dr_2}\right)^2 r_2 dr_2 . \tag{30}$$

When we require that the derivative at the origin be finite we obtain the condition

$$l^2 p^2 \cosh\psi_0 + 2i\sinh\psi_0 = 0 . \tag{31}$$

The final form of the integro-differential equation is

$$\frac{d\psi}{dr} = \frac{2}{r}\,\frac{\int_0^r e^{\delta}\left[\frac{\sinh(\psi_1-\psi_0)}{\cosh\psi_0} + \frac{1}{2}p^2 r_1^2 \sinh\psi_1\right]dr_1}{\int_0^r e^{\delta}\left[\frac{\cosh(\psi_1-\psi_0)}{\cosh\psi_0} - \frac{1}{2}p^2 r_1^2(1-\cosh\psi_1)\right]dr_1} . \tag{32}$$

Before proceeding to a numerical solution we will examine the weak-field approximation.

B. Weak-Field Approximation

The weak-field case was examined by Kurşunoğlu[2] where he introduced a current vector J_μ and found

$$\Box J_\mu - \kappa^2 J_\mu = 0 . \tag{33}$$

From this he deduced the spherically symmetric, time-independent charge density

$$\sigma = \frac{e^{-\kappa r}}{r} = J_4 \tag{34}$$

and the total energy

$$mc^2 = \sqrt{2}\,\kappa . \tag{35}$$

We linearize Eq. (32) by the approximations

$$\sinh\psi \simeq \psi, \quad \cosh\psi \simeq 1, \tag{36}$$

where ψ is small. This allows the reduction of Eq. (32) to

$$\frac{d\psi}{dr} \simeq \frac{2}{r}\,\frac{\int_0^r (\psi - \psi_0 + \frac{1}{2}p^2 r^2 \psi)dr}{\int_0^r dr} . \tag{37}$$

In the linearized problem the charge density is given by

$$r^2\psi \simeq 4\pi\int_0^r \sigma r^2 dr . \tag{38}$$

Substitution of Eq. (38) in (37) and differentiating yields

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\sigma}{dr}\right) - p^2\sigma = 0 ,$$

which we recognize as the result obtained previously by Kurşunoğlu.

C. Numerical Integration

Bonnor[11] objected to Kurşunoğlu's contention that the mass was totally derived from the field energy. Bonnor showed by asymptotic expansion that the mass and the charge were two apparently independent constants. The numerical solution resolves this difficulty by showing nonsingular solutions exist only when the mass is completely determined by the charge and the fundamental constant p. Indeed, if we attempt to place a mass or charge singularity at the origin we find ourselves unable to satisfy the equations with a continuous function; in this case, like Bonnor, we also have two independent constants.

Finite-difference methods were used to obtain numerical solutions of the integro-differential equation (32). The radial variable was rescaled to absorb the factor $2p^2$. Starting values for the derivative were selected and the corresponding charge, mass, and field distributions were found. In many cases the solution diverged in either a positive or negative direction (depending on the total charge). At a particular value of the charge the numerical solutions approached the Reissner-Nordström solution. The numerical method failed to reach the point at infinity, however, and this forced us to use the Reissner-Nordström solution at large distances. An asymptotic expansion was used in a previous paper[14] to show that the solution does indeed approach the Reissner-Nordström solution for large r.

Figure 1 shows the normalized radial field ($e^{\rho}\sinh\psi/\tanh\psi_0$) as a function of the radial variable X which keeps the corresponding Reissner-Nordström function constant [$X=R/(1+R)$, $R^2=e^{\rho}\cosh\psi$]. In the small-charge region [$\psi_r(0)\ll 1$] the numerical solution accurately reproduces the solution to the corresponding Klein-Gordon equation. For small X the numerical solution exhibits the behavior of Papapetrou's solution for $p^2=0$. Also, for large r the solution approaches the Reissner-Nordström solution. Thus we are in asymptotic agreement with the three related solutions.

Although the charges and masses of particles observed in nature are extremely small and lie in the asymptotic region [$\psi_r(0)\ll 1$], we have examined large values of $\psi_r(0)$ and have demonstrated

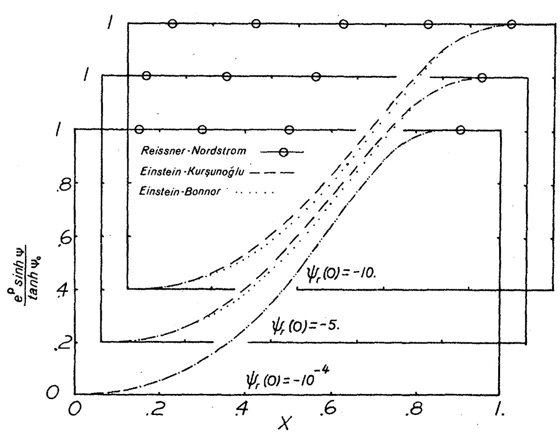

FIG. 1. Normalized radial field functions are displayed for starting derivative $\psi_r(0)=-10^{-4}$, -5, -10. The radial coordinate is chosen so that the comparison Reissner-Nordström (RN) function [the flux through a sphere of radius $(e^{\rho}\cosh\psi)^{1/2}$] is constant. The solutions approach the RN solution as $x\to 1$, where $x\equiv[1+(e^{\rho}\cosh\psi)^{-1/2}]^{-1}$; when $\psi_r(0)<1$, the solutions approach the solution to the Klein-Gordon equation.

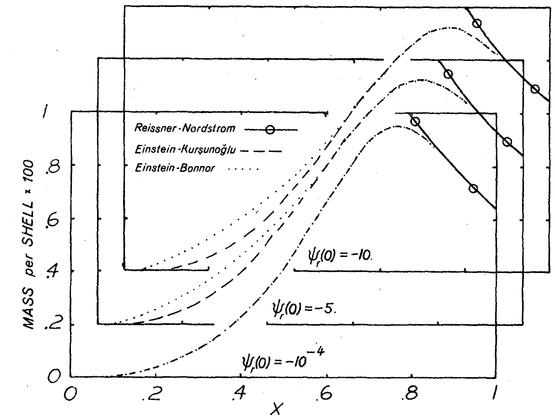

FIG. 2. The mass of the particle is identified as the integral of the field energy density. The contribution to the mass of the spherical layers of thickness $\Delta x=1/80$ is shown to approach the RN result as $x\to 1$ and to vanish as $x\to 0$. This allows the computation of a finite self-energy and the identification of the mass with the area under the curve. The curves are normalized to the charge squared.

a space distortion peculiar to our solution (i.e., $g_{11}g_{44}\neq -1$). Figure 2 shows the radial variation of $(-g_{11}g_{44})^{1/2}$ with X. Note that a volume computed from $\int\sqrt{-g}\,d^3x$ will be diminished in the presence of a large radial electric field.

Figure 3 displays the normalized mass in each spherical shell and compares it with the corresponding mass computed from the energy density for the Reissner-Nordström solution. Although the Reissner-Nordström solution may be found for any finite charge, the Hermitian solution limits the charge according to Eq. (31),

$$l^2=\frac{2i}{p^2}\tanh\psi_0, \quad -1\leq\tanh\psi_0\leq 1.$$

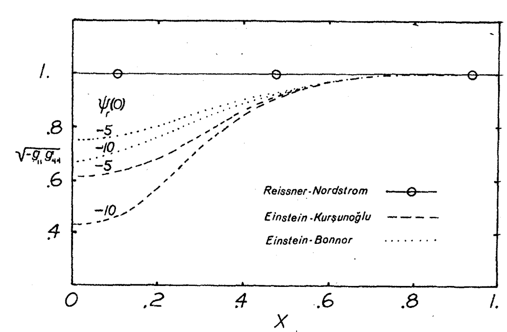

FIG. 3. As the starting derivative becomes more negative the volume element in the high-field region diminishes in the Hermitian form of the theories. The Reissner-Nordström solution does not show this behavior. The volume element in a sphere of radius $(e^{\rho}\cosh\psi)^{1/2}$ diminishes while the area of the sphere is maintained at $4\pi e^{\rho}\cosh\psi$.

If the Hermitian condition on g is dropped then another set of solution is found where the charge is not bounded,

$$l^2 = \frac{2}{p^2} \tan\psi_0 , \quad -\infty \leq \tan\psi_0 \leq \infty .$$

The linearization of this non-Hermitian theory yields the same small-charge solution but in the non-Hermitian case the volume in a sphere is greater than its Euclidean value.

The solutions for Kurşunoğlu's and Bonnor's theories are not identical for finite charge, but for the range of ψ considered neither is significantly different from the linearized solution. Between the two theories, taking into consideration charge conjugation, Hermitian and non-Hermitian solutions, we could form eight charged particles with the same charge-to-mass ratio. A neutral particle could also be formed by taking a difference between the terms of the theories. The neutral particle would apparently be much lighter than the charged particles.

IV. CONCLUSIONS

Using a method developed and applied to the Einstein-Bonnor theory in a previous paper the authors have solved the problem of the interaction of a charged particle with its own field in the Einstein-Kurşunoğlu unified field theory. Graphs of the numerical results are presented in three figures. The numerical results are in agreement in the small-charge region with the results of the corresponding Klein-Gordon equation.

The authors do not regard the theory as complete in that it does not appear to contain spin. Also, one would anticipate that a complete theory would have no independent constants and provide continuous fields with continuous first derivatives. The present theory provides fields which are continuous everywhere, but the first derivative is discontinuous at the origin due to the nonzero starting value necessary to provide the solution that approaches the Reissner-Nordström solution at large r.

If one assumes that the discontinuous derivatives at the origin are a result of some fundamental process not included in the present theory, then we have a theory that predicts a unique charge and charge-to-mass ratio. When the value of the universal constant p is given, the value of the starting derivative determines the solution completely. In other words, a solution that converges at large r exists only for a single charge and a single charge-to-mass ratio.

[1]A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1955), Appendix II.
[2]B. Kurşunoğlu, Phys. Rev. 88, 1369 (1952).
[3]W. B. Bonnor, Proc. R. Soc. London A226, 366 (1954).
[4]A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1953), 4th edition, p. 146.
[5]K. Borchsenius, Phys. Rev. D 13, 2707 (1976).
[6]J. W. Moffat, Phys. Rev. D 17, 1965 (1978).
[7]C. J. Isham, Abdus Salam, and J. Strathdee, Phys. Rev. D 3, 867 (1971).
[8]B. Kurşunoğlu, Phys. Rev. D 9, 2723 (1974).
[9]B. Kurşunoğlu, Phys. Rev. D 13, 1538 (1976).
[10]D. N. Pant, Nuovo Cimento 25B, 175 (1975).
[11]W. B. Bonnor, Phys. Rev. 92, 1067 (1953).
[12]J. W. Moffat and D. H. Boal, Phys. Rev. D 11, 1375 (1975).
[13]D. H. Boal and J. W. Moffat, Phys. Rev. D 11, 2026 (1975).
[14]R. Huerta and B. Parker, Phys. Rev. D 19, 2853 (1979).