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Spontaneously broken de Sitter symmetry and the gravitational holonomy group

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The local geometrical structure of general relativity is analyzed in detail from the standpoint of a formulation of gravity as a gauge theory of the de Sitter group SO(3,2). In order to reproduce the structure of the Einstein-Cartan theory, it is essential that the SO(3,2) gauge symmetry be spontaneously broken down to the Lorentz group. In the geometrical analysis of this spontaneously broken theory, the Goldstone field of the symmetry-breaking mechanism plays a central role, representing the coordinates of a point in an internal anti-de Sitter space where the motions induced by parallel transport across space-time take place. In order to establish the connection between the SO(3,2) gauge theory and the Einstein-Cartan theory, the gravitational vierbein and spin connection are derived from the original SO(3,2) gauge fields by passing over to a set of nonlinearly-transforming fields through a redefinition involving the Goldstone field. The original SO(3,2) gauge fields have a different but equally important role: they generate pseudotranslations and rotations in the internal anti-de Sitter space under a kind of parallel transport across space-time that is called "development." Development maps curves in space-time into image curves in the internal space, and vector fields along the curves in space-time into image vector fields along the image curves. Considering development along infinitesimal closed curves in space-time leads to the proper interpretation of the effects of torsion and of curvature in terms of the nonclosure of image curves and of the rotation of image vectors with respect to their original values.

I. INTRODUCTION

This paper is about the insights into the geometrical structure of the Einstein-Cartan theory of gravity that can be gained from a formulation of the theory as a Yang-Mills gauge theory of the de Sitter group SO(3, 2). The de Sitter symmetry must be spontaneously broken down to the Lorentz group SO(3, 1) in order to make contact with the usual four-dimensional geometry of gravity. In our discussion, we shall pay particular attention to the role of the Goldstone field of the theory, as well as to that of the SO(3, 2) Yang-Mills gauge fields. The latter do not directly represent the gravitational vierbein and spin connection as in other works on this subject, but, nonetheless, they are of central importance for the geometrical structure of the theory. Using them we shall make contact with a more precise formulation of the geometry of the Einstein-Cartan theory than is normally presented in the physics literature.

There is by now an extensive literature on the relation of general relativity to Yang-Mills gauge theories. Since the papers of Utiyama,[1] Kibble,[2] and Sciama,[3] there has been considerable interest in this topic and, rather than recount the history in detail, we refer the reader to the several available reviews of the subject.[4-7] The end result of this work has been to establish an analogy between general relativity, when written in first-order form in the vierbein formalism, and a gauge theory of the Poincaré group, with the spin connection representing the gauge field for the Lorentz rotations, and the vierbein field being considered as the compensating field for translations in space-time. Since the translations must be space-time dependent in order to have a local symmetry, they really represent infinitesimal general coordinate transformations considered actively, i.e., transforming fields but not the coordinates of space-time. Of course, the general coordinate group is an infinite-dimensional group, so the connection to the Poincaré group that is made in this way is really just an analogy based upon the flat-space limit of general relativity. The differences from the Poincaré group show up in the appearance of space-time-dependent functions in the algebra of commutators of the active general coordinate transformations. For a review of the various approaches that have been taken in considering general coordinate transformations as local translations, see Ref. 4.

In this paper, we take a quite different attitude to the Yang-Mills symmetry that underlies our discussion of gravity. We shall not place much emphasis on the connection to general coordinate transformations; this connection will emerge naturally, and is the same as that explained in Ref. 4. On the other hand, we will take the original Yang-Mills symmetry quite seriously, not discarding the "translational" parts in favor of general coordinate transformations, but preserving the full local symmetry rigorously throughout the work.

An important stimulus for the revival of interest in gauge-theoretic formulations of gravity has come from recent developments in supergravity.[8,9] In particular, the extended supergravity theories combine space-time and internal symmetries within one irreducible multiplet of symmetries. Pure supergravity is related to a gauge theory of the graded Poincaré group,[10] or, in the generalization of supergravity to include a matched pair of cosmological constant and spin-$\frac{3}{2}$ mass term[11], to a gauge theory of the graded group OSP(1, 4).[12] Both of the Refs. 10 and 12 considered the reduction to the corresponding purely gravitational theories, based respectively on the Poincaré group and on the group SP(4). Although these groups were used to motivate the construction of actions in Refs. 10 and 12, the end results were not strictly invariant under the respective groups. An extension to a superspace formulation of supergravity of the results of Ref. 12, which clarifies the relation to the work of Ref. 4, has recently been given in Ref. 13. The usefulness of the heuristic discussions of Refs. 10 and 12 was shown in the application of these techniques to study the SO(2) extended supergravity theory[14] and to construct the theory of conformal supergravity.[15]

In this paper, we base our discussion upon the de Sitter group SO(3, 2) which is locally isomorphic to the group SP(4). We take as a starting point the results of Ref. 16, where an action that is strictly SP(4) invariant was given, which in a particular gauge reduces to the gravitational action of Ref. 12. This work also introduced the five-vector nondynamical field that we shall study extensively here.

For our present purposes, we could equally well use the de Sitter group SO(4, 1); the choice of SO(3, 2) is made simply to retain the connection to the work on supergravity referred to above. The choice of the de Sitter group rather than the Poincaré group is not casually made, however. In order to make contact with the geometry of the Einstein-Cartan theory and still retain the Yang-Mills symmetry throughout the work, it is essential that this gauge symmetry be spontaneously broken down to the Lorentz group. We chose the de Sitter group because it is a semisimple group and in it the generators are initially all on an even footing. Spontaneous symmetry breaking causes four of the ten generators of the group to be broken, and thus to have a nonlinear action on field variables. The strength of the symmetry breaking is given by a parameter m^{-1} with the dimensions of length; in the usual fashion, the broken generators and their associated gauge fields are scaled by the factor m in order to normalize the Goldstone field. Letting the strength of the symmetry breaking tend to infinity leads back to the Poincaré group.

The symmetry breaking in our case is triggered by requiring a nondynamical SO(3, 2) five-vector field to be constrained to take its values in an internal anti-de Sitter space, a copy of which is associated to each point in space-time. This internal space is also a copy of the maximally symmetric solution to the theory's field equations. It is a four-dimensional submanifold of the five-dimensional space that the de Sitter group has a natural action in. The constrained theory can be generalized to a theory with a propagating Higgs scalar field in the usual way of spontaneously broken gauge theories; we shall give an example of how this may be done in the concluding section of this paper. Since the SO(3, 2) symmetry is spontaneously broken, the broken generators give rise to an inhomogeneous and nonlinear transformation law for the Goldstone field, which parametrizes the point in the internal anti-de Sitter space referred to above. The appropriate mathematical formalism for discussion of this situation is the theory of nonlinear realizations of groups,[17,18] which we shall use extensively.

The application of the theory of nonlinear realizations to gravity has been carried out in different ways in previous investigations. For example, the vierbein field has been considered as a Goldstone field related to a nonlinear realization of the group GL(4, R),[19] or of the affine and conformal groups.[20] For us, however, the Goldstone field is the nondynamical field referred to above. The approach which is most directly relevant to the present work is that due to Volkov and Soroka[21] (cf. also Ref. 22). These authors considered a nonlinear realization of the Poincaré group in the context of the spontaneous breakdown of supersymmetry. Unlike the present work, they did not associate a Goldstone field to the breakdown of Poincaré to Lorentz symmetry, but identified the corresponding coset parameters with the points of space-time itself. However, they did construct the vierbein and spin connection from the Poincaré gauge fields by passing to a set of redefined "nonlinear" fields as we shall do. Similar identifications of the vierbein and spin connection have subsequently been made also in Refs. 23, 24, which were primarily concerned with the application of nonlinear realizations of OSP(1, 4) to supergravity. None of these papers has treated the Goldstone field in the way that we shall do, however.

In the present work, the Goldstone field represents a point in an internal anti-de Sitter space, as we have mentioned. In describing the geometry of this internal space, we make use of some of the results of Refs. 25, 26 on the nonlinear realization of supersymmetry in anti-de Sitter space. In order to carry out parallel transport across space-time, we use the property of nonlinear realizations that they respect the linear representations of the stability subgroup, which in this case is the Lorentz group. The precise description of this parallel transport involves a map between the tangent space to space-time at a given point and the tangent space to the internal anti de Sitter space at the point whose coordinates are given by the Goldstone field. The vierbein field is the matrix of this map, which is formally known as the solder form. The underlying mathematics of this situation was presented in Ref. 27.

The role of the original SO(3, 2) gauge fields in our formulation is to generate a version of parallel transport of nonlinearly transforming fields that we call "development." Development differs from the usual version of parallel transport using the spin connection in that it involves calibrating transformations that use the broken de Sitter generators as well as the Lorentz generators. It is a purely gauge-theoretic notion which is carried out by solving a set of differential equations involving an operator Δ_μ, constructed from the original SO(3, 2) gauge fields, to find the result of developing the values of the Goldstone field and of nonlinear tensor fields along a given curve in space-time.

Since the broken generators act inhomogeneously on the Goldstone field, the result of developing this field from each point along a curve in space-time to some given point is to create an image curve in the internal space associated to that given point. It is because of this that we have named the process "development," for the resulting image curves and image tensor fields defined along them are identical to those obtained by a generalization, to our situation with internal anti-de Sitter spaces, of a purely geometrical construction known in differential geometry as development into the affine tangent spaces of a differentiable manifold. Although the details of this known geometrical construction, as discussed for example in Ref. 28, involve parallel transport of tensors alone using the spin connection, we show that the results agree with our form of development, which is based upon transport of the Goldstone field using the original SO(3, 2) gauge fields. The demonstration rests upon the equivalence between SO(3, 2) gauge transformations and parallel transport in the internal space.

The geometrical construction generated by development is important because it is the appropriate construction for the analysis of the effects of space-time torsion and curvature. It has not figured prominently in previous discussions of the Einstein-Cartan theory in the physics literature, although it is necessary for an accurate description of the effects of torsion. An analysis of the Einstein-Cartan theory using the formalism of modern differential geometry has been carried out in the work of Trautman,[29] although development is only obliquely referred to there through what is called a "radius vector field." In order for us to establish full contact with this theory, it is necessary to obtain the usual space-time torsion and curvature from the SO(3, 2) curvatures by a redefinition, involving the Goldstone field, which is analogous to that used to obtain the vierbein field and spin connection. From our discussion of development, torsion gives the gap by which an image curve in the internal space, corresponding to an infinitesimal closed curve in space-time, fails to close. The de Sitter curvature gives the relative rotation of an image vector developed around such an infinitesimal curve with respect to the original vector. The value of the de Sitter curvature is the difference between the usual Lorentz curvature of space-time and the curvature of anti-de Sitter space.

The paper is organized as follows. In Sec. II, we use the results of Ref. 16 to introduce the constrained field, which will become the Goldstone field upon passage to a nonlinear realization, as is discussed in Sec. III, where the usual features of the Einstein-Cartan theory are derived. Development is introduced in Sec. IV, and shown to generate the appropriate geometical construction to enable the torsion and curvature to be identified, as is done in Sec. V. Section VI concludes the paper with a look forward to applications of the present work to supergravity and to the global structure of general relativity. In this concluding section, we also give an example of how the constrained field that we use in the geometrical discussions can be replaced by a propagating Higgs field with a potential that causes spontaneous symmetry breakdown of SO(3, 2) to the Lorentz group. Some technical details on the calculations and on the description of anti-de Sitter space are presented in the Appendix.

II. SO(3,2) GAUGE THEORY OF GRAVITY

In order to give a concrete basis to the following geometrical discussion, in this section we recall the SO(3, 2)-invariant action for gravity which was given in Ref. 16. The initial intention is to construct the gravitational vierbein field, spin connection, and action using only the SO(3, 2) gauge fields; this aim will shortly be relaxed by the introduction of a nonpropagating field, as will be explained. Accordingly, we begin with just a four-dimensional differentiable space-time manifold $\mathfrak{M}$ and the SO(3, 2) gauge fields $\omega_\mu{}^{AB}(x)$, $A, B = 1, \ldots, 5$. With this minimum of mathematical apparatus, the only further objects that we may use to construct invariants are the numerically invariant SO(3, 2) tensor $\eta_{AB} = (1, 1, 1, -1, -1)$, the totally antisymmetric tensor ϵ^{ABCDE} (with $\epsilon^{12345} = 1$), and the Levi-Civita tensor density (with "world" indices) $\epsilon^{\mu\nu\rho\sigma}$. Invariants which could serve as the action of an SO(3, 2) gauge theory must be constructed from the field strengths

$$R_{\mu\nu}{}^{AB} = \partial_\mu \omega_\nu{}^{AB} - \partial_\nu \omega_\mu{}^{AB} + \omega_\mu{}^{AC} \omega_{\nu C}{}^B - \omega_\nu{}^{AC} \omega_{\mu C}{}^B , \tag{2.1}$$

where the index C has been lowered with η_{AB}.

The only invariant which can be built with the above elements that is polynomial in the field strengths is the topological invariant that gives the Pontryagin index, $\int d^4x \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{AB} R_{\rho\sigma AB}$. In Ref. 16 a nonpolynomial action was presented, which in the present notation is

$$I = \int d^4x [-(\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{AB} R_{\rho\sigma}{}^{CD} \epsilon_{ABCDE}) \times (\epsilon^{EFGHK} R_{\alpha\beta FG} R_{\lambda\tau HK} \epsilon^{\alpha\beta\lambda\tau})]^{1/2} . \tag{2.2}$$

If a local gauge choice is made corresponding to the $[a5]$ ($a = 1, \ldots, 4$) generators of the group such that

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd5} R_{\mu\nu}{}^{bc} R_{\rho\sigma}{}^{d5} = 0 \tag{2.3}$$

at every point $x^\mu \in \mathfrak{M}$, then (2.2) reduces to $\int d^4x (\Phi^2)^{1/2}$, where

$$\Phi = \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd} \epsilon_{abcd5} . \tag{2.4}$$

Taking the square root, the resulting action $\int d^4x \, \Phi$ has a residual SO(3, 1) gauge invariance. This is the action for gravity in the form given in Ref. 12.

The form (2.2) of the gravitational action is not the most convenient for our following discussion. The square root is somewhat unorthodox, and there is also a dimensional difficulty. In order to interpret the gauge field $\omega_{\mu a5}$ in the special gauge (2.3) as the vierbein $e_{\mu a}$, it must be scaled by a quantity with dimensions of length in order to give a dimensionless vierbein field. There is no indication of any scale of length in (2.2), however. In order to eliminate the square root and introduce a dimensional quantity, we introduce a constrained, nonpropagating SO(3, 2) five-vector field $y^A(x)$ with dimensions of length, together with a parameter m of dimension (length)$^{-1}$ and a scalar density Lagrange multiplier $\lambda(x)$. This five-vector field was first introduced in the work of Ref. 30, where it was used to construct OSP(1, 4)-invariant actions. The action now takes the form[16]

$$I = \int d^4x [m y^A \epsilon_{ABCDE} R_{\mu\nu}{}^{BC} R_{\rho\tau}{}^{DE} \epsilon^{\mu\nu\rho\tau} + \lambda(y^A y_A + m^{-2})] . \tag{2.5}$$

Notice that in order for this action to be even under parity transformations, it is necessary to make an O(3, 2) transformation of determinant minus one at the same time as the parity transformation is carried out on the spatial indices. This is of course just a generalization of the usual situation in the vierbein formalism with the Lorentz group. Elimination of first y^A, then λ by substituting the solutions to their algebraic field equations back into the action yields again (2.2). On the other hand, upon making the $[a5]$ gauge choice

$$y^a(x) = 0 , \tag{2.6}$$

and noting that the field $y^5(x)$ is then constrained to take the value m^{-1}, the integrand reduces directly to the form (2.4). Defining the scaled gauge fields

$$e_{\mu a} = m^{-1} \omega_{\mu a5} \tag{2.7}$$

and expanding the field strengths

$$R_{\mu\nu}{}^{ab} = B_{\mu\nu}{}^{ab} + m^2 (e_\mu{}^a e_\nu{}^b - e_\nu{}^a e_\mu{}^b) , \tag{2.8}$$

where

$$B_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_{\nu c}{}^b - \omega_\nu{}^{ac} \omega_{\mu c}{}^b \tag{2.9}$$

is the usual SO(3, 1) curvature tensor, the integrand (2.4) splits up into three pieces: $\mathcal{L}_{(BB)} + m^2 \mathcal{L}_{(eeB)} + m^4 \mathcal{L}_{(eeee)}$. The first of these is the integrand of the Gauss-Bonnet topological invariant, and so gives no contributions to the field equations, the second is the usual scalar curvature action of Einstein's theory, and the third is a cosmological constant.

The maximally symmetric solution to the field equations derived from (2.5) is an anti-de Sitter space (henceforth denoted AdS) satisfying

$$R_{\mu\nu}{}^{AB} = 0 . \tag{2.10}$$

In the special gauge (2.6), we may identify the scaled field $e_\mu{}^a$ in (2.7) as the vierbein field, and the remaining $\omega_\mu{}^{ab}$ as the spin connection. When these fields satisfy (2.10), they belong to a space that may be represented as a four-dimensional submanifold embedded in a five-dimensional pseudo-Euclidean space with metric η_{AB}, with the embedding equation

$$x^A x_A = x^i x^i - (x^4)^2 - (x^5)^2 = -m^{-2} . \tag{2.11}$$

Thus, the vacuum state of the theory (2.5) is an AdS space of the same curvature and signature as the space the y^A are constrained to lie in. This fact suggests the role that the y^A must play in the geometrical structure of the theory: They are the AdS analogs of the local inertial frames of general relativity. Each point of $\mathfrak{M}$ is provided with a field that lies in a local copy of the vacuum. If there is no cosmological constant, then the vacuum is just Minkowski space, and the local inertial frames are taken to be flat. With a cosmological constant, these local frames are taken to be curved. Note that in either case the frames are, in general, only truly inertial at one point [in our case, a frame is in fact only inertial at a point x^μ if its origin is chosen so that (2.6) holds there]. The deviations from inertial character in a vicinity of that point are minimized by making the appropriate choice of frame curvature.

In order to couple the theory (2.5) to matter fields, one needs to have an SO(3, 2)-invariant object that can serve as a metric tensor. This is constructed using the covariant derivative of y^A,

$$\nabla_\mu y^A = \partial_\mu y^A + \omega_\mu{}^A{}_C y^C . \qquad (2.12)$$

In the special gauge (2.6), this expression reduces to $\nabla_\mu y^a = e_\mu{}^a$, $\nabla_\mu y^5 = 0$. Consequently, matter couplings can be made using $\nabla_\mu y^A$ and the metric[30]

$$\tilde{g}_{\mu\nu} \equiv \nabla_\mu y^A \nabla_\nu y_A . \qquad (2.13)$$

Note that using $\tilde{g}_{\mu\nu}$, one may also construct other forms of the gravitational action than (2.5). The form (2.5) is, however, the unique form which permits elimination of the field y^A by an algebraic field equation.

The introduction of a constrained field y^A in (2.5) is an essential feature of the attempt to view gravity as a gauge theory of SO(3, 2). Although, for concreteness, we have introduced it with reference to a specific action principle, the reason why it arises is fundamental. General relativity in the form of the Einstein-Cartan theory is formulated in terms of spin connections and the canonical one-form (i.e., the vierbein field) defined on the bundle of orthonormal frames over the space-time manifold $\mathfrak{M}$.[29] Gravity is thus firmly linked to the tangent space structure of $\mathfrak{M}$. A general gauge group bundle, however, is only loosely attached to $\mathfrak{M}$, and need not have any rigid global linkage to the tangent spaces of $\mathfrak{M}$. For this reason, it is necessary for us to take a specific SO(3, 2) bundle which allows the construction of a one-form that *solders* the fibers with structure group SO(3, 2) to the tangent bundle of the space-time manifold $\mathfrak{M}$. The full mathematical details on the construction of this solder form can be found in Ref. 27.

In brief, the requirements for there to exist a solder form are as follows. There must exist a fiber bundle associated to the principal SO(3, 2) bundle on which the gauge connections are defined, the fibers of which can be identified with the coset spaces $\{SO(3,2)/SO(3,1)\}$, and which allows a point in each of these fibers to be smoothly associated with the underlying space-time point as one goes around the manifold $\mathfrak{M}$, i.e., this associated bundle must admit a global cross section. Note that the fiber $\{SO(3,2)/SO(3,1)\}$ has the same dimension as $\mathfrak{M}$. Once a cross section has been picked, the residual structure group is the Lorentz group SO(3, 1) and one says that the structure group has been reduced from SO(3, 2) to SO(3, 1). It is further required that the tangent spaces to the $\{SO(3,2)/SO(3,1)\}$ fibers at the points chosen by the cross section may be smoothly mapped onto the tangent spaces to $\mathfrak{M}$ at the underlying space-time points. That is to say that the associated bundle of $\{SO(3,2)/SO(3,1)\}$ fibers must be soldered to the space-time manifold $\mathfrak{M}$. The solder form is the map between the tangent spaces to $\mathfrak{M}$ and the tangent spaces to the fibers $\{SO(3,2)/SO(3,1)\}$ at the points selected by the cross section.

The coset spaces $\{SO(3,2)/SO(3,1)\}$ are just the local anti-de Sitter spaces that we have found it necessary to introduce. The cross section of the associated bundle of $\{SO(3,2)/SO(3,1)\}$ fibers is represented locally by the constrained five-vector field $y^A(x)$. The vierbein field of the theory is then the solder form. Note that in our discussion so far, we have only attempted to identify the vierbein field in the special gauge (2.6), where the constrained field points in the "5" direction everywhere. This will have to be generalized if we are to retain the full SO(3, 2) invariance in our discussion of gravity, and the vierbein field will have to be identified with more care, as we shall see in the next section. The importance of retaining the full SO(3, 2) invariance is illustrated by another gauge choice which can be made when the system is in its vacuum state. The vacuum equations (2.10) allow an SO(3, 2) gauge to be chosen in which $e_\mu{}^a = \omega_\mu{}^{ab} = 0$. The vacuum state is then described entirely by the constrained field $y^A(x)$ and the vacuum vierbein and spin connection will be given in terms of it and its derivatives.

III. SPONTANEOUS SYMMETRY BREAKING

The appearance of the constrained field $y^A(x)$ in our SO(3, 2) gauge formulation of gravity recalls the nonlinear σ model, and indicates that the de Sitter symmetry is spontaneously broken. Spontaneous symmetry breaking is also evidenced by the difference in propagation between the gauge fields $\omega_{\mu a5}$ and $\omega_{\mu ab}$; the former propagate, while the latter satisfy an algebraic field equation in the special gauge (2.6), which corresponds to the usual "unitary" gauge. The situation is analogous to that of the Higgs effect, in which gauge fields that were initially on an even footing end up with different propagation behavior. In this case the spontaneous symmetry breaking is triggered by the fact that the nondynamical field $y^A(x)$ is constrained to lie in an anti de Sitter space, $y^A y_A = -m^{-2}$. Note that although the difference in propagation character between the $\omega_{\mu a5}$ and $\omega_{\mu ab}$ is most clearly seen in the unitary gauge (2.6), a difference in propagation among the $\omega_{\mu AB}$ will occur in any gauge.

The fact that the SO(3, 2) symmetry is spontaneously broken plays a crucial role in the geometrical analysis of the theory. Its most important consequence is to allow us to construct the true vierbein and spin connection. The spin connection expresses the relative rotation of the orthonormal frames in the tangent spaces to the space-time manifold $\mathfrak{M}$ at two neighboring points x^μ and $x^\mu + dx^\mu$. It allows one to perform parallel transport of vectors that reside in these tangent spaces. In our SO(3, 2) gauge theory, we do not possess at the start any means of parallel transporting such SO(3, 1) four-vectors. However, we do have the gauge fields $\omega_{\mu AB}$, which permit us to parallel transport objects that transform according to the linear representations of SO(3, 2). What spontaneous symmetry braking of SO(3, 2) down to SO(3, 1) allows is the passage to a nonlinear realization[17,18] of SO(3, 2) in which the irreducible linear representations of SO(3, 2) go over into nonlinearly transforming sets of fields that transform independently according to their SO(3, 1) indices.

The importance of nonlinear realizations for gravity in the sense of this paper was realized by Volkov and Soroka[21] in connection with the spontaneous symmetry breaking of supersymmetry, where the Poincaré group is a broken subgroup and the Lorentz group is the actual stability subgroup. In Ref. 21, the vierbein and spin connection were identified with nonlinearly transforming fields derived from the original linear gauge fields as we shall do, but the coset parameters corresponding to the spontaneously broken translations were taken to represent the coordinates of points in space-time itself. In contrast, the present work takes the coset parameters corresponding to the broken generators to represent the coordinates in an *internal* anti-de Sitter space of the point determined by the constrained field $y^A(x)$. Moreover, although the Poincaré group is obtained from SO(3, 2) by a Wigner-Inönü contraction, from the standpoint of this paper, the de Sitter group is most natural. In it, the group generators are initially all on an even footing, and the scaling of the broken generators by the inverse radius m of the internal AdS space is a natural consequence of the symmetry-breaking mechanism. The value of the cosmological constant is thus determined by the strength of the symmetry breaking.

The passage from a linear to a nonlinear realization of SO(3, 2) requires choosing some parametrization of the coset space $\{SO(3,2)/SO(3,1)\}$, i.e., some set of coordinates in anti-de Sitter space. We shall adopt the standard exponential parametrization, as in the work of Keck[25] and of Zumino[26] on the nonlinear realization of supersymmetry in anti-de Sitter space. This work made use of rigid SO(3, 2) invariance in anti-de Sitter space-time. As we have noted, in the present paper, the AdS space is an internal space, not the space-time manifold $\mathfrak{M}$. Also, the SO(3, 2) invariance that we have is a local invariance, dependent upon the space-time point x^μ, although the effect of an SO(3, 2) transformation upon any given AdS space at some point x^μ is rigid throughout that AdS space. In this section, we shall make use of the results and calculational techniques of Ref. 26, with extensions to the case of local symmetry as indicated. One could equivalently use the formalism of Ref. 23 in which the extension to local symmetry was also given, but without proposing any particular interpretation for the coset parameter fields. All these calculations are carried out by straightforward application of the general scheme presented in Refs. 17, 18.

In the exponential parametrization, an element g of $G = SO(3,2)$ within some neighborhood of the identity may be uniquely represented in the form

$$g = e^{-i\zeta^a P_a} h\,, \tag{3.1}$$

where $h \in H = SO(3,1)$ is an element of the Lorentz group, the stability subgroup, and ζ^a parametrize the coset space $\{G/H\}$. The original generators M_{AB} of SO(3, 2) satisfy the Lie algebra

$$-i[M_{AB}, M_{CD}] = \eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD}\,, \tag{3.2}$$

and the generators P_a are defined by

$$P_a \equiv m M_{a5}\,. \tag{3.3}$$

These scaled generators satisfy the commutation relations

$$-i[P_a, P_b] = m^2 M_{ab}\,, \tag{3.4}$$

$$-i[M_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a\,. \tag{3.5}$$

The nonlinear action of $g_0 \in G$ on $\{G/H\}$ is given by

$$g_0 e^{-i\zeta^a P_a} = e^{-i\zeta'^a(\zeta, g_0) P_a} h_1(\zeta, g_0)\,, \tag{3.6}$$

where ζ'^a and h_1 are, in general, nonlinear functions of ζ^a and g_0. More precisely, if $g_0 = h_0 \in H$, then the action is linear,

$$e^{-i\xi'^a P_a} = h_0 e^{-i\xi^a P_a} h_0^{-1}\,, \tag{3.7}$$

$$h_1 = h_0\,, \tag{3.8}$$

since the P_a transform according to the four-vector representation of $H = SO(3,1)$. If $g_0 = \exp(-i\epsilon^a P_a)$, then the action is nonlinear and the transformation of ξ^a is inhomogeneous. The infinitesimal form of the transformation (3.6) is

$$e^{i\xi^a P_a}(g_0 - 1)e^{-i\xi^b P_b} - e^{i\xi^a P_a}\delta e^{-i\xi^b P_b} = h_1 - 1\,. \tag{3.9}$$

The coset parameters ξ^a in (3.1) are taken to be functions of the space-time coordinates x^μ. As in the usual treatments of spontaneous symmetry breaking, they represent the Goldstone fields of the theory. Their relation to the constrained field $y^A(x)$ may be deduced from their behavior under a transformation with $\epsilon^a(x)P_a = m\epsilon^a(x)M_{a5}$. Under infinitesimal transformations, the $\xi^a(x)$ transform by

$$\delta\xi^a = \epsilon^a + \left(\frac{z\cosh z}{\sinh z} - 1\right)\left(\epsilon^a - \frac{\epsilon^b \xi_b \xi^a}{\xi^2}\right), \tag{3.10}$$

where

$$z = m(\xi^a \xi_a)^{1/2} = m\xi\,, \tag{3.11}$$

and lower-case roman indices are raised and lowered with η_{ab}. Under finite transformations, (3.6) shows that the point $\xi^a = 0$ is transformed to $\xi^a = \epsilon^a$. On the other hand, $y^A = (0,0,0,0,m^{-1})$ is transformed by $\epsilon_{a5} = m\epsilon_a$ into $y^a = m^{-1}(\epsilon^a/\epsilon)\sinh(m\epsilon)$, $y^5 = m^{-1}\cosh(m\epsilon)$. Thus, the relations between the $y^A(x)$ and the $\xi^a(x)$ are given by

$$\begin{aligned} y^a &= m^{-1}(\xi^a/\xi)\sinh(m\xi)\,,\\ y^5 &= m^{-1}\cosh(m\xi)\,. \end{aligned} \tag{3.12}$$

These relations may also be expressed as

$$y^A(x) = \sigma_5 \left[e^{-i\xi^a(x)P_a} \right] \begin{pmatrix} 0\\0\\0\\0\\m^{-1} \end{pmatrix}, \tag{3.13}$$

where the symbol σ_5 denotes the five-vector representation of $SO(3,2)$. At this stage, the reason for scaling the broken M_{a5} generators by m to form P_a in (3.3) is clear: This scaling is necessary for the expansion of y^A in powers of ξ^a to have a first-order term that is just ξ^a itself:

$$y^A(x) = (0,0,0,0,m^{-1}) + (\xi^a(x),0) + O(\xi^2)\,. \tag{3.14}$$

Thus, the $\xi^a(x)$ have dimensions of length just like the $y^A(x)$, as is appropriate for coordinates in the internal AdS spaces, which we will interpret as local coordinate frames, or as maps of the space-time near a given point x^μ.

The relation (3.13) is the prototype of the general relation between linear and nonlinear fields, with the Goldstone field $\xi^a(x)$ acting as a bridge between the two types. Given a set of fields $\psi(x)$ transforming according to some representation σ of $SO(3,2)$, the corresponding nonlinear fields $\bar\psi(x)$ are defined by

$$\bar\psi(x) = \sigma[e^{i\xi^a(x)P_a}]\psi(x)\,. \tag{3.15}$$

Under the transformation (3.6), these fields transform according to their $SO(3,1)$ indices only, but with the nonlinear group element $h_1(\xi(x), g_0(x)) \in SO(3,1)$:

$$\bar\psi'(x) = \sigma[h_1(\xi(x), g_0(x))]\bar\psi(x)\,. \tag{3.16}$$

The explicit form of $h_1(\xi(x), e^{-i\epsilon(x)\cdot P})$ for infinitesimal P_a transformations with parameter $\epsilon^a(x)$ is calculated in the Appendix from Eq. (3.9). The result is

$$\begin{aligned}\tilde h_1(\xi(x),\epsilon(x)) &= h_1(\xi(x), 1 - i\epsilon(x)\cdot P) - 1\\ &= \frac{i}{2}(\epsilon^a\xi^b - \epsilon^b\xi^a)\frac{\cosh z - 1}{z\sinh z} M_{ab}\,. \end{aligned} \tag{3.17}$$

The tilde on $\tilde h_1$ is used to denote a Lie algebra element.

Except for the scalar representation, the irreducible representations of $SO(3,2)$ become reducible when considered as representations of $SO(3,1)$. According to (3.16), the nonlinear fields $\bar\psi(x)$ mix under general $SO(3,2)$ transformations only with fields belonging to the same irreducible linear representation of $SO(3,1)$, i.e., the nonlinear transformations (3.16) respect the fields' $SO(3,1)$ index types. Accordingly, one may consider smaller sets of nonlinear fields than are necessary to fill an irreducible $SO(3,2)$ representation, and (3.16) may then be read as the transformation law for such a set, with σ the $SO(3,1)$ representation the set belongs to. This is the key feature of nonlinear realizations that allows us to construct the vierbein and spin connection of general relativity from the $SO(3,2)$ gauge fields $\omega_{\mu AB}(x)$ and the Goldstone field $\xi^a(x)$, and to arrive at the usual geometrical interpretation of their role in the parallel transport of vectors that lie in the tangent spaces to the space-time manifold $\mathfrak{M}$.

In order to carry out parallel transport of a vector V^μ that lies in the tangent space to $\mathfrak{M}$ at x^μ, we must first convert this vector into an object $\bar V^a$ that transforms nonlinearly under $SO(3,2)$. Then it can be parallelly transported while respecting its $SO(3,1)$ index type. The conversion from V^μ to $\bar V^a$

is effected using the true vierbein $\bar{e}_\mu{}^a$, which is obtained together with the true spin connection $\bar{\omega}_\mu{}^{ab}$ from the SO(3,2) gauge fields $\omega_{\mu AB}$ and the Goldstone field ζ^a according to the general scheme (3.15), where in this case σ stands for the adjoint representation of SO(3,2). Since (3.15) obtains nonlinearly transforming fields from the linear fields by a P_a gauge transformation with minus the Goldstone field as parameter, it is necessary to take into account the inhomogenous transformation character of the gauge fields $\omega_{\mu AB}$. Thus, one obtains $\bar{e}_\mu{}^a(x)$ and $\bar{\omega}_\mu{}^{ab}$ from

$$\tfrac{1}{2}i\bar{\omega}_\mu{}^{ab}(x)M_{ab} - i\bar{e}_\mu{}^a(x)\mathrm{P}_a = e^{i\zeta^c(x)\mathrm{P}_c}[\partial_\mu + \tfrac{1}{2}i\omega_\mu{}^{ab}(x)\mathrm{M}_{ab} - ie_\mu{}^a(x)\mathrm{P}_a]e^{-i\zeta^d(x)\mathrm{P}_d}. \tag{3.18}$$

Note that in order to maintain consistency with (3.14), we must use the scaled gauge field $e_{\mu a} = m^{-1}\omega_{\mu a5}$, and $\bar{e}_{\mu a}$ is then correctly dimensionless. The details of the calculation of $\bar{e}_{\mu a}$ and $\bar{\omega}_\mu{}^{ab}$ are given in the Appendix; the result is

$$\bar{e}_\mu{}^a = e_\mu{}^a + (\cosh z - 1)\left(e_\mu{}^a - e_\mu{}^b\frac{\zeta_b\zeta^a}{\zeta^2}\right) + \frac{\sinh z}{z}\,\omega_\mu{}^a{}_c\zeta^c + \partial_\mu\zeta^a + \left(\frac{\sinh z}{z} - 1\right)\left(\partial_\mu\zeta^a - \frac{\zeta^c\partial_\mu\zeta_c}{\zeta^2}\,\zeta^a\right), \tag{3.19}$$

$$\bar{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab} - m^2(\zeta^a e_\mu{}^b - \zeta^b e_\mu{}^a)\frac{\sinh z}{z} + m^2(\zeta^a\omega_\mu{}^b{}_c\zeta^c - \zeta^b\omega_\mu{}^a{}_c\zeta^c)\left(\frac{\cosh z - 1}{z^2}\right) - m^2(\zeta^a\partial_\mu\zeta^b - \zeta^b\partial_\mu\zeta^a)\left(\frac{\cosh z - 1}{z^2}\right). \tag{3.20}$$

The expressions for $\bar{e}_\mu{}^a$ and $\bar{\omega}_\mu{}^{ab}$ are in agreement with those found in Ref. 23.

Under local SO(3,2) gauge transformations, these "barred" fields transform as follows:

$$-i\bar{e}'^a_\mu\mathrm{P}_a = h_1(-i\bar{e}_\mu{}^a\mathrm{P}_a)h_1{}^{-1}, \tag{3.21}$$

$$\tfrac{1}{2}i\bar{\omega}'^{ab}_\mu\mathrm{M}_{ab} = h_1(\tfrac{1}{2}i\bar{\omega}_\mu{}^{ab}\mathrm{M}_{ab})h_1{}^{-1} + h_1\partial_\mu h_1{}^{-1}. \tag{3.22}$$

Equations (3.21) and (3.22) show that the $\bar{e}_\mu{}^a$ and $\bar{\omega}_\mu{}^{ab}$ do not mix under SO(3,2) gauge transformations, in agreement with our general discussion. In addition, only $\bar{\omega}_\mu{}^{ab}$ has an inhomogeneous piece in its transformation law. The vierbein $\bar{e}_\mu{}^a$ transforms homogeneously, according to the four-vector representation of SO(3,1), but with the nonlinear group element $h_1(\zeta, g_0) \in H$, following the pattern of Eq. (3.16).

The procedure for parallel transport of a vector V^μ is now just the usual one in general relativity. In order to move a vector V^μ lying in the tangent space to $\mathfrak{M}$ at x_1^ν along some parametrized curve $x^\nu(t)$ in $\mathfrak{M}$ from $x^\nu(1) = x_1^\nu$ to $x^\nu(0) = x_0^\nu$, one first converts V^μ to the nonlinear components $\bar{V}^a$,

$$\bar{V}^a = V^\mu\bar{e}_\mu{}^a, \tag{3.23}$$

and then solves the differential equation

$$\frac{dx^\mu(t)}{dt}\,\bar{D}_\mu\bar{V}_\parallel{}^a(x(t); x_1) = 0, \tag{3.24}$$

where the covariant derivative $\bar{D}_\mu$ is defined by

$$\bar{D}_\mu \equiv \partial_\mu + \tfrac{1}{2}i\bar{\omega}_\mu{}^{ab}\mathrm{M}_{ab}, \tag{3.25}$$

so that

$$\bar{D}_\mu\bar{V}_\parallel{}^a = \partial_\mu\bar{V}_\parallel{}^a + \bar{\omega}_\mu{}^a{}_b\bar{V}^b. \tag{3.26}$$

The differential equation (3.24) must be solved subject to the initial condition

$$\bar{V}_\parallel{}^a(x(1); x_1) = \bar{V}^a. \tag{3.27}$$

The solution $V_\parallel{}^a(x(0); x_1)$ then gives the nonlinear components of the parallel transported vector at x_0^ν. To convert these nonlinear components back to ones with world indices, one must use the inverse vierbein field,

$$\bar{V}_\parallel{}^\mu(x_0; x_1) = \bar{e}^\mu{}_a(x_0)\bar{V}_\parallel{}^a(x_0; x_1), \tag{3.28}$$

where

$$\bar{e}^\mu{}_a\bar{e}_\nu{}^a = \delta^\mu{}_\nu, \quad \bar{e}_\mu{}^a\bar{e}^{\mu b} = \eta^{ab}. \tag{3.29}$$

In order to define $\bar{e}^\mu{}_a$, it is necessary that $\det(\bar{e}_{\mu a}) \neq 0$. Note that this condition is locally SO(3,2) invariant and is also invariant under world general coordinate transformations.

The covariant derivative $\bar{D}_\mu$ in (3.25) generates parallel transport according to the usual prescription. Note that by (3.22) and (3.16), $\bar{D}_\mu\bar{\psi}(x)$ has the same transformation law as $\bar{\psi}(x)$, i.e., $\bar{D}_\mu$ is a covariant derivative in the normal sense. Parallel transport of a vector around a small closed curve will result in a net SO(3,1) rotation between start and finish that is determined by the "barred" analog of the usual Lorentz curvature, which is defined by taking the commutator of two covariant derivatives $\bar{D}_\mu$:

$$\tfrac{1}{2}i\bar{B}_{\mu\nu}{}^{ab}\mathrm{M}_{ab} \equiv [\bar{D}_\mu, \bar{D}_\nu], \tag{3.30}$$

or explicitly,

$$\bar{B}_{\mu\nu}{}^{ab} = \partial_\mu\bar{\omega}_\nu{}^{ab} - \partial_\nu\bar{\omega}_\mu{}^{ab} + \bar{\omega}_\mu{}^{ac}\bar{\omega}_{\nu c}{}^b - \bar{\omega}_\nu{}^{ac}\bar{\omega}_{\mu c}{}^b. \tag{3.31}$$

This Lorentz curvature tensor gives the rotation that a vector's nonlinear components will undergo on parallel transport around a small closed curve. It is of course also possible to construct a connection $\Gamma_{\mu\nu}{}^{\lambda}$ with purely world indices from the vierbein $\bar{e}_{\mu}{}^{a}$ and spin connection $\bar{\omega}_{\mu}{}^{ab}$,

$$\Gamma_{\mu\nu}{}^{\lambda}=\bar{e}^{\lambda}{}_{a}(\partial_{\mu}\bar{e}_{\nu}{}^{a}+\bar{\omega}_{\mu}{}^{a}{}_{b}\bar{e}_{\nu}{}^{b})\,. \tag{3.32}$$

This connection will not in general be symmetric, nor will the curvature $\bar{B}_{\mu\nu\alpha\beta}$, obtained from (3.31) by changing the $[ab]$ indices to $[\alpha\beta]$ using vierbein fields, satisfy the cyclic identity.

These two conditions are the standard indications of the presence of torsion in a space-time, and we shall see in the following that the possibility of torsion is of great importance for the geometrical analysis of the theory.

We may rewrite the action (2.5) in terms of the barred fields $\bar{e}_{\mu}{}^{a}(x)$. As is usual with spontaneous symmetry breaking of a local symmetry, the local SO(3, 2) gauge invariance of the action (2.5) prevents the Goldstone field from making an explicit appearance. The action has the same form as (2.5) in the unitary gauge (2.6), but with all fields barred [note that in the gauge (2.6), $\zeta^{a}(x)=0$ everywhere, so that $\bar{e}_{\mu}{}^{a}=e_{\mu}{}^{a}$ and $\bar{\omega}_{\mu}{}^{ab}=\omega_{\mu}{}^{ab}$]:

$$I=\int d^{4}x\,\epsilon_{abcd}\bar{R}_{\mu\nu}{}^{ab}\bar{R}_{\rho\tau}{}^{cd}\epsilon^{\mu\nu\rho\tau}\,, \tag{3.33}$$

where

$$\bar{R}_{\mu\nu}{}^{ab}=\bar{B}_{\mu\nu}{}^{ab}+m^{2}(\bar{e}_{\mu}{}^{a}\bar{e}_{\nu}{}^{b}-\bar{e}_{\nu}{}^{a}\bar{e}_{\mu}{}^{b})\,. \tag{3.34}$$

In this form, the action is locally SO(3, 2) invariant because it is written in terms of barred fields and is constructed to be locally invariant under the linearly realized SO(3, 1) stability subgroup. The cosmological constant in (3.33) is proportional to m^{4}, and hence is determined by the strength of the symmetry breaking.

The nonlinear form (3.33) of the action is a straightforward consequence of the theory of nonlinear realizations. As such, it could have been written down immediately without starting from the linearly realized form (2.5). To do so, however, would give no indication of the role of the Goldstone field $\zeta^{a}(x)$ and the anti-de Sitter space it lives in, the interpretation of which we shall now address.

IV. DEVELOPMENT

In the last section, we have reproduced the usual local structure of general relativity in first-order form in the vierbein formalism.[1,2] The essential feature of the theory of nonlinear realizations that we have used is the respect for the nonlinear fields SO(3, 1) indices. The standard vierbein and spin connection are the nonlinearly transforming fields $\bar{e}_{\mu a}$ and $\bar{\omega}_{\mu ab}$ defined in (3.18). The spin connection $\bar{\omega}_{\mu ab}$ transforms in the required fashion (3.22) to construct the covariant derivative $\bar{D}_{\mu}$ which generates the usual parallel transport of vectors lying in the tangent spaces to the space-time manifold $\mathfrak{M}$. All of these standard features, however, do not fully describe the geometrical structure of the theory. The original linear gauge fields $e_{\mu}{}^{a}$ and $\omega_{\mu}{}^{ab}$ give rise to a second kind of differential operator and its associated notion of parallel transport, which is called development.

The development operator Δ_{μ}, defined by

$$\Delta_{\mu}=\partial_{\mu}+\tfrac{1}{2}i\omega_{\mu}{}^{ab}\mathsf{M}_{ab}-ie_{\mu}{}^{a}\mathsf{P}_{a}\,, \tag{4.1}$$

is understood to act on any field according to its transformation character. It has the same form as the covariant derivative ∇_{μ} introduced in (2.12), but when it is applied to a nonlinearly transforming field, the P_{a} generator produces an infinitesimal nonlinear transformation in accordance with (3.16). On a nonlinear field $\bar{\psi}(x)$, we shall write this action as $-i\mathsf{P}_{a}\bar{\psi}(x)$, considering P_{a} to be an operator. More specifically, on a vector field's nonlinear components $\bar{V}^{a}(x)$, we have

$$-i\epsilon^{b}(x)\mathsf{P}_{b}\bar{V}^{a}(x)=[\bar{h}_{1}(\zeta(x),\epsilon(x))]^{a}{}_{b}\bar{V}^{b}(x)\,, \tag{4.2}$$

where the value of $\bar{h}(\zeta,\epsilon)=\frac{1}{2}i[\bar{h}(\zeta,\epsilon)]^{ab}\mathsf{M}_{ab}$ is given in (3.7). We will consider M_{ab} also to be an operator, but of course its action is always linear.

The covariance properties of the operator Δ_{μ} require some explanation. Its action on a nonlinear field is not the same as that of applying Δ_{μ} to the fields of a linear embidding representation and then making the transformation to nonlinear form with $e^{i\zeta(x)\cdot\mathsf{P}}$. Nonetheless, the action of Δ_{μ} is covariant in the sense that the operator $[1+\rho^{\mu}(x)\Delta_{\mu}]$ preserves the nonlinear transformation character of the fields it acts on, with accuracy up to the first-order terms in the infinitesimal parameter $\rho^{\mu}(x)$. In order to establish this, we shall need the following result: When applied to fields $\bar{\psi}(x)$ that transform according to (3.16), with σ any representation of SO(3, 1), we have the equality

$$\Delta_{\mu}\bar{\psi}(x)=(\partial_{\mu}+\tfrac{1}{2}i\bar{\omega}_{\mu}{}^{ab}\mathsf{M}_{ab}+i\bar{e}_{\mu}{}^{a}\mathsf{P}_{a})\bar{\psi}(x)\quad. \tag{4.3}$$

Note the change of sign in the P_{a} term with respect to (4.1). Equation (4.3) may be proved by formal manipulations using the transformation law (3.6) and the definition (3.18), or more simply by substituting (3.19) and (3.20) into (3.17). Equation (4.3) holds by virtue of the existence of the automorphism $\mathsf{P}_{a}\to-\mathsf{P}_{a}$ of the SO(3, 2) Lie algebra. A corollary of (4.3) is that for any set of parameters $(\epsilon^{ab},\epsilon^{a})$, the following holds:

$$(\tfrac{1}{2}i\epsilon^{ab}\mathsf{M}_{ab}-i\epsilon^{a}\mathsf{P}_{a})\bar{\psi}(x)=(\tfrac{1}{2}i\bar{\epsilon}^{ab}\mathsf{M}_{ab}+i\bar{\epsilon}^{a}\mathsf{P}_{a})\bar{\psi}(x)\,, \tag{4.4}$$

where the barred parameters ($\bar{\epsilon}^{ab}, \bar{\epsilon}^{a}$) are defined according to (3.15).

Equation (4.3) may be rewritten

$$\Delta_\mu \bar{\psi}(x) = \bar{D}_\mu \bar{\psi}(x) + i\bar{e}_\mu{}^a \mathsf{P}_a \bar{\psi}(x) . \tag{4.5}$$

The first term transforms in the same way as $\bar{\psi}(x)$ because $\bar{D}_\mu$ is a covariant derivative, as we have already discussed. The second term just generates an infinitesimal P_a gauge transformation; in the operator $[1+\rho^\mu(x)\Delta_\mu]$, the contravariant world index of $\rho^\mu(x)$ is changed to that appropriate for the parameter of a P_a gauge transformation by the vierbein $\bar{e}_\mu{}^a$, which itself transforms homogeneously according to (3.21). Thus, we expect that up to first-order terms in $\rho^\mu(x)$, the quantity

$$\bar{\psi}_*(x) = [1+\rho^\mu(x)\Delta_\mu]\bar{\psi}(x) \tag{4.6}$$

has the same general nonlinear transformation character as $\bar{\psi}(x)$. This is certainly true for the M_{ab} transformations, which are linearly realized. In order to determine precisely the transformation properties of $\bar{\psi}_*(x)$, however, we must take into account the way in which its P_a gauge transformation law depends on the Goldstone field $\zeta^a(x)$.

The development operator Δ_μ may also be applied to the Goldstone field $\zeta^a(x)$. This field has its own special transformation law (3.10), which must be taken into account when performing the transformation $-ie_\mu{}^b(x)\mathsf{P}_b\zeta^a(x)$. Owing to the special transformation behavior of this field, Eq. (4.3) does not apply to it. It may be verified by explicit calculation, however, that under an infinitesimal P_a gauge transformation with parameter $\epsilon^a(x)$, the quantity

$$\zeta^a_*(x) = \zeta^a(x) + \rho^\mu(x)\Delta_\mu\zeta^a(x) \tag{4.7}$$

transforms by the same function of $\zeta^a{}_*(x)$ and $\epsilon^a(x)$ as the function (3.10) of $\zeta^a(x)$ and $\epsilon^a(x)$ that $\zeta^a(x)$ transforms by, keeping only up to first-order terms in $\rho^\mu(x)$. The transformation law of $\bar{\psi}_*(x)$ is also given up to first order in $\rho^\mu(x)$ by the same function (3.17) as that which occurs in the transformation of $\bar{\psi}(x)$, but, again, with the argument $\zeta^a(x)$ replaced by $\zeta^a_*(x)$. Specifically, taking the nonlinear components of a vector field $\bar{V}^a(x)$, and defining

$$\bar{V}^a_*(x) = \bar{V}^a(x) + \rho^\mu(x)\Delta_\mu\bar{V}^a(x) , \tag{4.8}$$

the transformation of $\bar{V}^a_*(x)$ is given by

$$\delta(\bar{V}^a_*(x)) = [\bar{h}_1(\zeta_*(x), \epsilon(x))]^a{}_b\bar{V}^b_*(x) + O(\rho^2) . \tag{4.9}$$

The quantities $\zeta^a_*(x)$ and $\bar{V}^a_*(x)$ in (4.7) and (4.8) are the result of bringing $\zeta^a(x+\rho(x))$ and $\bar{V}^a(x+\rho(x))$ an infinitesimal distance from the point $x^\mu + \rho^\mu(x)$ to the point x^μ using the development process which is generated by Δ_μ. The development of $\zeta^a(x_1)$ and $\bar{V}^a(x_1)$ along a finite curve $x^\mu(t)$ passing through x_1^μ at $t=1$ and through x_0^μ at $t=0$ is given by the solutions to the differential equations

$$\frac{dx^\mu(t)}{dt}\Delta_\mu\zeta^a_*(x(t);x_1) = 0 , \tag{4.10}$$

$$\frac{dx^\mu(t)}{dt}\Delta_\mu\bar{V}^a_*(x(t);x_1) = 0 , \tag{4.11}$$

subject to the initial conditions

$$\zeta^a_*(x(1);x_1) = \zeta^a(x_1) , \tag{4.12}$$

$$\bar{V}^a_*(x(1);x_1) = \bar{V}^a(x_1) . \tag{4.13}$$

The solutions $\zeta^a_*(x(0);x_1)$ and $\bar{V}^a_*(x(0);x_1)$ are the result of developing the values of $\zeta^a(x_1)$ and $\bar{V}^a(x_1)$ along the curve to the point x_0^μ. The expressions (4.7) and (4.8) give the solutions to (4.10) and (4.11) for development an infinitesimal distance from $x_0^\mu + \rho^\mu(x_0)$ to x_0^μ, because $\rho^\mu\Delta_\mu\zeta^a_* = 0$ can be written $\rho^\mu(\partial_\mu + (\Delta_\mu - \partial_\mu))\zeta_* = 0$, so to first order in ρ^μ,

$$\zeta^a_*(x_0+\rho) - \zeta^a_*(x_0) + \rho^\mu(\Delta_\mu - \partial_\mu)\zeta^a_*(x_0+\rho) + O(\rho^2) = 0$$

and then use of the initial condition $\zeta^a_*(x_0+\rho) = \zeta^a(x_0+\rho)$ gives

$$\zeta^a(x_0) + \rho^\mu\partial_\mu\zeta^a - \zeta^a_*(x_0) + \rho^\mu(\Delta_\mu - \partial_\mu)\zeta^a + O(\rho^2) = 0 ,$$

or

$$\zeta^a_*(x_0; x_0+\rho) = \zeta^a(x_0) + \rho^\mu\Delta_\mu\zeta^a + O(\rho^2) ,$$

which is just (4.7). A similar argument holds for (4.8).

In order to explain the geometrical meaning of development, we must first recall the discussion in Sec. II on the construction of the solder form that makes a rigid linkage between the reduced SO(3, 2) fiber bundle and the tangent spaces to the space-time manifold $\mathfrak{M}$. More precisely, the solder form is a smooth map between the tangent space to $\mathfrak{M}$ at a point P with coordinates x^μ and the tangent space to the internal AdS space at the point whose AdS coordinates are $\zeta^a(x)$, as the point P ranges over the whole manifold $\mathfrak{M}$. Note that in order to cover both the space-time manifold $\mathfrak{M}$ and the internal AdS space with coordinates, it will in general be necessary to use several coordinate charts, but we shall not be concerned with that here. Up until now, we have just treated the vierbein $\bar{e}_\mu{}^a(x)$ as a matrix that indicates how to convert the world components of a tensor into its nonlinear components. Now we must take into account the different spaces in which the two objects that have these two different kinds of components are to be found. Figure 1 illustrates the map between the tangent spaces $T_x(\mathfrak{M})$ and $T_{\zeta(x)}(\{G/H\}_x)$ whose matrix is $\bar{e}_\mu{}^a(x)$. The space $\{G/H\}_x$ is the internal AdS apce at x^μ.

In complement to the discussion in Sec. II on the

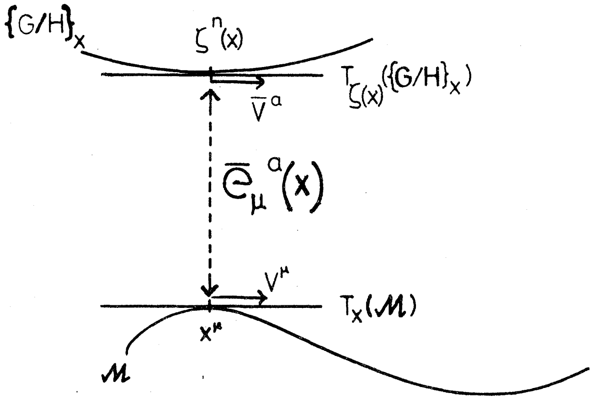

FIG. 1. The vierbein $\bar{e}_\mu{}^a(x)$ is the matrix of the map between the tangent space $T_x(\mathfrak{M})$ to the space-time manifold $\mathfrak{M}$ at x^μ and the tangent space $T_{\zeta(x)}(\{G/H\}_x)$ to the internal AdS space $\{G/H\}_x$ at the point $\zeta^n(x)$. The vierbein field changes the world components V^μ of a vector into the nonlinear components $\bar{V}^a$.

construction of the solder form, the interpretation of the vierbein field given here is also in full accord with the underlying mathematics of the theory of nonlinear realizations presented in Ref. 17 and particularly in Ref. 18. It can be seen from the definition (3.15) of the barred tensor field $\bar{\psi}(x)$ that its nonlinearly transforming components are obtained by projection of the components of the embedding linear field $\psi(x)$ into the tangent space $T_{\zeta(x)}(\{G/H\}_x)$. The independently transforming parts of $\bar{\psi}(x)$ are obtained by contracting different numbers of the indices of $\psi(x)$ with the normal to this tangent space; these parts consequently belong to different representations of the Lorentz group. Since a four-vector like $\bar{V}^a(x)$ belongs to $T_{\zeta(x)}(\{G/H\}_x)$, the vierbein $\bar{e}_\mu{}^a(x)$ is the matrix of a map into this space from $T_x(\mathfrak{M})$.

The vierbein matrix $\bar{e}_\mu{}^a(x)$ is expressed in components that are related to bases in $T_x(\mathfrak{M})$ and $T_{\zeta(x)}(\{G/H\}_x)$. Although the basis in $T_x(\mathfrak{M})$ will be taken to be coordinate induced (holonomic), this is not the case for the orthonormal basis in $T_{\zeta(x)}(\{G/H\}_x)$. The basis in this latter tangent space is determined by our choice of the exponential parametrization of the AdS space $\{G/H\}$ in Eq. (3.1) and the definition of $\bar{e}_\mu{}^a$ in (3.18). It is, however, a simple matter to relate this anholonomic basis to the $\{G/H\}$ coordinate basis; the relation is obtained by inspection of (3.19), setting $e_\mu{}^a = \omega_\mu{}^{ab} = 0$, and replacing $\partial_\mu \zeta^a$ by δ^a_μ, since the ζ^a are coordinates in $\{G/H\}$. This procedure makes use of the fact that the space $\mathfrak{M}_{\text{vac}}$ satisfying $R_{\mu\nu}{}^{AB}(x)=0$ is an AdS space identical to $\{G/H\}$, as we have already noted in Sec. II. Setting $e_\mu{}^a$ and $\omega_\mu{}^{ab}$ to zero everywhere picks a gauge throughout $\mathfrak{M}_{\text{vac}}$; this is necessary if $\mathfrak{M}_{\text{vac}}$ is to be identified with $\{G/H\}$ because we do not have the freedom of local gauge transformations at different points within the internal AdS space. There is only the freedom of choosing the origin of the AdS space and the orientation of the coordinate axes, corresponding to P_a and M_{ab} transformations that act rigidly throughout that space.

Defining $l_n{}^a(\zeta)$ to be the internal AdS vierbein field obtained by the above procedure, we have

$$l_n{}^a(\zeta) = \delta_n{}^a + \left(\delta_n{}^a - \frac{\zeta_n \zeta^a}{\zeta^2}\right)\left(\frac{\sinh z}{z} - 1\right). \qquad (4.14)$$

This expression can also be obtained from a result given in Ref. 26. The inverse AdS vierbein field is

$$l^n{}_a(\zeta) = \delta^n{}_a + \left(\delta^n{}_a - \frac{\zeta^n \zeta_a}{\zeta^2}\right)\left(\frac{z}{\sinh z} - 1\right). \qquad (4.15)$$

Further details on the description of AdS space in the chosen parametrization are given in the Appendix.

Henceforth, in order to distinguish between holonomic and anholonomic AdS tangent space indices, we shall reserve the letters n, r, s for the holonomic indices and $a, b, c, \ldots$ for the anholonomic ones. Using the AdS inverse vierbein field (4.15), we may combine the infinitesimal P_a and M_{ab} gauge transformations of $\zeta^n(x)$ with parameters $[\epsilon^a(x), \epsilon^{bc}(x)]$ from (3.7) and (3.10) into a single equation:

$$\delta\zeta^n(x) = \epsilon^n(x) + [\bar{h}_1(\zeta(x), \epsilon^a(x), \epsilon^{bc}(x))]^n{}_r \zeta^r(x), \qquad (4.16)$$

where $\epsilon^n(x) = \epsilon^a(x) l^n{}_a(\zeta(x))$ and $[\bar{h}_1]^n{}_r$ is the appropriate matrix for the transformation of $\zeta^n(x)$ according to (3.7), i.e., $[\bar{h}_1(\zeta, \epsilon^a, \epsilon^{bc})]^n{}_r = \delta^n{}_b \eta_{rc}([\bar{h}_1(\zeta, \epsilon^a)]^{bc} + \epsilon^{bc})$.

The result of applying the development operator Δ_μ to $\zeta^n(x)$ is

$$\Delta_\mu \zeta^n(x) = \theta_\mu{}^n(x), \qquad (4.17)$$

where

$$\theta_\mu{}^n(x) = \bar{e}_\mu{}^a(x) l^n{}_a(\zeta(x)) \qquad (4.18)$$

is a matrix that change holonomic indices in $T_x(\mathfrak{M})$ into holonomic indices in $T_{\zeta(x)}(\{G/H\}_x)$. A corollary of (4.17), which is analogous to the result (4.4), is that for any set of parameters $(\epsilon^{ab}, \epsilon^a)$, the following holds:

$$(\tfrac{1}{2} i \epsilon^{ab} \mathsf{M}_{ab} - i \epsilon^a \mathsf{P}_a)\zeta^n(x) = \bar{\epsilon}^a l^n{}_a(\zeta(x)), \qquad (4.19)$$

where $\bar{\epsilon}^a$ is defined as before, according to (3.15).

Equation (4.17) is the key to the geometrical interpretation of the development process generated by Δ_μ. For an infinitesimal development from $x_0^\mu + \rho^\mu(x_0)$ to x_0^μ, we have

$$\zeta^n_*(x_0; x_0 + \rho) = \zeta^n(x_0) + \rho^\mu(x_0)\theta_\mu{}^n(x_0). \qquad (4.20)$$

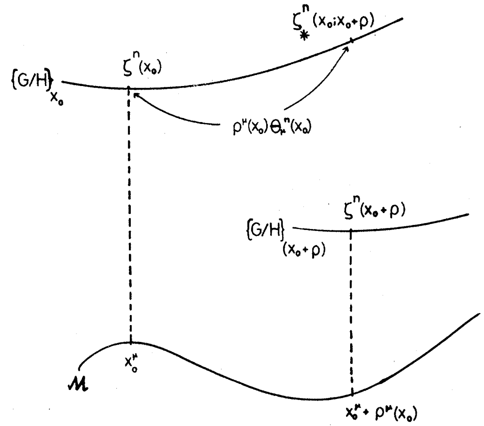

FIG. 2. Infinitesimal development of the Goldstone field $\zeta^n(x)$ with $\rho^\mu(x_0)\Delta_\mu$. The point $\zeta^n(x+\rho)$ in $\{G/H\}_{(x+\rho)}$ is mapped into the point $\zeta^n_*(x_0;x_0+\rho)=\zeta^n(x_0)+\rho^\mu(x_0)\theta_\mu{}^n(x_0)$ in $\{G/H\}_{x0}$.

Thus, an infinitesimal development of $\zeta^n(x_0+\rho)$ from $x_0^\mu+\rho^\mu(x_0)$ to x_0^μ yields a point $\zeta^n_*(x_0;x+\rho)$ in $\{G/H\}_{x_0}$ that is separated from $\zeta^n(x_0)$ by $\rho^\mu(x_0)\theta_\mu{}^n(x_0)$, since for infinitesimal ρ^μ, $\rho^\mu\theta^n_\mu$ may be thought of as lying in $\{G/H\}_{x_0}$ itself. The point $\zeta^n_*(x_0;x_0+\rho)$ in $\{G/H\}_{x_0}$ may be associated with the point $x_0^\mu+\rho^\mu(x_0)$ in $\mathfrak{M}$. The situation is illustrated in Fig. 2. Note that the interpretation of the vierbein field given earlier and illustrated in Fig. 1 is essential for consistency with the above interpretation of (4.20). Considering infinitesimal vectors ρ^μ in $T_x(\mathfrak{M})$ as representing infinitesimal translations, there are corresponding vectors in $T_{\zeta(x_0)}(\{G/H\}_{x_0})$ that are obtained by development, as given in (4.20). These latter vectors $\rho^\mu\theta^n_\mu(x_0)$ are the same as those obtained by straightforwardly mapping with the vierbein $\bar{e}_\mu{}^a(x_0)$, and then changing to the AdS holonomic basis, as is shown in (4.18).

The association of $x_0^\mu+\rho^\mu(x_0)$ and $\zeta^n_*(x_0;x_0+\rho)$ is especially natural when the space-time $\mathfrak{M}$ is in its vacuum state $\mathfrak{M}_{\rm vac}$, so that it is an AdS space just like $\{G/H\}_{x_0}$. Since in this state, $R_{\mu\nu}{}^{AB}=0$ everywhere, there exists a gauge in which $e_\mu{}^a=\omega_\mu{}^{ab}=0$ everywhere, so that $\Delta_\mu=\partial_\mu$, and the developed value $\zeta^n_*(x_0;x_0+\rho)$ is just $\zeta^n(x_0+\rho)$. In the vacuum state $\mathfrak{M}_{\rm vac}$, development over a finite distance is given in the indicated gauge by $\zeta^n_*(x_0;x)=\zeta^n(x)$. Thus, each point x^μ in $\mathfrak{M}_{\rm vac}$ is associated with the point in $\{G/H\}_{x_0}$ that has the coordinates $\zeta^n(x)$. In addition, one can choose coordinates in $\mathfrak{M}_{\rm vac}$ such that $x^\mu=\delta^\mu_n\zeta^n(x)$, and then we have just the situation we used to obtain the AdS vierbein field (4.14), where $\mathfrak{M}_{\rm vac}$ and $\{G/H\}_{x_0}$ are not only geometrically identical, but are coordinatized in the same way; accordingly we also have $\bar{e}_\mu{}^a(x)=\delta_\mu{}^n l_n{}^a(\zeta(x))$, and $\theta_\mu{}^n=\delta_\mu{}^n$.

In order to interpret development in $\mathfrak{M}_{\rm vac}$ as above, it is not essential to make any special gauge or coordinate choices. If a different gauge choice is made in the vicinity of x_0^μ, but keeping the gauge at x_0^μ itself unchanged, then neighboring points x^μ will be identified with the same points $\zeta^n_*(x_0;x)$ in $\{G/H\}_{x_0}$ as before, but the relation to $\zeta^n(x)$ will be different. In this more general case, the linear gauge fields $e_\mu{}^a$ and $\omega_\mu{}^{ab}$ are nonvanishing, and their contribution to Δ_μ just corrects for the different local choice of gauge. Similarly, the choice of coordinates in $\mathfrak{M}_{\rm vac}$ does not affect the association of points in $\mathfrak{M}_{\rm vac}$ with points in $\{G/H\}_{x_0}$.

In general space-times $\mathfrak{M}$, it is not possible to associate unambiguously every point in $\mathfrak{M}$ with a point in $\{G/H\}_{x_0}$. What the development process does, however, is to unambiguously map a curve in $\mathfrak{M}$ passing through x_0 into an *image curve* in $\{G/H\}_{x_0}$ which passes through $\zeta^n(X_0)$. Moreover, if the curve in $\mathfrak{M}$ is an autoparallel, satisfying

$$\frac{d^2x^\mu(t)}{dt^2}+\Gamma_{\rho\sigma}{}^\mu(x(t))\frac{dx^\rho(t)}{dt}\frac{dx^\sigma(t)}{dt}=0\,, \qquad (4.21)$$

then the image curve will be a geodesic in $\{G/H\}_{x_0}$. In order to see this, we need to know what happens to vectors lying in the tangent spaces to $\mathfrak{M}$ along $x^\mu(t)$ under development.

The development of a vector $\bar{V}^a(x_1)$ along the curve $x^\mu(t)$ from x_1^μ to x_0^μ is given by the solution to Eq. (4.11) subject to the boundary condition (4.13). In order to interpret the solution to this equation, we return to the infinitesimal solution $\bar{V}_*(x_0;x_0+\rho)$ in (4.8) and note the decomposition (4.5) of Δ_μ into $\bar{D}_\mu+i\bar{e}_\mu{}^aP_a$. The quantity $\bar{V}^a(x_0)+\rho^\mu(x_0)\bar{D}_\mu\bar{V}^a(x_0)$ just represents the result of ordinary parallel transport in $\mathfrak{M}$ of $\bar{V}^a(x_0+\rho)$ to the point x_0^μ with the usual covariant derivative $\bar{D}_\mu$, as discussed in Sec. III. The remaining term $+i\rho^\mu(x_0)\bar{e}_\mu{}^b(x_0)P_b\bar{V}^a(x_0)$ then generates an internal parallel transport in $\{G/H\}_{x_0}$ from $\zeta^n(x_0)$ to $\zeta^n_*(x_0;x_0+\rho)$, where ζ^n_* is given in (4.7). In order to show this, we need to have the internal AdS spin connection corresponding to the AdS vierbein field (4.14). This can be obtained by inspection from (3.20), following the procedure we used to obtain $l_n{}^a(\zeta)$ in (4.14). The result is

$$\Omega_n{}^{ab}(\zeta)=-m^2(\zeta^a\delta_n{}^b-\zeta^b\delta_n{}^a)\left(\frac{\cosh z-1}{z^2}\right). \qquad (4.22)$$

This AdS spin connection satisfies the standard relation to $l_n{}^a(\zeta)$, i.e.,

$$\Omega_n{}^{ab}(\zeta)=\tfrac{1}{2}\,[l^{ra}(\partial_n l_r{}^b-\partial_r l_n{}^b)+l^{ra}l^{sb}(\partial_s l_{rc})l_n{}^c]-(a\leftrightarrow b)\,, \qquad (4.23)$$

where $\partial_n=\partial/\partial\zeta^n$, and $a,b,c,\ldots$ are raised and lowered with η_{ab}.

It follows from (3.17), (4.15), and (4.22) that for a P_a gauge transformation with parameter ϵ^a,

$$[\bar{h}_1(\zeta,\epsilon)]^{ab} = \epsilon^c l^n{}_c(\zeta)\Omega_n{}^{ab}(\zeta)\,, \tag{4.24}$$

so that the operation $+i\rho^\mu \bar{e}_\mu{}^c \mathrm{P}_c$ gives the same change in the components $\bar{V}^a$ as does parallel transport in $\{G/H\}_{x_0}$ from $\zeta^n(x_0)$ to

$$\zeta^n_*(x_0;x_0+\rho) = \zeta^n(x_0) + \rho^\mu(x_0)\bar{e}_\mu{}^c(x_0) l^n{}_c(\zeta(x_0))\,.$$

The positive sign of the $i\bar{e}_\mu{}^a \mathrm{P}_a$ term in (4.3) is essential for obtaining this result. The net effect of an infinitesimal development from $x_0^\mu + \rho^\mu(x_0)$ to x_0^μ on a vector $\bar{V}^a(x_0+\rho)$ is to parallel transport it along the infinitesimal trajectory shown in Fig. 3, first using $\bar{D}_\mu$ in $\mathfrak{M}$ from $x_0^\mu + \rho^\mu(x_0)$ to x_0^μ and then using $\partial_n + i\Omega_n{}^{ab}(\zeta)\mathrm{M}_{ab}$ in $\{G/H\}_{x_0}$ from $\zeta^n(x_0)$ to $\zeta^n_*(x_0;x_0+\rho)$.

Development along a finite curve $x^\mu(t)$ in $\mathfrak{M}$ maps a vector $\bar{V}^a(x_1)$ lying in $T_{\zeta(x_1)}(\{G/H\}_{x_1})$ into its image $\bar{V}^a_*(x_0;x_1)$ lying in $T_{\zeta(x_0;x_1)}(\{G/H\}_{x_0})$. At an intermediate point x_i^μ along the curve, the internal space $\{G/H\}_{x_i}$ contains a partially developed image curve with $\bar{V}^a_*(x_i;x_1)$ lying in the internal tangent space $T_{\zeta_*}(x_i;x_1)(\{G/H\}_{x_i})$. The process is summarized in Fig. 4. If the curve $x^\mu(t)$ is thought of as being broken up into many short segments $\rho^\mu(x_i)$, development along each new segment ending at a point x_i^μ consists in moving the previously developed image curve rigidly from $\{G/H\}_{x_{i-1}}$ into $\{G/H\}_{x_i}$ so that the previous end point $\zeta^n_*(x_{i-1};x_{i-1}) = \zeta^n(x_{i-1})$ is mapped into $\zeta^n_*(x_i;x_{i-1}) = \zeta^n(x_i) + \rho^\mu(x_i)\theta_\mu{}^n(x_i) + O(\rho^2)$. At the same time, vectors previously developed into the tangent spaces along the image curve in $\{G/H\}_{x_i}$ must be moved rigidly together with the image curve into their appropriate new tangent spaces to $\{G/H\}_{x_i}$.

The rigid character of the mapping of the image curve and of the vectors defined along it means that the previously developed curve is not distorted during development along subsequent segments. The mapping is rigid because it is brought about by M_{ab} and P_a transformations which are the isometries of AdS space and act rigidly throughout it,

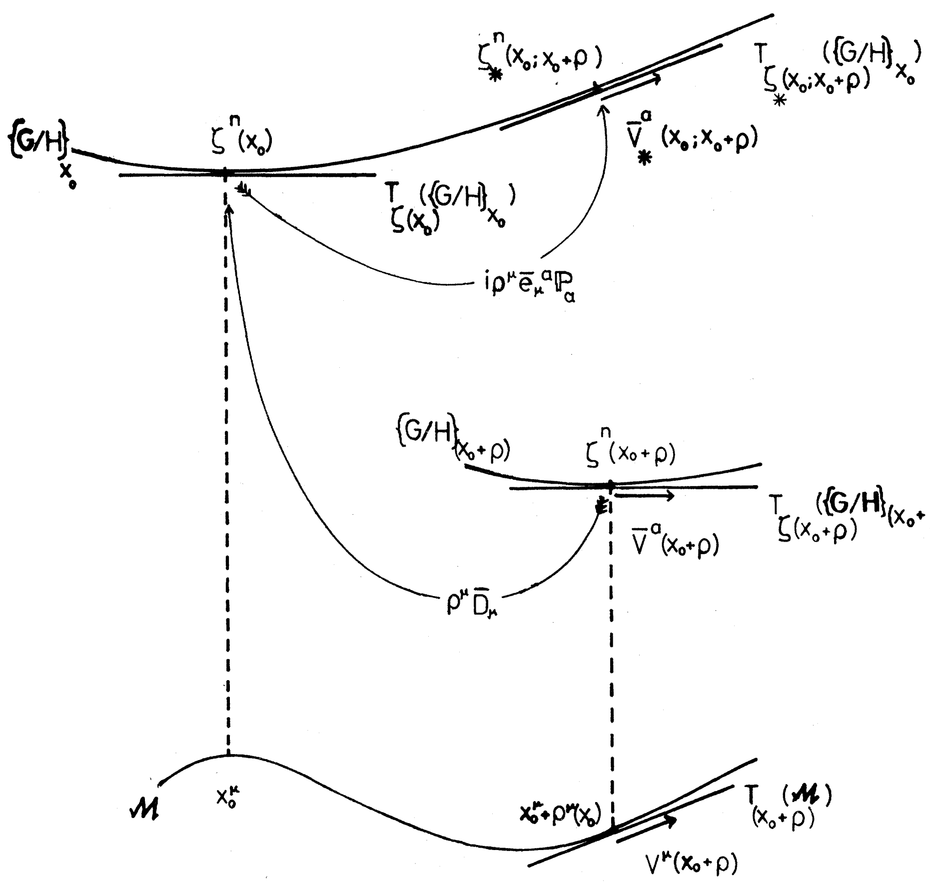

FIG. 3. Infinitesimal development of a vector $\bar{V}^a(x_0+\rho) = V^\mu(x_0+\rho)\bar{e}_\mu{}^a(x_0+\rho)$ with $\rho^\mu(x_0)\Delta_\mu$. The vector $\bar{V}^a(x_0+\rho)$ in $T_{\zeta(x_0+\rho)}(\{G/H\}_{(x_0+\rho)})$ is mapped into the vector $\bar{V}^a_*(x_0;x_0+\rho)$ in $T_{\zeta_*(x_0;x_0+\rho)}(\{G/H\}_{x_0})$.

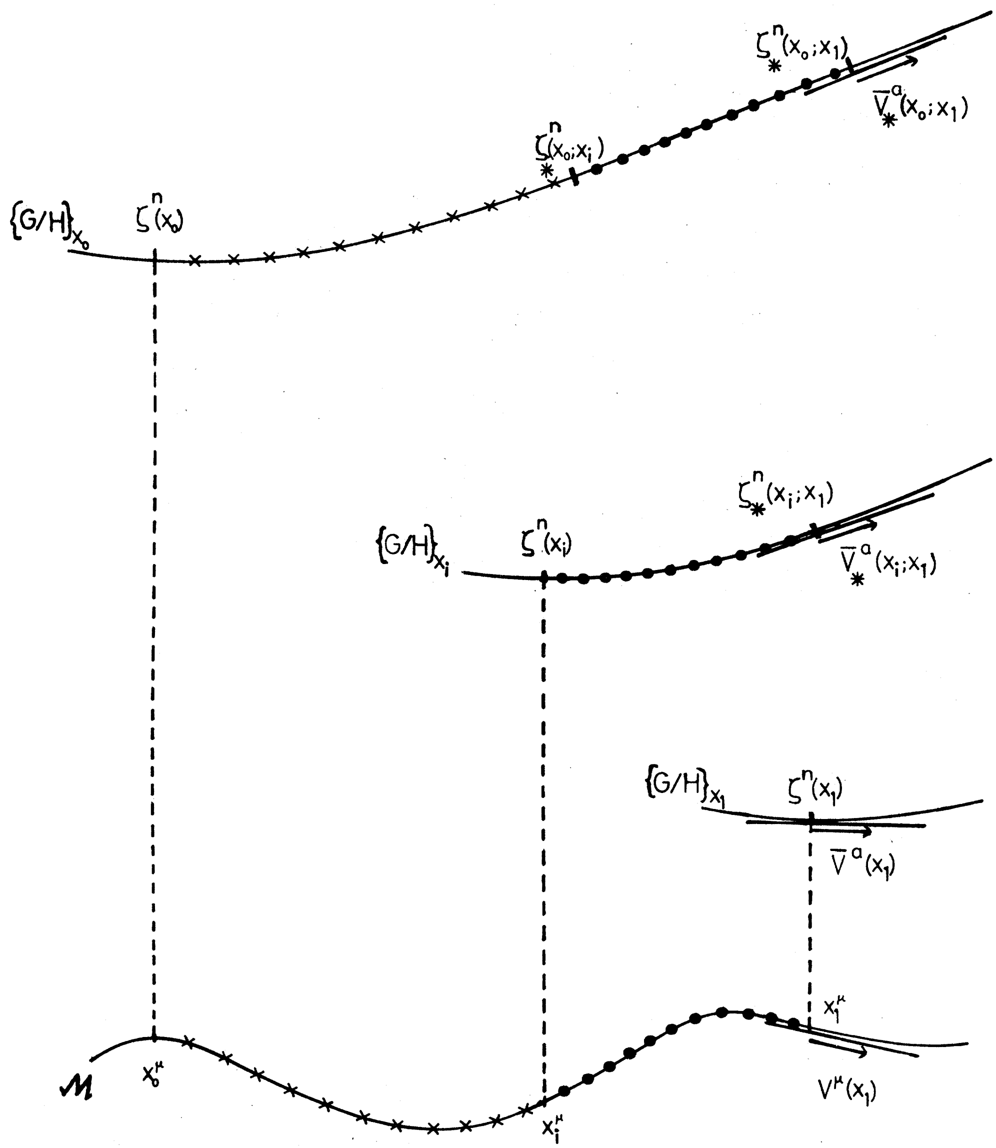

FIG. 4. Development from x_1^μ to x_0^μ proceeds by adding new segments without distorting the previously developed curve, moving it rapidly in $\{G/H\}$ to accommodate the new segments. An intermediate state of this development is shown at the point x_i^μ. The portion of the image curve in $\{G/H\}_{x_0}$ indicated by the dotted line is congruent to the image curve in $\{G/H\}_x$.

analogously to the rigid rotations of a sphere. Although the P_a gauge part of the mapping acts rigidly throughout the Ads space in this sense, its detailed effect upon the coordinates of the points and the components of the vectors along the image curve is highly AdS coordinate dependent, according to (3.10) and (3.17). Algebraically, the rigidity of the mapping is enforced by the covariance property (4.9) of the development operator Δ_μ when acting on $\overline{V}^a(x)$ and the corresponding property for its action on $\zeta^n(x)$, which, using (4.16), can now be written

$$\delta\zeta_*^n(x)=\epsilon^a(x)l^n{}_a(\zeta_*(x))+[\tilde{h}_1(\zeta_*(x),\epsilon(x))]^n{}_r\zeta_*^r(x)+O(\rho^2)\,. \tag{4.25}$$

Equations (4.25) and (4.9) ensure that $\zeta_*^n(x_i;x_1)$ and $\overline{V}_*^a(x_i;x_1)$ keep in step with each other under the isometric P_a and M_{ab} transformations induced in $\{G/H\}$ by subsequent development from x_i^μ to x_0^μ.

The process of development may be visualized by considering the curve $x^\mu(t)$ and the vector field $\overline{V}^a(x(t))=V^\mu(x(t))\overline{e}_\mu{}^a(x(t))$ defined along it to be inked onto the space-time manifold $\mathfrak{M}$. An AdS space, which is a local copy of the vacuum $\mathfrak{M}_{\text{vac}}$, is rolled along the curve $x^\mu(t)$ without slipping or twisting, and as it rolls, the curve $x^\mu(t)$ and vector fields $\overline{V}^a(x(t))$ are printed onto it. It is the function of the linear gauge fields $e_\mu{}^a$ and $\omega_\mu{}^{ab}$ to determine that there is no slipping or twisting of the AdS space as it rolls, despite the fact that it will in general be coordinatized differently at each point of $\mathfrak{M}$ it passes over. The local choices of origin and coordinate axis orientation of the AdS space correspond to the freedom of local P_a and M_{ab} gauge choices in $\mathfrak{M}$.

If the space-time $\mathfrak{M}$ is in its vacuum state $\mathfrak{M}_{vac}$, then there is no rolling to be done, since the internal AdS space and $\mathfrak{M}_{vac}$ fit together exactly. Curves and vector fields in $\mathfrak{M}_{vac}$ are just printed directly onto the overlying part of the AdS space. As we have seen, there is a gauge choice in $\mathfrak{M}_{vac}$ such that $e_\mu{}^a=\omega_\mu{}^{ab}=0$, so $\Delta_\mu=\partial_\mu$ and development of the nonlinear components of vectors in this case involves no change in their values at all—they are simply reassigned to the overlying tangent spaces to the AdS space. In other words, when space-time is in its vacuum state $\mathfrak{M}_{vac}$, it and all of the internal AdS spaces over each of the points of $\mathfrak{M}_{vac}$ may be unambiguously identified with each other, and similarly for their respective tangent spaces. In general space-times, the development process carries out such an identification for points and tangent spaces along a particular curve, but, as we shall see, when different curves are taken, the identifications are in general different.

The description of the development process given above makes clear another important feature, that the image curve reflects as accurately as possible in $\{G/H\}$ the shape of the original curve in $\mathfrak{M}$. This is due to the fact that the tangent vector $\dot{x}^\mu(t)$ to the curve in $\mathfrak{M}$ is mapped by the vierbein field into $\dot{x}^\mu(t)e_\mu{}^a(x(t))$ in $T_{\zeta(x(t))}(\{G/H\}_{x(t)})$, which in the AdS holonomic basis is $\dot{x}^\mu(t)\theta_\mu{}^n(x(t))$, which is also the tangent to the image curve, by (4.20). Consequently, autoparallels in $\mathfrak{M}$ have image curves in $\{G/H\}$ that are geodesics. Note that we must make a distinction between autoparallels and geodesics in $\mathfrak{M}$ because the space-time may have torsion. The mapping of autoparallels in $\mathfrak{M}$ passing through x_0^μ into their corresponding image geodesics in $\{G/H\}_{x_0}$ provides the means to construct a local coordinate system in $\mathfrak{M}$ around the point x_0^μ, letting $x_0^\mu=\delta^\mu{}_n\zeta^n(x_0)$ and giving nearby points the coordinates of their images in $\{G/H\}_{x_0}$ on the corresponding image geodesics. The change from an original set of coordinates to the new set $\zeta_*^n(x_0;x)$ has an associated transformation matrix which at the point x^μ is just the matrix $\theta_\mu{}^n(x_0)$ from (4.18).

If $\zeta^n(x_0)=0$, so that at the point x_0^μ the "unitary" gauge choice (2.6) is made and the center of the local coordinate system is the origin, then the coordinate system induced in $\mathfrak{M}$ is a normal coordinate system. It inherits this property from the exponential parametrization (3.1) of $\{G/H\}$, which gives a normal coordinate system about the AdS origin, since the AdS geodesics passing through the origin take the simple form $\zeta^n=tc^n$, where the c^n are fixed and t is an affine parameter (in this regard, cf. Ref. 18). The lines $x^\mu=tc^\mu$ in these coordinates consequently are autoparallels in $\mathfrak{M}$. Thus, as can be seen from (4.21), the symmetric part $\Gamma^\lambda{}_{(\mu\nu)}$ of the connection (3.32) vanishes at the origin. Taking into account the possible presence of torsion, these normal coordinates are the inertial frames of the theory.

The process of development that we have introduced and discussed in this section is an essentially gauge-theoretic process, since it is a form of parallel transport of the nonlinearly transforming fields using the original SO(3, 2) Yang-Mills gauge fields for calibration. The geometrical interpretation of development that we have derived provides the link to a generalization, to our situation with internal AdS spaces, of a purely geometrical construction known in differential geometry as development into the flat affine tangent spaces of a differentiable manifold, as discussed, e.g., in Ref. 28.

It is due to this link that we have named the process generated by Δ_μ "development": The image curves and image tensor fields are identical for the two processes, although the process of development using Δ_μ on the Goldstone and tensor fields is ostensibly quite different from the geometrical construction described in 28. This latter construction involves ordinary parallel transport, using the spin connection, of the tangent vectors to a curve in space-time which is parametrized by a parameter t. These tangent vectors are all parallely transported up to some given point x_0^μ. The resulting set of parametrized vectors are expressed in anholonomic components using the vierbein field at x_0^μ and used to generate the image curve in the internal space, a parametrized vector being carried out along the image curve until it reaches the corresponding value of t, at which point it must be tangent to the image curve. Of course, in our situation, the internal space is not flat, but curved, and upon carrying the parametrized vectors along the image curve, the AdS spin connection (4.22) must be used to generate AdS parallel transport of these vectors. The equivalence of this process to the development that we have described relies upon the geometrical interpretation of the vierbein field as illustrated in Fig. 1 and the fact that the SO(3, 2) transformations induced by development with Δ_μ generate precisely the necessary space-time and AdS parallel transport, as is shown by (4.17) for the operation of Δ_μ on the Goldstone field $\zeta^n(x)$, and by (4.3) and (4.24) for its operation on tensor fields like $\overline{V}^a(x)$.

The process of development is important for the geometrical structure of the Einstein-Cartan theory because the geometrical construction which it is equivalent to is the appropriate one for the proper interpretation of the effects of torsion and curvature in space-time.[28] This construction, however, has not been widely used in the physics literature on the Einstein-Cartan theory, although it was obliquely referred to in the work of Trautman,[29] as we have mentioned in the Introduction.

V. THE HOLONOMY GROUP

The mapping of curves in the space-time manifold $\mathfrak{M}$ into the internal AdS space by the development process that we have discussed does not in general yield an unambiguous assignment of points in $\mathfrak{M}$ to points in $\{G/H\}$. Two different curves leaving a point x_0^μ but crossing again at some other point x_1^μ will give different assignments of that point to points in $\{G/H\}_{x_0}$. In addition, a vector in the tangent space to $\mathfrak{M}$ at x_1^μ will in general be mapped into different vectors tangent to $\{G/H\}_{x_0}$ by development along the two different curves. The nonintegrability of the development process may be investigated by considering a curve in $\mathfrak{M}$ that returns to its starting point x_0^μ. Then the result of developing the Goldstone field $\zeta^n(x_0)$ and a nonlinearly transforming field such as a vector $\overline{V}^a(x_0)$ around the given curve can be compared with the original values.

The group of transformations induced in the fibers of a gauge theory by such round trips is known as the holonomy group. This group contains important topological information about the fiber bundle over the space-time manifold $\mathfrak{M}$ which represents the global structure of a particular solution to the theory. In addition, the elements of the holonomy group for infinitesimal closed curves in $\mathfrak{M}$ describe the local structure of a solution, i.e., they determine the local values of the SO(3,2) curvatures. It is with the infinitesimal holonomy group that we are mainly concerned in this paper. In order to fix the relation of the spontaneously broken SO(3,2) gauge theory to the Einstein-Cartan theory, we must relate the effects of nonvanishing SO(3,2) curvature to the effects of nonvanishing SO(3,1) curvature and torsion in the standard theory.

In order to compare the developed values $\zeta_*^n(x_0;x_0)$ and $\overline{V}_*^a(x_0;x_0)$ with the original values at x_0^μ of the Goldstone and vector fields after travelling around an infinitesimal closed curve, we need to have solutions of the differential equations (4.10) and (4.11) that are valid up to second order in the displacement $[x^\mu(t)-x_0^\mu]$. To obtain these solutions, we follow a standard procedure and first replace the differential equations (4.10) and (4.11) by the integral equations

$$\zeta_*^n(x(t);x_0)=\zeta^n(x_0)+\int_t^1[\tfrac{1}{2}i\omega_\mu{}^{bc}(x(t))\mathrm{M}_{bc}-ie_\mu{}^b(x(t))\mathrm{P}_b]\zeta_*^n(x(t);x_0)\frac{dx^\mu(t)}{dt}dt \tag{5.1}$$

and

$$\overline{V}_*^a(x(t);x_0)=\overline{V}^a(x_0)+\int_t^1[\tfrac{1}{2}i\omega_\mu{}^{bc}(x(t))\mathrm{M}_{bc}-ie_\mu{}^b(x(t))\mathrm{P}_b]\overline{V}_*^a(x(t);x_0)\frac{dx^\mu(t)}{dt}dt\,. \tag{5.2}$$

Note that the initial conditions (4.12) and (4.13) are incorporated into (5.1) and (5.2), and that the point we are developing from is $x_0{}^\mu$, as well as the point we shall end up at.

To first order in $[x^\mu(t)-x_0^\mu]$, the solutions to (5.1) and (5.2) are, as in (4.7) and (4.8),

$$\zeta_{*(1)}^n(x(t);x_0)=\zeta^n(x_0)-[\tfrac{1}{2}i\omega_\mu{}^{bc}(x_0)\mathrm{M}_{bc}-ie_\mu{}^b(x_0)\mathrm{P}_b]\zeta^n(x_0)(x^\mu(t)-x_0{}^\mu) \tag{5.3}$$

and

$$\overline{V}_{*(1)}^a(x(t);x_0)=\overline{V}^a(x_0)-[\tfrac{1}{2}i\omega_\mu{}^{bc}(x_0)\mathrm{M}_{bc}-ie_\mu{}^b(x_0)\mathrm{P}_b]\overline{V}^a(x_0)(x^\mu(t)-x_0{}^\mu)\,. \tag{5.4}$$

These expressions may be substituted back into (5.1) and (5.2) to obtain the second-order solutions $\zeta_{*(2)}^n(x(t);x_0)$ and $\overline{V}_{*(2)}^a(x(t);x_0)$. For the situation we are interested in, the result is simplified by the fact that the starting point and end point of the curve in $\mathfrak{M}$ are the same, $x^\mu(1)=x_0^\mu$, so that

$$\int_0^1\frac{dx^\mu(t)}{dt}dt=0 \tag{5.5}$$

and also

$$\oint x^\mu dx^\nu\equiv\int_0^1 x^\mu(t)\frac{dx^\nu(t)}{dt}dt=-\int_0^1 x^\nu(t)\frac{dx^\mu(t)}{dt}dt\,, \tag{5.6}$$

as can be seen by integrating by parts. As a result of (5.6), the terms in the integrals for $\zeta^n_{*(2)}(x_0;x_0)$ and $\overline{V}^a_{*(2)}(x_0;x_0)$ that are of second order in $[x^\mu(t)-x_0{}^\mu]$ may be antisymmetrized in their world indices. Since these terms also involve application of two SO(3, 2) generators to either $\zeta^n(x_0)$ or $\overline{V}^a(x_0)$, the antisymmetrization yields commutators of the SO(3, 2) generators, permitting application of the commutation relations (3.2), (3.4), and (3.5). The results are

$$\zeta^n_{*(2)}(x_0;x_0)-\zeta^n(x_0)=\tfrac{1}{2}[\tfrac{1}{2}iR_{\mu\nu}{}^{bc}(x_0)\mathrm{M}_{bc}-iR_{\mu\nu}{}^{b}(x_0)\mathrm{P}_b]\zeta^n(x_0)\oint x^\mu dx^\nu \tag{5.7}$$

and

$$\overline{V}^a_{*(2)}(x_0;x_0)-\overline{V}^a(x_0)=\tfrac{1}{2}[\tfrac{1}{2}iR_{\mu\nu}{}^{bc}(x_0)\mathrm{M}_{bc}-iR_{\mu\nu}{}^{b}(x_0)\mathrm{P}_b]\overline{V}^a(x_0)\oint x^\mu dx^\nu , \tag{5.8}$$

where the SO(3, 2) curvatures have been given before in Eq. (2.1).

The results (5.7) and (5.8) are of course just what one expects in a gauge theory: The gauge transformations induced by travel around an infinitesimal closed curve are given by the SO(3.2) curvatures contracted with the antisymmetric expression $\oint x^\mu dx^\nu$, which for an infinitesimal parallelogram with sides $(dx^\mu, \delta x^\mu)$ just takes the value $(dx^\mu\delta x^\nu - dx^\nu \delta x^\mu)$. Of course, the SO(3, 2) transformations are applied here to nonlinearly transforming fields, but the transformation character of $\zeta^n(x_0)$ and $\overline{V}^a(x_0)$ was never explicitly used in the derivation of (5.7) and (5.8). The SO(3.2) commutation relations which were used hold regardless of the transformation character of the fields that the generators are applied to. In agreement with equations (5.7) and (5.8), we also have the commutation relations for the development operator Δ_μ, which may be verified directly:

$$[\Delta_\mu,\Delta_\nu]=\tfrac{1}{2}iR_{\mu\nu}{}^{ab}\mathrm{M}_{ab}-iR_{\mu\nu}{}^{a}\mathrm{P}_a . \tag{5.9}$$

The results (5.7) and (5.8) also have a straightforward geometric interpretation. Using the relations (4.19) and (4.4), they can be recast as

$$\zeta^n_{*(2)}(x_0;x_0)-\zeta^n(x_0)=\tfrac{1}{2}\overline{R}_{\mu\nu}{}^{b}(x_0)l^n{}_b(\zeta(x_0))\oint x^\mu dx^\nu \tag{5.10}$$

and

$$\overline{V}^a_{*(2)}(x_0;x_0)-\overline{V}^a(x_0)=\tfrac{1}{2}[\tfrac{1}{2}i\overline{R}_{\mu\nu}{}^{bc}(x_0)\mathrm{M}_{bc}+i\overline{R}_{\mu\nu}{}^{b}(x_0)\mathrm{P}_b]\overline{V}^a(x_0)\oint x^\mu dx^\nu , \tag{5.11}$$

where the barred de Sitter curvature $\overline{R}_{\mu\nu}{}^{bc}$ has been given before in Eq. (3.34). The barred torsion tensor is

$$\overline{R}_{\mu\nu}{}^{b}=\partial_\mu \overline{e}_\nu{}^b-\partial_\nu\overline{e}_\mu{}^b+\overline{\omega}_\mu{}^b{}_c\overline{e}_\nu{}^c-\overline{\omega}_\nu{}^b{}_c\overline{e}_\mu{}^c . \tag{5.12}$$

Both $\overline{R}_{\mu\nu}{}^{bc}$ and $\overline{R}_{\mu\nu}{}^{b}$ are obtained from the expression for the unbarred SO(3, 2) curvatures in (2.1) by putting bars onto all the fields occurring there.

The geometrical interpretation of (5.10) now follows immediately from the discussion of development given in Sec. IV: The result of development of $\zeta^n(x_0)$ around an infinitesimal closed curve beginning and ending at x_0^μ is to produce an image curve in $\{G/H\}_{x_0}$ that fails to close by an amount determined by the torsion tensor $\overline{R}_{\mu\nu}{}^b(x_0)$. The P_a gauge term in (5.11) gives the corresponding correction to the value of $\overline{V}^a(x_0)$, which is equal to the change in the components $\overline{V}^a$ upon parallel transport across the gap in the image curve indicated by (5.10), using the AdS spin connection $\Omega_n{}^{bc}(\zeta)$ given in (4.22). The equivalence of the P_a gauge transformation with parameter $-\frac{1}{2}\overline{R}_{\mu\nu}{}^b(x_0)\oint x^\mu dx^\nu$ to parallel transport across the gap is shown by the relation (4.24).

The term in (5.11) involving the de Sitter curvature $\overline{R}_{\mu\nu}{}^{bc}$ gives the SO(3,1) rotation of the vector $\overline{V}^a_*(x_0;x_0)$ with respect to its starting value $\overline{V}^a(x_0)$ before development, exclusive of any rotation induced by parallel transport across the gap in the image curve, as discussed above. The situation is illustrated in Fig. 5. This rotation is not the same as that induced by ordinary parallel transport using the covariant derivature $\overline{D}_\mu$ in (3.25). The difference is due to the fact that at all points $x(t)$ on the curve in $\mathfrak{M}$, the image $\overline{V}^a_*(x(t);x_0)$ lies in $\{G/H\}_{x(t)}$ at the image point $\zeta^n_*(x(t);x_0)$. As we have discussed, and illustrated in Fig. 3, development of $\overline{V}^a(x_0)$ into $\overline{V}^a_*(x(t);x_0)$ involves parallel transport in $\mathfrak{M}$ from $x_0{}^\mu$ to $x^\mu(t)$, and then back in $\{G/H\}_{x(t)}$ from $\zeta^n(x(t))$ to $\zeta^n_*(x(t);x_0)$. The two parallel transports produce rotations of $\overline{V}^a_*(x(t);x_0)$ with respect to $\overline{V}^a(x_0)$ that tend to cancel, and in fact do cancel exactly when space-time is in the state $\mathfrak{M}_{\rm vac}$ and the gauge where $e_\mu{}^a=\omega_\mu{}^{ab}=0$ is

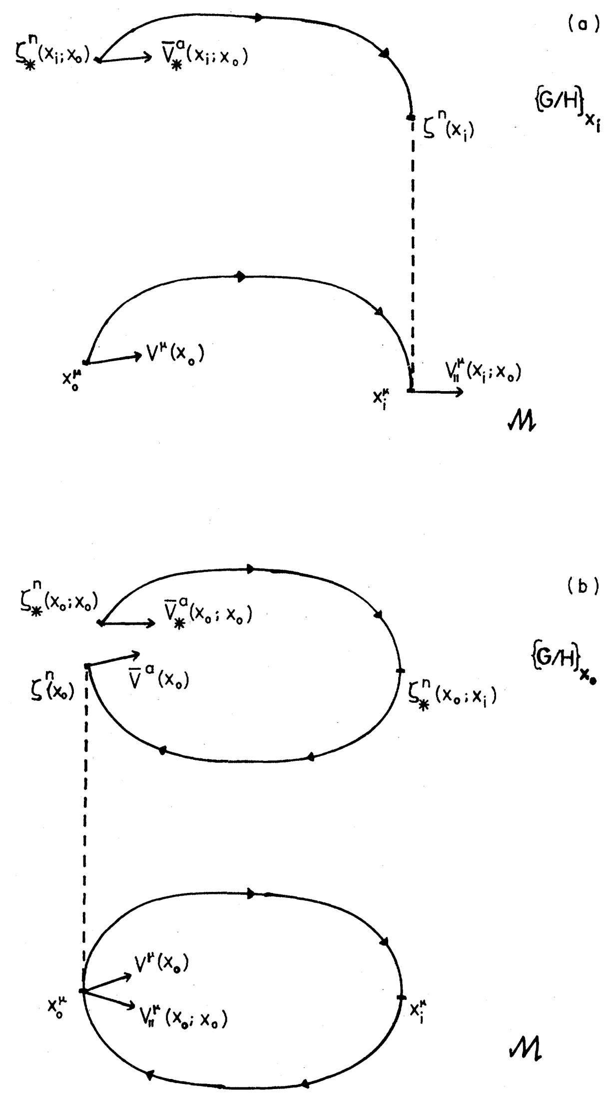

FIG. 5. Development around a closed curve in $\mathfrak{M}$. In (a), an intermediate stage is shown at a point x_i^μ, with the paritally developed curve extending in $\{G/H\}_{x_i}$ back to $\zeta_*^n(x_i;x_0)$. In (b), the final result after development all the way around the curve and back to x_0^μ is shown. For an infinitesimal curve $\mathfrak{M}$, the gap $\delta\zeta^n$ in the image curve in $\{G/H\}_{x_0}$ is determined by the torsion $\bar{R}_{\mu\nu}{}^a(x_0)$. The rotation of $V^\mu(x_0)$ into $V_\parallel^\mu(x_0;x_0)$ is determined by $\bar{B}_{\mu\nu\alpha\beta}(x_0)=\bar{B}_{\mu\nu}{}^{ab}\bar{e}_{\alpha a}\bar{e}_{\beta b}$, while the rotation of $\bar{V}^a(x_0)$ into $\bar{V}_*^a(x_0;x_0)$, exclusive of that induced by closing the gap in the image curve, is given by $\bar{R}_{\mu\nu}{}^{ab}(x_0)$.

chosen, so that $\Delta_\mu=\partial_\mu$.

In general space-times, the net rotation of $\bar{V}_*^a(x_0;x_0)$ with respect to $\bar{V}^a(x_0)$ after development around an infinitesimal closed curve is given by the *difference* of the usual Lorentz curvature tensor $\bar{B}_{\mu\nu}{}^{ab}(x_0)$ defined in (3.31), which gives the rotation due to parallel transport in $\mathfrak{M}$ using $\bar{D}_\mu$, and the curvature tensor of the internal AdS space, which gives the rotation due to parallel transport in $\{G/H\}_{x_0}$ using $\partial_n+i\Omega_n{}^{ab}(\zeta)\mathsf{M}_{ab}$. In AdS holonomic indices, the AdS curvature tensor is $-m^2(l_n{}^a(\zeta)l_r{}^b(\zeta)-l_r{}^a(\zeta)l_n{}^b(\zeta))$; conversion of the AdS holonomic indices to world indices using $\theta_\mu{}^n\theta_\nu{}^r$, evaluating the result at $\zeta^n(x_0)$, and subtracting it from $\bar{B}_{\mu\nu}{}^{ab}(x_0)$ gives $\bar{R}_{\mu\nu}{}^{ab}(x_0)$, as is shown in (3.34).

VI. CONCLUSIONS AND PROSPECTS

In this paper, we have started from a spontaneously broken SO(3, 2) gauge field theory and proceeded through a detailed investigation of the role of the Goldstone field $\zeta^n(x)$ to analyze the theory's local geometrical structure. The theory limits precisely to the Einstein-Cartan theory of gravity as the strength of the symmetry breaking tends to infinity, both at the level of the geometry and at that of the dynamics, since the size of the cosmological constant in the action (3.33) is determined by m^4. It should be noted that in order to carry out this limit on the action (3.33), it is necessary to first discard the Gauss-Bonnet integral (thus confining the discussion to spaces where its value is zero), and then scale the entire action by m^{-2} before letting $m\to 0$. This procedure carries out the well-known Wigner-Inönü contraction on the group SO(3, 2) to yield the Poincaré group.

Unfortunately, after the Wigner-Inönü contraction has been carried out, the traces of the original Yang-Mills gauge invariance are rather obscure. In the Poincaré limit, the P_a gauge transformations do not affect any of the nonlinearly transforming tensor fields $\bar{\psi}(x)$, since in this case $\bar{h}_1(\zeta,\epsilon)=0$ for all ϵ^a. The only field that a P_a gauge transformation does affect in this limit is the Goldstone field itself. While the resulting transformations in $\{G/H\}$, which in this case is just Minkowski space, do reproduce exactly the transformations in the flat affine tangent spaces of the Einstein-Cartan theory, their relation to a spontaneously broken gauge-theoretic formulation of gravity does not appear to have been recognized in the literature. In the present work, this connection is made naturally using the process of development generated by the differential operator Δ_μ. This process plays a pivotal role in the connection between the gauge-theoretic and geometric aspects of the theory, since on the one hand it uses the original linear SO(3, 2) gauge fields, and on the other, it generates the appropriate geometrical construction for the analysis of the effects of torsion and curvature.

The way in which the image curves of the development process have been generated in this paper is markedly different from that of the straightforward geometrical construction known in differential geometry. Here, the image curves have been generated by a form of parallel transport of the Goldstone field $\zeta^n(x)$ using the SO(3, 2) Yang-Mills gauge fields for calibration. In the usual geometrical construction, no such Goldstone field appears, and development is carried out simply by parallel transport of parametrized tangent vectors. This is first done using the ordinary spin connection in space-time, and then using the spin connection for anti-de Sitter space in the internal space associated to the space-time point toward which the development is being carried out. We have shown the complete equivalence of these two different procedures in Sec. IV. In order to connect the Yang-Mills gauge-theoretic and geometrical aspects of gravity, the presence of the Goldstone field $\zeta^n(x)$, which takes its values in a space upon which the gauge group SO(3, 2) acts transitively, is an essential feature that has not figured in other works on this subject.

We have mentioned in the introduction that the emphasis in previous investigations of the relation between general relativity and Yang-Mills gauge theories has been on the sense in which general coordinate transformations are the extensions to general space-times of the translational gauge transformations of the Poincaré group in flat space-time. In Ref. 4, it was emphasized that in order to recognize this relation, the general coordinate transformations should be taken actively, and should be taken together with specific local Lorentz transformations determined by the spin connection. The net effect of an infinitesimal "gauge" transformation in this sense is to perform an infinitesimal parallel transport using the spin connection.

In the present work, the above relation between infinitesimal parallel transports in space-time and infinitesimal P_a gauge transformations emerges automatically from our discussion of development. Considering, for example, a vector field $\overline{V}^a(x)$, development from $x_0^\mu+\rho^\mu$ to x_0^μ produces an image vector $\overline{V}_*^a(x_0;x_0+\rho)$ lying in $T_{\zeta_*(x_0;x_0+\rho)}(\{G/H\}_{x_0})$. By construction, the result of parallel transporting $\overline{V}^a(x_0+\rho)$ from $x_0^\mu+\rho^\mu$ to x_0^μ across space-time gives the same result as an internal AdS parallel transport of $\overline{V}_*^a(x_0;x_0+\rho)$ from $\zeta_*^n(x_0;x_0+\rho)$ to $\zeta^n(x_0)$. Furthermore, this latter parallel transport is equivalent to a P_a gauge transformation by our discussion in Sec. IV, i.e.,

$$\begin{aligned}\overline{V}^a(x_0)+\rho^\mu\overline{D}_\mu\overline{V}^a(x_0) &=\overline{V}^a(x_0)+\rho^n(\partial_n+\tfrac{1}{2}i\Omega_n{}^{bc}\mathsf{M}_{bc})\overline{V}_*^a(\zeta_*)\\ &=\overline{V}^a(x_0)+\rho^n(\partial_n-il_n{}^b\mathsf{P}_b)\overline{V}_*^a(\zeta_*)\\ &=(1-i\rho^\mu\overline{e}_\mu{}^b\mathsf{P}_b)\overline{V}_*^a(x_0;x_0+\rho)\,, \quad (6.1)\end{aligned}$$

where $\rho^n=\rho^\mu\theta_\mu{}^n$ and the image vector field $\overline{V}_*^a$ is considered to be a function of ζ_*^n in taking the AdS derivative ∂_n.

An infinitesimal "gauge" transformation in the sense of Ref. 4 consists, when applied to a vector field such as $\overline{V}^a(x)$, in making an infinitesimal parallel transport of $\overline{V}^a(x)$ using $\overline{D}_\mu$ along an infinitesimal displacement ρ^μ. Equation (6.1) shows that the transformed value of $\overline{V}^a(x)$ at the point x_0^μ is equal to the result of making a P_a gauge transformation of parameter $\rho^\mu\overline{e}_\mu{}^b(x_0)$ on $\overline{V}_*^a(x_0;x_0+\rho)$. Note that since the transformations in space-time are taken actively, the corresponding P_a gauge transformations in the AdS space must also be taken actively, so the image vector field must be rigidly shifted in the AdS space by the isometric P_a gauge transformations. Accordingly, the transformed value of $\overline{V}^a(x_0)$ is given by the effect of a P_a transformation on $\overline{V}_*^a(x_0;x_0+\rho)$, just as the transformation in space-time parallel transports $\overline{V}^a(x_0+\rho)$ to the point $x_0{}^\mu$.

Of course, the connection between general coordinate transformations and P_a gauge transformations cannot be maintained beyond the first infinitesimal order. This is shown by the occurrence of space-time-dependent functions (i.e., curvature and torsion) in the commutation relations of the "gauge" transformations considered in Ref. 4, or by our discussion of the gravitational holonomy group given in Sec. V. It seems to us that the degree to which one may wish to consider an equivalence between general coordinate transformations and P_a gauge transformations is really just a matter of taste. We have seen in this paper that the proper interpretation of the P_a gauge transformations is in terms of the isometries of the internal spaces, each of which is associated to a particular point in space-time, and in which the Goldstone field corresponding to the spontaneously broken P_a symmetries takes its values.

Another way to view the relation between P^a gauge transformations and general coordinate transformations is to make a gauge choice which correlates the coordinates of the point $\zeta^n(x)$ in the internal space to those of its associated point x^μ in space-time. For example, we can choose $x^\mu=\zeta^n\delta_n{}^\mu$. In order to maintain this condition upon performing a P^a gauge transformation, a compensating general coordinate transformation must be made. However, the commutator algebra of these "compensated P^a gauge transformations" involves derivatives in the "structure constants." Thus we see that the group structure of finite transformations of this kind is just that of the general coordinate group plus the local Lorentz group.

Part of the motivation for undertaking the analysis carried out in this paper came from the heuristic relations between supergravity and gauge theories of the graded Poincaré or de Sitter groups that were given in Refs. 10, 12. The full local geometrical structure of supergravity is still unclear, and we expect that extensions of the present work will play an important part in the final understanding. One unsolved problem of considerable importance is to deduce the auxiliary field structure of the extended supergravity theories, generalizing the known results for pure supergravity.[31] This could be done by explicit constructions using the component fields of supermultiplets, or by an extension of one of the superspace approaches. A geometrically motivated approach to a first-order formulation of SO(2) extended supergravity in superspace has recently been given in Ref. 13.

An interesting connection between the present work and the constrained superspace approach of Wess and Zumino[32] is suggested by a reformulation of their constraints in terms of the graded Poincaré curvatures. Supergravity is formulated in a four-Bose plus four-Fermi dimensional curved superspace, which is modelled after the flat coset space resulting from a spontaneous breakdown of graded Poincaré to Lorentz symmetry. We shall not pursue the details of this generalization of the geometry discussed in the present paper, but merely note that the graded Poincaré group defined on the (4+4)-dimensional superspace has curvatures associated with rotations and translations which generalize the curvatures of the Poincaré group in that they contain terms bilinear in the supersymmetric gauge fields. In addition, there is also a curvature associated with the generators of supersymmetry transformations.

The superspace constraints of Ref. 32 involve the components of the superspace torsion, and read

$$T_{\alpha\dot\beta}{}^{c}=2i(\sigma^{c})_{\alpha\dot\beta}\,, \tag{6.2a}$$

$$T_{\dot\alpha\dot\beta}{}^{c}=T_{\alpha\beta}{}^{c}=T_{\underline{\alpha}a}{}^{c}=T_{ab}{}^{c}=0\,, \tag{6.2b}$$

$$T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}}=0\,, \tag{6.3}$$

where the notation of Ref. 32 is used, except for the underlining of the spinorial indices, which indicates that the indices shown can correspond to either dotted or undotted two-component spinorial indices. Using the graded Poincaré curvatures in superspace, Eqs. (6.2) may be reformulated[33] in terms of the translational graded Poincaré curvature as

$$R_{AB}{}^{a}=0\,, \tag{6.4}$$

where A and B are superspace "world" indices, and the transition from internal indices in (6.2) to the "world" indices has been effected using the superspace vielbein field. In order to obtain the expression (6.4), the components of the vielbein field must be identified with the nonlinear fields obtained from the graded Poincaré gauge fields by a redefinition which generalizes (3.18). Thus $E_A{}^a$ must be identified with $\bar{e}_A{}^a$ and $E_A{}^\alpha$ with $\bar{\psi}_A{}^{\underline{\alpha}}$ (where the superscript bar on $\bar{\psi}_A{}^{\underline{\alpha}}$ is meant to indicate a nonlinearly transforming field). Using the spinorial graded Poincaré curvature and the inverse vielbein field, the constraints (6.3) may be reformulated as

$$E_{\underline{\alpha}}{}^{A}E_{\underline{\beta}}{}^{B}R_{AB}{}^{\underline{\gamma}}=0\,. \tag{6.5}$$

The particularly simple form (6.4), which includes both (6.2a) and (6.2b) in one formula, suggests that an understanding of the role of the graded Poincaré symmetry in pure supergravity may considerably facilitate the superspace analysis of the extended supergravity theories. An understanding of the constraint structure for these extended theories would essentially solve the problem of how to express them in a covariant locally supersymmetric formulation.

In this paper, we have been primarily concerned with the relation between the gauge-theoretic and geometrical aspects of gravity. The formulation of gravity as a spontaneously broken gauge theory of the de Sitter group may have interesting implications for gravitational dynamics as well. The formulation (2.5) of the gravitational action, with $y^A y_A$ constrained to take the value $-m^{-2}$, is sufficient to motivate the anlysis of the geometrical role of the Goldstone field as we have done. Moreover, (2.5) generates precisely the field equations of general relativity with a cosmological constant. At a dynamical level, however, the action (2.5) is suggestive of a generalized unconstrained theory with a Higgs field and a potential that produces spontaneous symmetry breaking in the normal fashion.

It is a simple matter to replace the constraint in (2.5) by a Higgs field and an appropriate potential for the symmetry breaking. In order to construct an action for the Higgs field, we need to have an SO(3,2)-invariant object that generalizes the "metric" that we wrote down for the constrained theory in (2.13). The appropriate object is

$$g_{\mu\nu}\equiv(m^2y^Ay_A)^{-1}[(y^By_B)^{-1}(y^C\nabla_\mu y_C)(y^D\nabla_\nu y_D)-\nabla_\mu y^E\nabla_\nu y_E]\,, \tag{6.6}$$

where the quantity m^{-1} is now the vacuum expectation value of the Higgs field. An action for the Higgs field is then

$$I_H=k^{-4}\int d^4x\sqrt{-g}\,[\tfrac{1}{2}(y^Ay_A)^{-1}(y^B\nabla_\mu y_B)(y^C\nabla_\nu y_C)g^{\mu\nu}-V(y^Dy_D)]\,, \tag{6.7}$$

where k is a coupling constant with the dimensions of length and $V(y^A y_A)$ is a potential that produces the desired strength of symmetry breaking. Together with (6.7), we take the gravitational action

$$I_G=(m\kappa^2)^{-1}\int d^4x\,[y^A\epsilon_{ABCDE}R_{\mu\nu}{}^{BC}R_{\rho\tau}{}^{DE}\epsilon^{\mu\nu\rho\tau}]\,, \tag{6.8}$$

where κ is the usual gravitational coupling constant.

Since the SO(3, 2) symmetry is spontaneously broken down to SO(3,1) by the potential in (6.7), it is appropriate to make the field redefinition

$$y^A(x)=\quad \sigma_5(e^{-i\zeta^n(x)P_n})\begin{pmatrix}0\\0\\0\\0\\ \overline{\phi}(x)\end{pmatrix}\quad , \tag{6.9}$$

where P_n is defined by scaling M_{n5} by m as before in (3.3). The Goldstone field $\zeta^n(x)$ then enters into the definition of the vierbein and spin connection exactly as given before in (3.18). Upon passage to these nonlinearly transforming fields, the value of the metric $g_{\mu\nu}$ given in (6.6) does not change, since it is an SO(3, 2) invariant. In terms of the nonlinear fields, the functional form of $g_{\mu\nu}$ is just

$$g_{\mu\nu}=\overline{e}_\mu{}^a\overline{e}_{\nu a}\,. \tag{6.10}$$

Thus, the Higgs action reduces to the familiar expression

$$I_H=k^{-4}\int d^4x\sqrt{-g}\,[-\tfrac{1}{2}\partial_\mu\overline{\phi}\partial_\nu\overline{\phi}g^{\mu\nu}-V(-\overline{\phi}^2)]\,. \tag{6.11}$$

Upon shifting the Higgs scalar field $\overline{\Phi}$ by its vacuum expectation value,

$$\overline{\phi}=\overline{\phi}'+m^{-1}\,, \tag{6.12}$$

we obtain a theory for a self-interacting scalar field $\overline{\Phi}'(x)$ coupled to gravity, both in the usual way through the occurrence of $g_{\mu\nu}$ in (6.11) and through a nonminimal coupling term arising from the gravitational action (6.8). Besides this nonminimal coupling term, (6.8) produces a Gauss-Bonnet term, a scalar curvature term, and cosmological constant term exactly as before in (2.5). In addition, there may be a contribution to the cosmological constant from (6.7).

The theory described by (6.7) and (6.8) is a scalar-tensor theory of gravity with a cosmological constant. It is reminiscent of the Brans-Dicke theory, with which it shares the property of limiting to the pure tensor theory of general relativity as the scalar coupling constant k tends to zero. Generally, the mass of the Higgs field will be of order m, and since this value determines the scale of the cosmological constant, the Higgs field will be essentially massless, although the exact value of its mass obviously depends upon the particular potential chosen. Note that the field $\overline{\Phi}'$ does not have the customary dimensions; in order to correct for this and normalize the kinetic term in (6.11), a factor of k^{-2} should be absorbed into $\overline{\Phi}'$. The coupling constant k then appears in all the interaction terms involving the $\overline{\Phi}'$ field.

Although the theory described here limits to general relativity as $k\to 0$, there is an interesting difference in the theory's dynamics that shows up in strong-field regions where the Higgs field $\overline{\Phi}'$ is not constant. Varying $\overline{\omega}_{\mu ab}$ in (6.8) and using the theory's Bianchi identities, we have

$$2m(1+m\overline{\phi}')\overline{e}_\mu{}^c\epsilon_{abcd}\epsilon^{\mu\nu\rho\tau}\overline{R}_{\rho\tau}{}^d=\partial_\mu\overline{\phi}'\epsilon_{abcd}\epsilon^{\mu\nu\rho\tau}\overline{R}_{\rho\tau}{}^{cd}\,. \tag{6.13}$$

Instead of having the torsion $\overline{R}_{\rho\tau}{}^d$ vanish in the absence of spinning matter as in the Einstein-Cartan theory, (6.13) shows that there will be torsion in regions where $\overline{\Phi}'$ is not constant and the de Sitter curvature $\overline{R}_{\rho\tau}{}^{cd}$ is nonzero.

The fact that torsion need not be zero when gravity is generalized to include the Higgs field may be of particular interest in connection with solutions to the theory that have nontrivial topology. Indeed, we have emphasized that if the Einstein-Cartan theory is to be formulated as a gauge theory of the de Sitter group, a very particular global structure must be chosen for the SO(3, 2) fields in relation to the tangent bundle of the space-time manifold, permitting the construction of a solder form. Local physics does not reflect the necessity of such a particular global structure, however, so a natural generalization of the Einstein-Cartan theory is obtained simply by relaxing the requirements on the global structure for the SO(3, 2) fields. For example, we could retain the requirement that a cross section of the bundle of AdS spaces may be chosen, but give up the requirement that the SO(3, 2) bundle be reduced to the tangent bundle of the space-time manifold.

A potentially more interesting possibility than simply abandoning the reduction to the tangent bundle of space-time would be to abandon also the necessity of being able to construct a cross section in the bundle of AdS spaces. In this case, inclusion of the Higgs field would be necessary, for the obstruction to constructing a cross section of AdS spaces would force the local representative field $y^A(x)$ to pass through zero on some submanifold of space-time. The AdS spaces have the topology[34] $S^1 \times R^3$, and it is the S^1 part that will permit an obstruction to be set up on suitably chosen space-times. The classification of a bundle of circles over a four-dimensional space-time manifold is given by the bundle's first Chern class.[35] For example, the maximally extended Schwarzschild/Kruskal space-time has topology[34] $S^2 \times R^2$, and for this topology the bundle of circles is classified by the integers.[35] For any such nontrivial bundle, the local representative field $y^A(x)$ would have to pass through zero on some submanifold of space-time.

As we have seen in (6.13), there will be torsion in regions where the $\overline{\Phi}$ field is nonconstant. Since this field gives the radius of the local internal AdS spaces, it will vary near the places where $y^A(x)$ must pass through zero. Consequently, we may expect to find solutions to the field equations derived from (6.7) and (6.8) that have torsion in localized regions due to such topological effects. There are many interesting questions concerning the physical effects of such topologically nontrivial solutions, such as the effect on the singularity structure of black holes or of cosmological models, or whether some of the analogies between gravitation and superconductivity suggested recently by Hanson and Regge[36] may be realized in this way.

Work on the questions mentioned in this section is in progress and will be reported elsewhere.

Note added in proof. A brief outline of the results of this paper may be found in K. S. Stelle and P. C. West, J. Phys. A: Math. Nucl. Gen. 12, L205 (1979).

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APPENDIX

In this appendix, we discuss how to derive some of the formulas given in the text, in particular the expressions for $\overline{e}_\mu{}^a$, $\overline{\omega}_\mu{}^{ab}$, and $\tilde{h}_1(\zeta, \epsilon)$. We use the techniques of Ref. 26, which we summarize here for convenience.

For any two quantities X and Y involving the SO(3,2) Lie algebra generators, define

$$X \wedge Y = [X, Y]\,,\qquad X^2 \wedge Y = [X, [X, Y]]\,,\ \text{etc.}. \tag{A1}$$

An expression $f(X) \wedge Y$ is defined by expanding $f(X)$ into a power series in X and then using the above formulas. In particular, we have

$$e^X Y e^{-X} = e^X \wedge Y \tag{A2}$$

and

$$e^X \delta e^{-X} = \frac{(1 - e^X)}{X} \wedge \delta X\,. \tag{A3}$$

Also, the equation

$$f(X) \wedge Y = Z \tag{A4}$$

can be solved in the form

$$Y = [f(X)]^{-1} \wedge Y\,. \tag{A5}$$

When written in the above notation, Eq. (3.9) becomes, with $g_0 = 1 - i\epsilon^a P_a$,

$$e^{i\epsilon^a P_a} \wedge (-i\epsilon^b P_b) - \frac{(1 - e^{i\zeta^a P_a})}{i\zeta^b P_b} \wedge (i\delta\zeta^c P_c) = h_1(g_0) - 1\,. \tag{A6}$$

Evaluating the part of the left-hand side of (A6) that is proportional to the P_a generators allows the evaluation of $\delta\zeta^a$, with the result given in (3.10). The calculation is easily carried out using (A5) and the identity

$$(i\zeta^a P_a)^{2n} \wedge \epsilon^b P_b = (m^2\zeta^2)^n \left(\epsilon^a P_a - \frac{\zeta^a \epsilon_a \zeta^b}{\zeta^2} P_b\right), \tag{A7}$$

which follows from the commutation relations (3.4) and (3.5). Evaluating the part of the left-hand side of (A6) that is proportional to the generators M_{ab} yields the expression for $\tilde{h}_1(\zeta, \epsilon) = h_1 - 1$ given in (3.17).

The nonlinear fields $\overline{e}_\mu{}^a(x)$ and $\overline{\omega}_\mu{}^{ab}(x)$ are evaluated from their definition in Eq. (3.18), which can be written

$$\tfrac{1}{2} i \overline{\omega}_\mu{}^{ab}(x) M_{ab} - i\overline{e}_\mu{}^a(x) P_a = \frac{(1 - e^{i\zeta^a P_a})}{i\zeta^b P_b} \wedge \partial_\mu (i\zeta^c P_c) + e^{i\zeta^c P_c} \wedge (-i e_\mu{}^a P_a) + e^{i\zeta^c P_c} \wedge (\tfrac{1}{2} i \omega_\mu{}^{ab} M_{ab})\,. \tag{A8}$$

The expressions for $\overline{e}_\mu{}^a$ and $\overline{\omega}_\mu{}^{ab}$ given in Eqs. (3.19) and (3.20) are easily found from (A8) using (A7) and the identity

$$[(-i\zeta^c P_c)^{2n+1}, \tfrac{1}{2} i M_{ab}] = -\tfrac{1}{2} i (m^2\zeta^2)^n (\zeta_a P_b - \zeta_b P_a)\,, \tag{A9}$$

which also follows from (3.4) and (3.5).

As explained in the text, the vierbein and spin connection of the internal anti-de Sitter space may be found from the expressions for $\overline{e}_\mu{}^a$ and $\overline{\omega}_\mu{}^{ab}$ given in (3.19) and (3.20) by setting $e_\mu{}^a = \omega_\mu{}^{ab} = 0$.

The resulting AdS vierbein field is given in (4.14). It can be used to calculate the metric S_{nm} of AdS space, with the result

$$S_{nm}(\zeta)=l_n{}^a\eta_{ab}l_m{}^b=\left(\eta_{nm}-\frac{\zeta_n\zeta_m}{\zeta^2}\right)\frac{\sinh^2 z}{z^2}+\frac{\zeta_n\zeta_m}{\zeta^2}. \qquad \text{(A10)}$$

Alternatively, we may calculate $S_{nm}(\zeta)$ by using the embedding equation for the AdS space into the pseudo-Euclidean five-dimensional space,

$$y^A y_A=-m^{-2}. \qquad \text{(A11)}$$

The metric of the AdS space in the coordinates y^a is found by using (A11) to eliminate y^5, giving

$$S_{ab}(y)=\eta_{ab}-\frac{y_a y_b}{(y^c y_c+m^{-2})}\ . \qquad \text{(A12)}$$

Then, the metric $S_{nm}(\zeta)$ in ζ^n coordinates is given by

$$S_{nm}(\zeta)=\frac{\partial y^a}{\partial\zeta^n}\frac{\partial y^b}{\partial\zeta^m}S_{ab}(y)\,. \qquad \text{(A13)}$$

The result obtained in this way from (A13) agrees with (A10).

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