Semiclassical relativity: The weak-field limit

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Semiclassical relativity is a theory in which quantum matter fields interact with a classical gravitational field via the semiclassical Einstein equation $G_{ab} = 8\pi\langle T_{ab}\rangle$. We consider, for the case of massless quantum fields, the weak-field limit of this theory. It is shown that the linearized quantum stress energy can be determined in a manner which is essentially independent of the details of any regularization prescription. Using this expression for the quantum stress energy, we solve the linearized semiclassical Einstein equation. The solutions found include (1) perturbations that satisfy the linearized classical Einstein equation $\dot{G}_{ab} = 0$, (2) perturbations that grow exponentially in time, and (3) perturbations that (in some sense) travel faster than the speed of light.

I. INTRODUCTION

General relativity is a classical theory of gravity. It incorporates neither the quantum nature of matter nor the quantum nature of the gravitational interaction. Direct attempts to extend general relativity to a full quantum theory of gravity have encountered difficulties.[1] For this reason, some interest has focused on a semiclassical theory in which gravity is still treated classically (according to general relativity) but matter is now treated quantum mechanically (according to quantum field theory). One hopes that in certain regimes this theory will be a valid approximation to a full quantum theory of gravity, as well as perhaps yield some insight into the full quantum theory.

Semiclassical relativity can be divided into two parts. The first consists of describing the effect of curved spacetime on quantum fields, and the second of describing the effect of quantum fields on curved spacetime. The first part of this theory is now fairly well understood.[2] The most important new feature which arises is that gravitational fields can create particles. In particular, Hawking[3] has shown that a black hole creates particles with an exactly thermal spectrum. This prediction is important not just for its possible astrophysical consequences,[4] but also for the deep links with thermodynamics which seem to be implied.[5] Particle creation in cosmological models has also been investigated.[6] The latter part of semiclassical relativity describes the back reaction of these created particles on the gravitational field. Unfortunately, relatively little is known about the physical implications of this part of the theory. One expects statements of the form: "Black holes decrease in size as they radiate particles" and, perhaps, "anisotropies and inhomogeneities are damped out by particle creation in the early universe." However, difficulties of (among other things) computing the stress energy of a quantum field in a curved spacetime have hindered predictions of this type (although preliminary results have been obtained[7]).

In this paper we consider the back reaction of quantum fields on a classical gravitational field. In order to get more insight into the nature of this back reaction, we investigate semiclassical relativity (with massless quantum fields) in the limit of *weak* gravitational fields. In this limit the back reaction is described by perturbations off Minkowski spacetime whose linearized Einstein tensor is equal to the linearized created stress energy of the quantum field (i.e., perturbations satisfying the linearized semiclassical Einstein equation).

At first thought it might appear that this weak-field limit of semiclassical relativity is equivalent to the weak-field limit of classical relativity: Since the stress energy is quadratic in the matter field, and a linearized gravitational perturbation creates a first-order change in that field, the change in the stress energy would seem to be second order. Indeed, the stress energy of a classical field coupled to gravity *does* vanish to first order. However, it turns out that there *is* a first-order contribution to the quantum stress energy which can be thought of as resulting from the cross term (matrix element) between the perturbed field and unperturbed field configurations. Thus, although the weak-field limit of semiclassical relativity does not include the energy density of the created particles, it nevertheless includes *some* effects of particle creation (as well as first-order "vacuum polarization" effects).

Unfortunately, it turns out that for a given gravitational perturbation, there is a family of mathematical candidates to represent the physical created stress energy of a massless quantum field. There does not appear to be any way at present to select one member of this family as the "correct" one to include in the back-reaction calculations. Thus, we are forced to consider the back reaction determined by each member of this family.

In order to solve the linearized semiclassical Einstein equation, one needs an expression for a first-order quantum stress energy (selected from the above family) in terms of the gravitational perturbation. Such an expression was obtained by Capper *et al.*[12] using dimensional regularization. Here, we derive an equivalent result using Wald's axiomatic approach.[14] This approach has the advantage of being less dependent on ambiguities in the regularization procedure, and also yields a convenient expression for the stress energy in terms of an integral over the gravitational perturbation. This expression is sufficiently simple that the back-reaction problem can be solved quite easily.

We find three main classes of solutions to the linearized semiclassical Einstein equation. The first class consists of all solutions to the linearized classical Einstein equation. These solutions can be interpreted as "weak gravitational waves traveling through spacetime which do not interact with the quantum field." The second class of solutions consists of perturbations which grow exponentially in time. These solutions might be interpreted as "gravitational fields which create particles, which in turn create more gravitational fields, etc." The existence of this second class of solutions is perhaps surprising. It indicates that flat spacetime is unstable in semiclassical relativity. The third class of solutions (which only occur for certain choices of the stress energy) is even more surprising. In consists of linear superpositions of plane waves traveling in *spacelike* directions. These solutions represent "coupled gravitational-quantum field disturbances traveling faster than the speed of light." Since these solutions are presumably unphysical, we may conclude that the correct candidate for the quantum stress energy must be one which does not admit solutions of this kind.

Finally, we consider the weak-field limit of semiclassical relativity in the presense of a classical stress-energy source. As an example, we take the classical stress energy to represent that of our sun, and estimate the first-order quantum corrections to the bending of light (i.e., the corrections due to the gravitational effects of the virtual particle-antiparticle pairs in the space around the sun).

The organization of this paper is as follows. In Sec. II we review some of the mathematical formalism of semiclassical relativity. In Sec. III, the weak-field limit is considered and our expression for the first-order quantum stress energy is derived. Finally, in Sec. IV, we solve the linearized semiclassical Einstein equation. Some technical issues concerning distributions in general, and our Green's function in particular, are discussed in the Appendix.

II. SEMICLASSICAL RELATIVITY

Let M, g_{ab} be a spacetime, ϕ be a quantum field[8] on that spacetime (which is an operator on a free Fock space of states $\mathcal{F}$), and ξ be a state in the Fock space $\mathcal{F}$. In semiclassical relativity, a physical system is described by a collection (M, g_{ab}, ϕ, ξ) which satisfies the semiclassical Einstein equation

$$G_{ab} = 8\pi \langle \xi | T_{ab} | \xi \rangle , \tag{1}$$

where T_{ab} denotes a "stress-energy operator of the quantum field ϕ." Roughly speaking, one can interpret the semiclassical Einstein equation as follows. One starts with a quantum matter field in state ξ and a classical gravitational field in some configuration (determined, say, by initial data). One now evolves this system by requiring that at each point the classical Einstein tensor be equal to the average value of the quantum stress-energy operator.

Unfortunately, it turns out that there is no obvious expression for the stress energy of a quantum field in an arbitrary curved spacetime. We now review this difficulty and discuss a possible resolution.

Recall that the stress energy of a classical field is quadratic in the field, e.g., for a massless scalar field:

$$T_{ab} = \nabla_a \psi \nabla_b \psi - \tfrac{1}{2} g_{ab} \nabla_m \psi \nabla^m \psi . \tag{2}$$

As is well known, in quantum field theory, ψ becomes an operator-valued distribution. The problem in defining a quantum stress energy is that if one naively substitutes the quantum field into the formula for T_{ab} [e.g., Eq. (2)], one obtains an expression involving the product of distributions, which is not defined. Furthermore, in order for the semiclassical Einstein equation to make sense, each expectation value of the quantum stress-energy operator must be a smooth tensor field (not an arbitrary distribution) on spacetime. Thus there appears to be no natural quantum analog of the classical stress-energy tensor.

The standard procedure which has been adopted for dealing with this problem is the following. One formally forces the quantum field operator to be defined at each point of spacetime, and substitutes into the classical expression for the stress energy. The result is an "operator" which, however, has in-finite expectation values. One now removes the singular part, a process which is called regularization. In flat spacetime there is a natural regularization prescription: normal ordering. Physically, this prescription corresponds to adjusting the vacuum expectation value of T_{ab} to be zero. In an arbitrary curved spacetime, this prescription is no longer available since there is no unique vacuum state of the system.

In recent years several regularization prescriptions for any curved spacetime have been proposed, e.g., point-separation,[9,10] dimensional regularization,[11,12] and zeta-function regularization.[13] The question naturally arises as to which regularization prescription should be used to obtain the physical stress energy of a quantum field in curved spacetime. This question has, to some extent, been answered by Wald.[14] In short, his answer consists of first specifying a list of axioms which any reasonable candidate for the stress energy must satisfy, and then displaying a class of prescriptions which yields all stress-energy operators that satisfy these axioms. We now discuss this result in a bit more detail.

Consider the following axioms on a stress-energy operator T_{ab}:

(1) Matrix elements between orthogonal states are given by the formal expression (which now gives finite, unambiguous results).

(2) If the spacetime is flat (with standard R^4 topology), then the vacuum expectation value is zero.

(3) Expectation values are conserved: $\nabla_a\langle T^{ab}\rangle = 0$.

(4) Expectation values are causal: For a fixed in-state $\langle T_{ab}(p)\rangle$ depends only on the geometry to the past of p. Similarly, for a fixed out-state $\langle T_{ab}(p)\rangle$ depends only on the geometry to the future of p.

Notice the absence of an energy condition for $\langle T_{ab}\rangle$. Indeed, the existence of particle creation from the vacuum suggests that $\langle T_{ab}\rangle$ cannot satisfy an energy condition.[15] *A priori*, it is not obvious that there exists any stress-energy operator that satisfies these axioms, i.e., the axioms might be inconsistent. However, Wald has shown[16] that for a large class of spacetimes, there indeed exists a stress-energy operator T_{ab} satisfying these axioms. Furthermore, it is easy to show that the class of all stress-energy operators that satisfy these axioms consists precisely of those that can be obtained by adding a conserved local curvature tensor[17] times the identity operator to T_{ab}. [Adding a multiple of the spacetime metric violates axiom (2) and hence is excluded.] Thus the axioms do not uniquely determine a quantum stress-energy operator, but given any one, the remaining freedom is well understood. Unfortunately, this remaining freedom is quite large. We now impose the additional condition that the quantum stress energy cannot depend on sixth-order derivatives (or higher) of the metric. The motivation for this condition is twofold. On the one hand, we desire *some* additional condition to make the theory of back reaction more manageable, and on the other, it is known[18] that this condition is the greatest restriction on the order of derivatives of the metric which is consistent with the axioms.[19]

We now return to the semiclassical Einstein equation [Eq. (1)]. In principle, to answer any question regarding the interaction of quantum fields and gravity, one would simply solve this equation (with some choice of the stress-energy operator) for the situation of interest, and study the resulting solution. In practice, however, it has proved difficult to proceed in this manner. The regularization prescription which relates a given stress-energy operator to the curvature of spacetime turns out to be very complicated, making solutions to Eq. (1) difficult to find. In fact, I am aware of only two classes of solutions to the semiclassical Einstein equation (for any choice of stress-energy operator). The first class consists of simply Minkowski spacetime, any quantum field, and the in-vacuum state of the quantum field, i.e., $(M,\eta_{ab},\phi,0)$. This solution can be viewed as the vacuum solution of semiclassical relativity. Note that for any nonvacuum in-state ξ, (M,η_{ab},ϕ,ξ) is *not* a solution. (The expectation value in-state ξ of the normal-ordered stress energy is nonzero.) The second class of solutions to the semiclassical Einstein equation consists of a Robertson-Walker spacetime, a conformally invariant quantum field, and the in-vacuum state of the field: $(M,a^2(t)\eta_{ab},\phi,0)$. Substituting this form of a solution into Eq. (1) one obtains a fourth-order ordinary differential equation on the conformal factor $a(t)$ which can be solved, e.g., numerically.

We now linearize the semiclassical Einstein equation about its vacuum solution,

$$\dot{G}_{ab}=8\pi\langle 0|\dot{T}_{ab}|0\rangle, \qquad (3)$$

where a dot over a tensor denotes the corresponding linearized tensor. What is the freedom in the right-hand side of this equation? We have seen that if T^1_{ab} and T^2_{ab} are two stress-energy operators satisfying Wald's axioms and our additional condition, then $\langle 0|T^1_{ab}|0\rangle-\langle 0|T^2_{ab}|0\rangle$ is a conserved local curvature tensor with less than six derivatives of the metric. Therefore $\langle 0|\dot{T}^1_{ab}|0\rangle-\langle 0|\dot{T}^2_{ab}|0\rangle$ must be a linearized curvature tensor with these properties. But the *only* linearized curvature tensors with these properties are[20] the linearized Einstein tensor $\dot{G}_{ab}$ and the linearized form of

$$A_{ab} \equiv \frac{\delta}{\delta g^{ab}} \int C_{mnrs} C^{mnrs} \epsilon_{cdef}\,, \tag{4}$$

$$B_{ab} \equiv \frac{\delta}{\delta g^{ab}} \int R^2 \epsilon_{mnrs}\,, \tag{5}$$

that is,[21]

$$\dot{A}_{ab} = -2\nabla^2 \dot{G}_{ab} - \tfrac{2}{3}\nabla_a \nabla_b \dot{G}^m{}_m + \tfrac{2}{3}\eta_{ab}\nabla^2 \dot{G}^m{}_m\,, \tag{6}$$

$$\dot{B}_{ab} = 2\eta_{ab}\nabla^2 \dot{G}^m{}_m - 2\nabla_a \nabla_b \dot{G}^m{}_m\,. \tag{7}$$

Thus there exists a three-parameter family of candidates for the linearized stress-energy operator of a quantum field. However, we see from Eq. (3) that the change in the theory produced by adding a multiple of $\dot{G}_{ab}$ to the stress energy is of a trivial nature (essentially, one is just changing the value of the gravitational coupling constant). Therefore, we shall concentrate on a two-parameter family of stress-energy tensors that differ by multiples of $\dot{A}_{ab}$ and $\dot{B}_{ab}$.

III. LINEARIZED QUANTUM STRESS ENERGY

In this section we derive an expression for the linearized quantum stress energy in terms of the gravitational perturbation. (This expression is equivalent to one previously obtained by Capper *et al.*[12]) Our approach consists of introducing a Green's function and expressing[22] $\langle \dot{T}_{ab}\rangle$ in terms of an integral over the gravitational perturbation. Surprisingly, it turns out that the appropriate Green's function can be essentially determined from its general properties. Notice that we are using a Green's function in a manner which is different from most other applications in physics. Typically, Green's functions are employed to solve a given linear differential equation. Although symmetry properties (e.g., Poincaré invariance) are often invoked to help determine the Green's function, the essential ingredient in its determination is that it satisfy the given differential equation. By contrast, the Green's function for the linearized quantum stress energy will be determined without the use of a differential equation relating $\langle \dot{T}_{ab}\rangle$ to γ_{ab}. We now proceed to carry out this analysis.

Let M, η_{ab} be Minkowski spacetime and let γ_{ab} be an arbitrary linear perturbation which has compact support.[23] Consider a massless quantum field on the perturbed spacetime and let $\langle \dot{T}_{ab}\rangle$ denote one of the possible candidates for the in-vacuum expectation value of the first-order stress energy of this field. Clearly, $\langle \dot{T}_{ab}\rangle$ depends linearly on γ_{ab}. If we make the reasonable assumption that $\langle \dot{T}_{ab}\rangle$ depends continuously (in a suitable sense) on γ_{ab} as well, then we conclude that there must exist a tensor distribution $H_{ab}{}^{m'n'}(x,x')$ (see Appendix) such that

$$\langle \dot{T}_{ab}(x)\rangle = \hbar \int_M H_{ab}{}^{m'n'}(x,x')\gamma_{m'n'}(x')\,, \tag{8}$$

where the indices ab denote tensors at the point x and $m'n'$ denote tensors at the point x'. Notice that the distribution $H_{ab}{}^{m'n'}$ *is* our desired Green's function, i.e., it yields the first-order quantum stress energy in terms of the gravitational perturbation.

We claim that $H_{ab}{}^{m'n'}(x,x')$ satisfies the following five properties:

(1) Poincaré invariance (i.e., invariant under any Poincaré transformation of both x and x' simultaneously);

(2) symmetry under interchange of a and b, and similarly of m' and n';

(3) vanishing divergence on any index;

(4) support on the past light cone of x [i.e., for fixed x, $H_{ab}{}^{m'n'}(x,x')=0$ for x' off the past light cone of x];

(5) dimension cm^{-8}.

The first property is immediate ($H_{ab}{}^{m'n'}$ is independent of the perturbation γ_{ab}). Symmetry under interchange of a and b, and m' and n', reflects the symmetry of $\langle \dot{T}_{ab}\rangle$ and γ_{ab}, respectively. Vanishing divergence on the index a or b follows from conservation of $\langle \dot{T}_{ab}\rangle$. Vanishing divergence on the index m' or n' follows from gauge invariance of $\langle \dot{T}_{ab}\rangle$: Consider a perturbation which is pure gauge, i.e., $\gamma_{m'n'} = \nabla_{(m'}\xi_{n')}$ for some vector field $\xi_{n'}$ of compact support. Then

$$0 = \int_M H_{ab}{}^{m'n'}\nabla_{(m'}\xi_{n')} = -\int (\nabla_{m'} H_{ab}{}^{m'n'})\xi_{n'}\,. \tag{9}$$

Since this must be true for all $\xi_{n'}$, $H_{ab}{}^{m'n'}$ must have vanishing divergence on m' (and similarly n'). For the fourth property, fix x and consider a perturbation whose support does not intersect the past light cone of x. Since the retarded Green's function for a massless field at x has support on this past light cone, we see that the field operator at x will be the flat spacetime field operator, and hence the stress energy must be the flat spacetime stress energy, i.e., it must vanish. The final property follows from noting that the stress energy has dimensions cm^{-2}, $\hbar$ has dimensions cm^2, the integral has dimensions cm^4, and the perturbation is dimensionless.

Recall that $\langle \dot{T}_{ab}\rangle$ in Eq. (8) was chosen arbitrarily from the family of candidates for the quantum stress energy. Therefore, *any* stress energy in this family must be expressible in terms of a distribution $H_{ab}{}^{m'n'}$ which satisfies the above five properties. We now determine all such distributions. The approach will be to use properties (1), (2), and (3) to convert $H_{ab}{}^{m'n'}$ into a *scalar* Green's function, and then use properties (1), (4) and (5) to determine the allowed class of scalar Green's functions.

As shown in the Appendix, translation invariance alone implies that there exists a distribution H_{abcd} such that

$$\int_M H_{ab}{}^{m'n'}(x,x')\gamma_{m'n'}(x') = \int_M H_{abcd}(x-x')\gamma^{cd}(x') \qquad (10)$$

for any smooth γ_{ab} of compact support. We now fix an origin in Minkowski spacetime and take the Fourier transform of Eq. (8), obtaining

$$\langle \hat{T}_{ab}(k)\rangle = \hbar \hat{H}_{abcd}(k)\hat{\gamma}^{cd}(k)\,, \qquad (11)$$

where the caret denotes the Fourier transform. Since $H_{ab}{}^{m'n'}(x,x')$ was Poincaré invariant, $H_{abcd}(x-x')$ is Lorentz invariant, and therefore $\hat{H}_{abcd}(k)$ must be Lorentz invariant as well, i.e., invariant under Lorentz transformations in momentum space. Consequently, $\hat{H}_{abcd}(k)$ can be expressed (see Appendix) as a sum of terms, each consisting of a Lorentz-invariant scalar distribution times a tensor product of the position vector field in momentum space k^a, and the metric η_{ab}. Using this fact, together with property (2), i.e., $\hat{H}_{abcd} = \hat{H}_{(ab)(cd)}$, we conclude that

$$\hat{H}_{abcd} = (\tfrac{2}{3}g_1 + 2g_2)k_ak_bk_ck_d + h_1k^2k_ak_b\eta_{cd} + h_2k^2k_{(a}\eta_{b)(c}k_{d)} + h_3k^2\eta_{ab}k_ck_d + h_4k^4\eta_{ab}\eta_{cd} + g_1k^4\eta_{a(c}\eta_{d)b}\,, \qquad (12)$$

where $g_1, g_2, h_1, \ldots, h_4$ are distributions which have yet to be determined. The divergence-free property of $H_{ab}{}^{m'n'}$ translates into the condition that $\hat{H}_{abcd}$ be orthogonal to k^a and k^c. Imposing this condition in Eq. (12) we obtain four linearly independent equations which can be used to express the h_i's in terms of the g_i's. We obtain

$$\hat{H}_{abcd} = g_1[\tfrac{2}{3}k_ak_bk_ck_d + \tfrac{1}{3}k^2(\eta_{ab}k_ck_d + k_ak_b\eta_{cd}) - 2k^2k_{(a}\eta_{b)(c}k_{d)} - \tfrac{1}{3}k^4\eta_{ab}\eta_{cd} + k^4\eta_{a(c}\eta_{d)b}] + g_2[2k_ak_bk_ck_d - 2k^2(\eta_{ab}k_ck_d + k_ak_b\eta_{cd}) + 2k^4\eta_{ab}\eta_{cd}]\,. \qquad (13)$$

Substituting back into Eq. (11) we notice a surprising simplification. The two tensors multiplying g_1 and g_2 are precisely the Fourier transform of the linearized conserved local curvature tensors $\dot{A}_{ab}$ and $\dot{B}_{ab}$ given by Eqs. (6) and (7).[24] Therefore, taking the inverse Fourier transform we obtain[25]

$$\langle \dot{T}_{ab}(x)\rangle = \hbar\int_M G_1(x-x')\dot{A}_{ab}(x') + \hbar\int_M G_2(x-x')\dot{B}_{ab}(x')\,, \qquad (14)$$

where G_1 and G_2 (the Fourier transforms of g_1 and g_2, respectively) have yet to be determined.

We have thus replaced the original tensor distribution by two scalar distributions G_1 and G_2. We must now determine the class of scalar distributions which satisfy the remaining properties. These properties are Lorentz invariance (1), support on the past light cone of x(4), and dimensions cm^{-4}(5). To begin, fix a point x in Minkowski spacetime. Define a scalar field σ on M by setting for each point p, $\sigma(p)$ equal to one-half the square of the geodesic distance from x to p. Consider the distribution H defined by its action on a test function f (i.e., C^∞ function of compact support) as follows. For each real number $\alpha \le 0$, let $F(\alpha)$ denote the value of the integral of f over the past hyperbola consisting of all points satisfying $\sigma = \alpha$.[26] Thus F is a real-valued function defined on $(-\infty, 0]$. Now define

$$\int_M H(x-x')f(x') \equiv \lim_{\alpha\to 0^-}[F'(\alpha) + 2\pi\ln(-\alpha)f(x)]\,. \qquad (15)$$

In coordinates, if u, v are the standard retarded and advanced time coordinates with origin x, and $d\Omega$ is the standard unit of solid angle,

$$\int_M H(x-x')f(x') = \int_0^{4\pi}\int_{-\infty}^0 \left[\frac{\partial f}{\partial u}\bigg|_{v=0}\ln(-u) + \frac{1}{2}\frac{\partial f}{\partial v}\bigg|_{v=0}\right] \times du\,d\Omega\,. \qquad (16)$$

It is not hard to verify that H is indeed a continuous linear map from test functions to the real numbers and hence defines a distribution. It follows from a theorem due to Methée[27] (see Appendix) that each of the scalar distributions G_1 and G_2 appearing in Eq. (14) must be some multiple of H plus some multiple of the Dirac δ distribution at x, i.e., $G_1 = aH + \alpha\delta_x$, $G_2 = bH + \beta\delta_x$ for some constants a, b, α, β.

We have now determined the scalar distributions G_1 and G_2. Substituting into Eq. (14) we obtain the result

$$\langle \dot{T}_{ab}(x)\rangle = \hbar\left\{\int_M H(x-x')[a\dot{A}_{ab}(x') + b\dot{B}_{ab}(x')] + \alpha\dot{A}_{ab}(x) + \beta\dot{B}_{ab}(x)\right\}, \qquad (17)$$

where H is the Green's function given by Eq. (16), $\dot{A}_{ab}$ and $\dot{B}_{ab}$ are linearized conserved local curvature tensors given by Eqs. (6) and (7), and $a, b, \alpha,$

β are constants.

To summarize, using only general properties of the quantum stress energy and none of the details of a specific regularization prescription, we have shown that any candidate for the linearized stress energy of a massless quantum field must be given by Eq. (17) for certain values of the constants a, b, α, and β. Thus there is a four-parameter family of candidates for the first-order quantum stress energy which satisfies our five properties.

There appears, however, to be a problem with Eq. (17). Recall that any two stress-energy tensors for a given quantum field on a given spacetime satisfying Wald's axioms (discussed in Sec. II) can differ by at most a *local* curvature tensor. However, it is clear that the stress energy defined by Eq. (17) for one choice of the constants a and b will differ in a *nonlocal* way from the stress energy defined by this equation (with the same perturbation) for another choice of a and b. The resolution of this apparent contradiction is that the five properties we have used to obtain our Green's function formula for the stress energy do not fully capture the content of Wald's axioms. For example, we have considered only the in-vacuum expectation value of the stress energy and have not required that the out-vacuum expectation value of the stress energy be causal. Since the difference between any two stress-energy tensors satisfying Wald's axioms must be a local curvature tensor, the subset of stress-energy tensors (for each quantum field) given by Eq. (17) which are consistent with the axioms must consist of precisely those with a fixed value of a and b.

To determine the constants a and b we proceed as follows. Fix a perturbation γ_{ab} on Minkowski spacetime (M, η_{ab}) and a massless quantum field on this perturbed spacetime. First compute the stress energy $\langle \dot{T}_{ab} \rangle$ created by the perturbation γ_{ab}, and then compute the stress energy $\langle \dot{\tilde{T}}_{ab} \rangle$ created by the perturbation $\tilde{\gamma}_{ab} = 4\gamma_{ab}$ on the spacetime with metric $\tilde{\eta}_{ab} = 4\eta_{ab}$. We find that $\dot{\tilde{A}}_{ab} = \frac{1}{16}\dot{A}_{ab}$ and $\dot{\tilde{B}}_{ab} = \frac{1}{16}\dot{B}_{ab}$. However, $\langle \dot{\tilde{T}}_{ab} \rangle \neq \frac{1}{4}\langle \dot{T}_{ab} \rangle$. The presence of the logarithm in the definition of H [see Eq. (15)] implies that

$$\langle \dot{\tilde{T}}_{ab} \rangle = \tfrac{1}{4}\langle \dot{T}_{ab} \rangle + \pi\hbar \ln 2 (a\dot{A}_{ab} + b\dot{B}_{ab}) . \tag{18}$$

Now let $g_{ab}(t)$ be any one-parameter family of metrics such that $g_{ab}(0) = \eta_{ab}$ and $\dot{g}_{ab}(0) = \gamma_{ab}$. Take any regularization prescription satisfying Wald's axioms and compute for each t the stress energy $\langle T_{ab}(t) \rangle$. Then compute for each t the stress energy $\langle \tilde{T}_{ab}(t) \rangle$ resulting from the metric $\tilde{g}_{ab}(t) = 4g_{ab}$. One finds

$$\langle \tilde{T}_{ab}(t) \rangle = \tfrac{1}{4}\langle T_{ab}(t) \rangle + \pi\hbar \ln 2 [A A_{ab}(t) + B B_{ab}(t)] , \tag{19}$$

where A and B are constants (independent of both the parameter t and spacetime point) which are now known. Therefore linearizing this equation and comparing with the one above [Eq. (18)] we find that the stress energy given by Eq. (17) will be consistent with Wald's axioms if and only if $a \equiv A$ and $b \equiv B$. Fortunately, the values of A and B and hence a and b for several quantum fields have already been determined.[9,12] In Table I we list the values of these coefficients for massless quantum fields of spin 0, $\frac{1}{2}$, and 1. The first-order stress energy of two or more massless quantum fields interacting only with gravity is given by Eq. (17) with a and b equal to the sum of the appropriate coefficients for each field.

Strictly speaking, the fact that $\langle \dot{\tilde{T}}_{ab} \rangle \neq \frac{1}{4}\langle \dot{T}_{ab} \rangle$ means that the stress energy defined by Eq. (17) does not have the correct physical dimensions of cm^{-2}. In order to correct this, we introduce a preferred length λ into the theory.[12,28] Define a new distribution H_λ by replacing $\ln(-\alpha)$ by $\ln(-\alpha/\lambda^2)$ in Eq. (15). In terms of coordinates the action of H_λ on a test function f is

$$\int_M H_\lambda(x-x')f(x') = \int_{-\infty}^{0}\int_{0}^{4\pi} \left[\frac{\partial f}{\partial u}\bigg|_{v=0} \ln(-u/\lambda) + \frac{1}{2}\frac{\partial f}{\partial v}\bigg|_{v=0} \right] \times d\Omega\, du . \tag{20}$$

We now replace the distribution H in Eq. (17) by H_λ. It is easy to verify that the stress energy so defined now has the correct dimensions. Since the length scale λ is arbitrary, we can view it as another free parameter of the theory. However, it is not independent of the parameters α and β. Notice from Eq. (20) that

$$\int_M H_{\lambda'}(x-x')f(x') - \int_M H_\lambda(x-x')f(x') = 4\pi \ln(\lambda/\lambda')f(x) . \tag{21}$$

Therefore, by adjusting the value of λ we can rewrite Eq. (17) so that $\alpha\dot{A}_{ab}(x)$ does not explicitly

TABLE I. Values of the coefficients "a" and "b" appearing in Eq. (17) for several choices of massless quatum field.

Quantum field	a	b
Klein-Gordon field	$\frac{1}{(4\pi)(960\pi^2)}$	$\frac{1}{(4\pi)(576\pi^2)}$
Conformally invariant scalar field	$\frac{1}{(4\pi)(960\pi^2)}$	0
Neutrino field	$\frac{3}{(4\pi)(960\pi^2)}$	0
Maxwell field	$\frac{12}{(4\pi)(960\pi^2)}$	0

appear.

We thus obtain our final expression yielding the first-order stress energy in terms of the gravitational perturbation[29]:

$$\langle\dot{T}_{ab}(x)\rangle = \hbar\left\{\int_M H_\lambda(x-x')[a\dot{A}_{ab}(x')+b\dot{B}_{ab}(x')]+\beta\dot{B}_{ab}(x)\right\}, \tag{22}$$

where λ and β are arbitrary parameters, H_λ is the Green's function given by Eq. (20), a and b are the fixed coefficients (depending on the quantum field) given by Table I, and $\dot{A}_{ab}$ and $\dot{B}_{ab}$ are the linearized conserved curvature tensors given by Eqs. (6) and (7).

Equation (22) has the following immediate consequence: *For a perturbation in Minkowski spacetime satisfying the linearized vacuum Einstein equation, the first-order stress energy of any massless quantum field vanishes everywhere.* To see this, one simply observes [from Eqs. (6) and (7)] that if $\dot{G}_{ab}=0$, then $\dot{A}_{ab}=\dot{B}_{ab}=0$ as well. Physically, this fact means that for perturbations satisfying the linearized Einstein equation, there is neither particle creation nor vacuum polarization to first order.[30]

It is easy to verify that Eq. (22) is equivalent to an expression obtained by Capper *et al.*[12]: If one takes the Fourier transform of Eq. (22) [see Appendix, Eq. (A23)], and restricts γ_{ab} to be in an appropriate gauge, then one finds that this expression, e.g., for a quantum Maxwell field, is equivalent to the expression for the one-loop photon contribution to the graviton self-energy computed by Capper *et al.* using dimensional regularization. In particular, our expression incorporates the standard trace anomaly[31,32] for conformally invariant quantum fields.

IV. LINEARIZED SOLUTIONS

In this section we use our Green's function formula [Eq. (22)] for the linearized stress energy of a massless quantum field to investigate the solutions to the linearized semiclassical Einstein equation

$$\dot{G}_{ab} = 8\pi\langle\dot{T}_{ab}\rangle. \tag{23}$$

Notice one important difference between this equation and the linearized classical Einstein equation. Unlike $\dot{G}_{ab}=0$, Eq. (23) is nonlocal. That is, given a perturbation γ_{ab} defined only in a neighborhood of a point, one cannot ask whether γ_{ab} is a solution to Eq. (23) in this neighborhood. For the quantum stress energy evaluated at a point x involves an integral of the perturbation over the entire past light cone of x. Physically, the nonlocal character of the quantum stress energy can be interpreted (in part) as resulting from particles created by the curvature at a point x' propagating through space and contributing to the stress energy at the point x.

We now begin our investigation of the linearized back-reaction problem. That is, we fix a quantum field, fix values for β and λ, and seek perturbations γ_{ab} which satisfy the linearized semiclassical Einstein equation [Eq. (23)], where $\langle\dot{T}_{ab}\rangle$ is given by Eq. (22).

Recall that the local curvature tensors $\dot{A}_{ab}$ and $\dot{B}_{ab}$ appearing in the equation for the linearized quantum stress energy can both be expressed in terms of the linearized Einstein tensor [see Eqs. (6) and (7)]. Thus the semiclassical Einstein equation can be viewed as an equation on a symmetric, conserved tensor $\dot{G}_{ab}$. Therefore, we adopt the following program for finding metric perturbations γ_{ab} satisfying Eq. (23). We first solve Eq. (23) for $\dot{G}_{ab}$ and then (imposing, say the Coulomb gauge) solve

$$-\tfrac{1}{2}\nabla^2(\gamma_{ab}-\tfrac{1}{2}\gamma^m{}_m\eta_{ab}) = \dot{G}_{ab} \tag{24}$$

for γ_{ab}. Since Eq. (24) is easily solved once $\dot{G}_{ab}$ is known, we concentrate on solving Eq. (23) for $\dot{G}_{ab}$.

One solution to Eq. (23) for any choice of λ,β and quantum field is clearly $\dot{G}_{ab}=0$. Therefore, *every solution to the linearized classical Einstein equation is also a solution to the linearized semiclassical Einstein equation.* Physically, these solutions can be interpreted as weak gravitational waves which do not interact with the quantum field. To find additional solutions to the semiclassical Einstein equation we consider special choices of the quantum field and parameters λ and β.

For simplicity we consider first a quantum Maxwell field and we set $\beta=0$, $\lambda\neq 0$ in the formula for the linearized stress energy. In this case the semiclassical Einstein equation becomes

$$\dot{G}_{ab}(x) = \frac{\hbar}{40\pi^2}\int_M H_\lambda(x-x')\dot{A}_{ab}(x'). \tag{25}$$

Taking the trace of both sides of this equation we find $\dot{G}^m{}_m=0$. Therefore using the expression for $\dot{A}_{ab}$ in terms of $\dot{G}_{ab}$ [Eq. (6)] we can rewrite Eq. (25) in the form

$$\dot{G}_{ab}(x) = \frac{-\hbar}{20\pi^2}\int_M H_\lambda(x-x')\nabla'^2\dot{G}_{ab}(x'), \tag{26}$$

where ∇'^2 is the wave operator in x'. Equation (26) thus describes the interaction of a quantum Maxwell field with a weak gravitational field in semiclassical relativity with $\beta=0$. We now investigate its solutions.

We first find all asymptotically well-behaved solutions to Eq. (26), i.e., all solutions $\dot{G}_{ab}$ which can be Fourier transformed. Taking the Fourier transform of Eq. (26) we obtain

$$\hat{\dot{G}}_{ab}=\frac{\hbar}{20\pi^2}k^2\hat{H}_\lambda\hat{\dot{G}}_{ab}\,, \tag{27}$$

where a caret denotes the Fourier transform. It turns out that the Fourier transform of the distribution H_λ can be represented (see Appendix) by the function

$$\hat{H}_\lambda=-2\pi[\ln\lambda^2|k^2|+2\gamma-1+i\pi\theta_-(k)]\,, \tag{28}$$

where γ is Euler's constant and θ_- is the step function that has the value -1 inside the future light cone in momentum space, $+1$ inside the past light cone, and 0 elsewhere. Substituting (28) into (27) we obtain

$$f(k^2)\hat{\dot{G}}_{ab}(k)=0\,, \tag{29}$$

where

$$f(k^2)=1+\frac{\hbar}{10\pi}k^2[\ln\lambda^2|k^2|+2\gamma-1+i\pi\theta_-(k)]\,.$$

For $\hat{\dot{G}}_{ab}$ to satisfy this equation, it can be nonzero only when $f=0$. It turns out that the zeros of f depend upon the relative values of λ and a critical value on the order of the Planck length: $\lambda_{\rm crit}\equiv e^{-\gamma}(\hbar/10\pi)^{1/2}$. If $\lambda>\lambda_{\rm crit}$, one finds that f is nonzero everywhere in momentum space. Thus the only solution to Eq. (29) is $\dot{G}_{ab}=0$. Consequently, *for* $\lambda>\lambda_{\rm crit}$, *the only asympotically well-behaved perturbations* γ_{ab} *which satisfy the linearized semiclassical Einstein equation* (*with a quantum Maxwell field and* $\beta=0$) *are those satisfying* $\dot{G}_{ab}=0$, i.e., solutions to the linearized classical Einstein equation. For $\lambda=\lambda_{\rm crit}$, one finds that there exists precisely one value of k^2 at which $f=0$. It turns out that this value is positive (i.e., k^a is spacelike) and therefore f vanishes on a timelike hyperbola $\mathcal{H}$ in momentum space. Now recall that $\dot{G}_{ab}$ must be conserved, so $\hat{\dot{G}}_{ab}$ must be transverse to the hyperbola, i.e., $k^a\hat{\dot{G}}_{ab}=0$. Therefore, for $\lambda=\lambda_{\rm crit}$, the general Fourier analyzable solution to Eq. (26) is obtained by specifying on the hyperbola $\mathcal{H}$ any symmetric, transverse, trace-free tensor field L_{ab} (which is suitably well behaved asymptotically) and setting

$$\dot{G}_{ab}(x)=\int_{\mathcal{H}}L_{ab}(k)e^{ik\cdot x}\,. \tag{30}$$

[If we want $\dot{G}_{ab}$ to be real, we must additionally choose L_{ab} such that $\bar{L}_{ab}(k)=L_{ab}(-k)$, where a bar denotes complex conjugation.] Finally, for $\lambda<\lambda_{\rm crit}$, one finds that there exist precisely two values of k^2 at which $f=0$. Both of these values turn out to be positive, and therefore f vanishes on two timelike hyperbolas $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus the solutions to Eq. (26) with $\lambda<\lambda_{\rm crit}$ are given by a formula identical to Eq. (30) with $\mathcal{H}$ replaced by $\mathcal{H}_1\cup\mathcal{H}_2$. Notice that since k^a is always *spacelike* in Eq. (30), these solutions represent gravitational waves which (in some sense) travel faster than the speed of light. Thus for $\lambda\leq\lambda_{\rm crit}$, semiclassical relativity admits "tachyonlike" solutions.

We now ask whether there exist solutions to Eq. (26) which are initially well behaved, but grow exponentially in time. (Solutions of this type would not be Fourier analyzable and hence would not be included in the analysis of the above paragraph.) Let t^a be a constant unit timelike vector field in Minkowski spacetime and define a time coordinate by $t_a=\nabla_a t$. We try a solution of the form

$$\dot{G}_{ab}=L_{ab}\,e^{\omega t}\,, \tag{31}$$

where L_{ab} is a constant, symmetric, trace-free, spatial (i.e., $t^aL_{ab}=0$) tensor field and ω is a complex constant (with positive real part). Note that the $\dot{G}_{ab}$ given by Eq. (31) is indeed conserved. We now substitute this form of $\dot{G}_{ab}$ into Eq. (26) and evaluate the integral using Eq. (20) for the Green's function H_λ. We obtain the following equation on ω:

$$1=-\frac{\hbar}{10\pi}\omega^2(\ln\lambda^2\omega^2+2\gamma-1)\,. \tag{32}$$

Are there values of ω which satisfy this equation? The answer is yes (for any value of λ). Qualitatively, the dependence of the solutions on λ is the following. In general, there exist two solutions. For $\lambda\ll\lambda_{\rm crit}$, one solution is much less than $\omega_{\rm crit}\equiv(10\pi/\hbar)^{1/2}\sim10^{43}\ {\rm sec}^{-1}$ and the other is much greater than this frequency. As λ increases toward $\lambda_{\rm crit}$, the two solutions approach each other, and when $\lambda=\lambda_{\rm crit}$, they coincide at $\omega=\omega_{\rm crit}$. As λ increases past $\lambda_{\rm crit}$, both solutions become complex, but in the limit $\lambda\to\infty$, they both approach $\omega=0$. In short, for *any* value of λ, there exist exponentially growing solutions to Eq. (26). Physically, the existence of these exponentially growing perturbations indicates that the vacuum solution of semiclassical relativity (i.e., Minkowski spacetime with the in-vacuum state of the quantum field) is unstable.

To summarize, we have been considering a quantum Maxwell field interacting with a weak gravitational field in the framework of semiclassical relativity. We have set the free parameter β in the definition of the quantum stress energy equal to zero. For all values of the remaining parameter λ, we found that there exist both well-behaved solutions (perturbations satisfying $\dot{G}_{ab}=0$) and badly behaved solutions (perturbations growing exponentially in time). In addition, we found that if λ is less than a critical value (on the order of the Planck length) then there exist tachyonlike solutions (perturbations that are a superposition of plane waves traveling in spacelike directions). Since this last class of solutions is presumably unphysical, it is fortunate that semiclassical relativity provides a natural way to eliminate them. We simply restrict the free parameter λ to values greater than this critical value.

The existence of a critical value of λ should perhaps be emphasized. Recall that in the definition of the quantum stress energy, λ was completely undetermined. We have now found, however, that when this stress energy is substituted into the semiclassical Einstein equation, there emerges a preferred value of λ (or at least a restriction on the allowed range of λ). In other words, the requirement that semiclassical relativity have reasonable dynamics can be used to help determine the appropriate stress energy of a quantum field.

We now set the free parameter β equal to a *nonzero* constant, and investigate its effect on the linearized back-reaction problem for a quantum Maxwell field. The semiclassical Einstein equation now becomes

$$\dot{G}_{ab}(x) = 8\pi\hbar\left[\frac{1}{320\pi^3}\int_M H_\lambda(x-x')\dot{A}_{ab}(x') + \beta\dot{B}_{ab}(x)\right]. \tag{33}$$

Recall that $\dot{B}_{ab}$ depends only on the trace of $\dot{G}_{ab}$:

$$\dot{B}_{ab} = 2\eta_{ab}\nabla^2\dot{G}^m{}_m - 2\nabla_a\nabla_b\dot{G}^m{}_m. \tag{7}$$

Further, recall that every solution $\dot{G}_{ab}$ to the semiclassical Einstein equation with $\beta=0$ has vanishing trace. Therefore we immediately conclude that every solution to the semiclassical Einstein equation with $\beta=0$ is again a solution to this equation with $\beta\neq 0$. In particular, there exist exponentially growing solutions and (for certain values of λ) tachyonlike solutions to Eq. (33). However, it turns out that there are additional solutions to Eq. (33) which we now investigate.

To find all asymptotically well-behaved solutions to Eq. (33), we again take the Fourier transform. Proceeding as before, we find that in addition to the trace-free solutions we found earlier, there now exist the following "pure trace" solutions. Let $\mathcal{H}_\beta$ be the hyperbola in momentum space consisting of all points satisfying $k^a k_a = -(48\pi\hbar\beta)^{-1}$. Let h_{ab} be the induced metric on $\mathcal{H}_\beta$, and let $\hat{\phi}$ be any asymptotically well-behaved function on $\mathcal{H}_\beta$. Then

$$\dot{G}_{ab} = \int_{\mathcal{H}_\beta}\hat{\phi}h_{ab}e^{ik\cdot x} = \phi\eta_{ab} - (48\pi\hbar\beta)\nabla_a\nabla_b\phi \tag{34}$$

is a solution to Eq. (33) (where ϕ is the Fourier transform of $\hat{\phi}$). In fact, this exhausts the class of additional well-behaved solutions to Eq. (33). That is, every Fourier analyzable solution to Eq. (33) can be expressed as the sum of a solution given by Eq. (30) and one given by Eq. (34). What is the physical interpretation of these new solutions? It is easy to check that $\dot{A}_{ab}=0$ for any perturbation given by Eq. (34). (They can be generated by conformally flat metric perturbations.) Thus the quantum stress energy is purely local.[33] Hence one might interpret these solutions as weak gravitational waves which induce vacuum polarization, but not particle creation. Notice that if $\beta<0$, then the vectors k^a in Eq. (34) are again spacelike. Thus we are once again led to restrict the values of the parameters in semiclassical relativity (in this case $\beta\geq 0$) in order to rule out unphysical solutions. (Surprisingly, the most natural regularization prescriptions usually involve negative values of β.) If one considers exponentially growing solutions to Eq. (33), then one finds—in addition to the exponentially growing solutions for the case $\beta=0$—additional solutions of this type when $\beta<0$, e.g.,

$$\dot{G}_{ab} = (\eta_{ab} + t_a t_b)e^{\omega t}, \tag{35}$$

where $\omega = (-48\pi\hbar\beta)^{-1/2}$.

So far we have been considering the interaction of gravity with a particular quantum field—a Maxwell field—in the context of semiclassical relativity. How does gravity interact with other quantum fields in this framework? For conformally invariant scalar fields or neutrino fields, there is essentially no change. This can be seen immediately from the expression for the stress-energy tensor [Eq. (22)] and the values of the coefficients in Table I. The only change in the semiclassical Einstein equation when passing from Maxwell to, e.g., neutrino fields is in the value of the numerical coefficient in front of the curvature tensor $\dot{A}_{ab}$. Thus the solutions are qualitatively the same. For the massless scalar field, however, the situation is slightly different. One now has a nonlocal contribution to the stress energy from the curvature tensor $\dot{B}_{ab}$ as well as $\dot{A}_{ab}$. One can analyze the semiclassical Einstein equation for this field in exactly the same manner as for the Maxwell field. The most important difference in the character of the solutions turns out to be the following. There now exist tachyonlike solutions *for all values of the parameter* λ. Thus for a massless scalar field, unlike a conformally invariant field, one cannot eliminate these unphysical solutions by restricting λ in some way.

We conclude this section with a brief discussion of semiclassical relativity in the presense of a classical stress energy. Although all matter can be expressed in terms of *some* quantum state, it is often convenient to approximate part of a physical system (e.g., a star) as being classical. Thus instead of trying to solve

$$G_{ab}=8\pi\langle\psi\,|\,T_{ab}\,|\psi\rangle \tag{36}$$

for some complicated in-state ψ of the quantum field, one considers

$$G_{ab}=8\pi(\langle\xi\,|\,T_{ab}\,|\,\xi\rangle+\mathcal{T}_{ab})\,, \tag{37}$$

where ξ is a relatively simple (e.g., vacuum) in-state and $\mathcal{T}_{ab}$ is a classical stress-energy tensor.

We now linearize Eq. (37) off the solution consisting of Minkowski spacetime, the in-vacuum state of the field, and zero classical stress energy. The linearized equation is

$$\dot{G}_{ab}=8\pi(\langle\dot{T}_{ab}\rangle+\dot{\mathcal{T}}_{ab})\,. \tag{38}$$

Equation (38) might be used in the following manner. One fixes a classical stress energy $\dot{\mathcal{T}}_{ab}$ on Minkowski spacetime and asks for perturbations γ_{ab} which satisfy this equation for a given quantum field and given values of λ and β. If the classical stress energy $\dot{\mathcal{T}}_{ab}$ has a Fourier transform, then these solutions can be found quite easily. [We again follow the program of first solving Eq. (38) for $\dot{G}_{ab}$ and then solving Eq. (24) for γ_{ab}.]

For convenience we consider a quantum Maxwell field and set $\beta=0$, $\lambda>\lambda_{\rm crit}$ in the formula for the quantum stress energy. Taking the Fourier transform of Eq. (38) we obtain

$$f(k^2)\hat{\dot{G}}_{ab}(k)=8\pi\hat{\dot{\mathcal{T}}}_{ab}\,, \tag{39}$$

where the function f is given by Eq. (29). Since we have chosen $\lambda>\lambda_{\rm crit}$ (to rule out unphysical solutions in the theory), f is nonzero everywhere. Therefore, one solution to Eq. (38) is simply

$$\dot{G}_{ab}=8\pi\int\frac{\hat{\dot{\mathcal{T}}}_{ab}}{f}\,e^{ik\cdot x}\frac{d^4k}{(2\pi)^4}\,. \tag{40}$$

In fact, this is the only asymptotically well-behaved solution to Eq. (38). [If there were two, their difference would satisfy Eq. (38) with $\dot{\mathcal{T}}_{ab}=0$. But we have shown that the only asymptotically well-behaved solution to this equation—with $\lambda>\lambda_{\rm crit}$—is the zero solution.] If $\dot{\mathcal{T}}_{ab}$ does not have a Fourier transform, then in general it is more difficult to solve Eq. (38). In fact, for some choices of $\dot{\mathcal{T}}_{ab}$ there may exist no solutions at all (e.g., if $\dot{\mathcal{T}}_{ab}$ grows exponentially near past null infinity).

The form of Eq. (38) suggests a natural recursive procedure for approximating solutions (which is valid for $\dot{\mathcal{T}}_{ab}$ asymptotically well behaved in the past). To zeroth order we set $\dot{G}^0_{ab}=8\pi\dot{\mathcal{T}}_{ab}$. To first order, we set $\dot{G}^1_{ab}=8\pi\langle\dot{T}_{ab}\rangle_0$ where $\langle\dot{T}_{ab}\rangle_0$ is defined by substituting $\dot{G}^0_{ab}$ into the expression for $\langle\dot{T}_{ab}\rangle$ [Eq. (22)]. To second order, we set $\dot{G}^2_{ab}=8\pi\langle\dot{T}_{ab}\rangle_1$, etc. One thus obtains an expansion of the solution in powers of $\hbar$.

As an example of this recursive procedure, we estimate the first-order quantum correction to the gravitational field around the sun (which will yield, e.g., the quantum correction to the bending of light). For $\dot{\mathcal{T}}_{ab}$ we take the linearized form of the stress energy for a static, spherically symmetric perfect fluid, i.e.,

$$\dot{\mathcal{T}}_{ab}=\rho t_a t_b\,, \tag{41}$$

where the density ρ is a function only of r and the 4-velocity t_a is a constant unit timelike vector field in Minkowski spacetime. (The linearized pressure vanishes.) Following the above approximation procedure, to zeroth order we have $\dot{G}^0_{ab}=8\pi\rho t_a t_b$. This is simply the classical general relativistic description of the gravitational field around the sun. To first order, we find (say, for a quantum Maxwell field and $\beta=0$)

$$\dot{G}^1_{ab}=\frac{\hbar}{5\pi}\left\{\int_M H_\lambda[-2\nabla^2\rho t_a t_b+\tfrac{2}{3}\nabla_a\nabla_b\rho-\tfrac{2}{3}\eta_{ab}\nabla^2\rho]\right\}. \tag{42}$$

Notice from Eqs. (21) and (42) that changing the free parameter λ changes $\dot{G}^1_{ab}$ by terms involving second derivatives of ρ, e.g., $\nabla_a\nabla_b\rho$. One can show, however, that these changes do not affect the description of the gravitational field (i.e., the appropriate metric perturbation γ_{ab}) in the region where $\rho=0$. Similarly, one can show that including a nonzero value of β in Eq. (42) does not affect this description.

Thus, the prediction (in semiclassical relativity) of the first-order quantum correction to the gravitational field outside the sun is independent of the free parameters of the theory. What is its magnitude? If we crudely estimate $\nabla\rho\sim\rho/R_\odot$ where $R_\odot$ is the radius of the sun and $\int H_\lambda\rho\sim\rho$ for points near the sun, then we find that the relative size of this quantum correction is

$$\frac{\dot{G}^1}{\dot{G}^0}\sim\frac{\hbar}{{R_\odot}^2}\sim10^{-87}$$

(rather small indeed).

V. CONCLUSIONS

For what physical systems might one expect a semiclassical theory of matter and gravity to yield interesting predictions? Certainly, these systems do not include ordinary stars such as the sun. In fact, it might appear from Eq. (42) that there exist no physical systems for which a semiclassical theory is appropriate. For this equation seems to imply that the first-order quantum correction to a classical gravitational field will be negligible unless that gravitational field changes on the scale of the Planck length. And one expects that the quantum effects of gravity itself will be important on these scales. Nevertheless, there may still be an interesting regime of applicability for a semiclassical theory, as we now indicate. Consider a system in which particles are created by a gravitational field and accumulate over a long period of time (e.g., the age of the universe). Although initially the effect of the particles is negligible, eventually the accumulated stress energy could become significant and lead to macroscopic changes in the system. (This is in fact what is believed to occur in the evaporation of a black hole.)

Given that there may exist physical systems for which the quantum nature of matter is significant but the quantum nature of gravity is still negligible, one can now ask for a theory that yields a valid description of these systems. The most natural candidate for such a theory is semiclassical relativity. However, we have encountered two unpleasant features of this theory. The first is that it predicts the existence of weak gravitational waves that seem to travel faster than light. The second is that it predicts the instability of the vacuum. Fortunately, for most quantum fields, the first difficulty can be eliminated by suitably restricting some of the parameters in the theory, i.e., $\lambda > \lambda_{crit}, \beta \geq 0$. (In fact, the only known massless field for which one cannot eliminate these solutions—the massless scalar field—does not appear to be present in nature.) However, the instability of the vacuum state of semiclassical relativity (described by Minkowski spacetime and the in-vacuum state of the quantum field) is a more serious difficulty of this theory. (Clearly, the space around us is not unstable to suddenly becoming strongly curved.) There are a variety of viewpoints that one might take toward this difficulty, which we now discuss.

Assume that there exists *some* reasonable theory describing the interaction of a quantum matter field and a classical gravitational field. Then one might adopt any one of at least four different viewpoints toward the fact that the vacuum state of semiclassical relativity is unstable. The first viewpoint may be expressed as follows: Semiclassical relativity yields a valid description of nature. The existence of linearized exponentially growing solutions gives very little information about the actual stability of the vacuum in the full theory. When higher-order effects are included, one will find that the vacuum state of semiclassical relativity is in fact stable. The second viewpoint is the following[36]: Semiclassical relativity is valid, but its parameter λ is very large. Recall that the time scale for the instability of the vacuum increases with λ. Thus although the vacuum is in principle unstable, in practice one never notices this fact because the time scale is so large (e.g., greater than the age of the universe). The third viewpoint one might adopt is: Semiclassical relativity, as discussed here, is not quite the correct semiclassical theory. One must now add some restrictions (e.g., future boundary conditions) to rule out these irrelevant exponentially growing solutions. Finally, one has: The appropriate semiclassical theory is simply not semiclassical relativity. Although this theory is the most natural extension of Einstein's theory, it does not accurately describe nature.

It might, however, turn out that none of the above viewpoints are correct. There may exist a reasonable semiclassical theory of matter and gravity only in regions of strong gravitational fields. If this is indeed the case, the appropriate semiclassical theory may turn out to be a suitable strong field limit of semiclassical relativity.

Finally, one is left with the possibility that there is simply no reasonable semiclassical theory of matter and gravity. The only consistent way to describe gravity coupled to quantum fields is by taking into account the quantum nature of gravity itself.

It is hoped that future research in this area will shed more light on the status of semiclassical relativity, and perhaps lead to new insights into a full theory of quantum gravity.

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APPENDIX: TENSOR DISTRIBUTIONS IN MINKOWSKI SPACETIME

This appendix is divided into three parts. First, we review some basic definitions and properties of tensor distributions in Minkowski spacetime. Next, we characterize distributions which are invariant under the action of the Poincaré group. Finally, we discuss the distribution H_λ which appears in the formula [Eq. (22)] for the linearized quantum stress energy. Only sketches of proofs will be provided. For more details see, e.g., Schwartz[34] or Friedlander.[35]

1. Basic definitions and properties

Let $V_{(r,s)}$ denote the vector space of all smooth tensor fields of rank (r,s) on Minkowski spacetime (M,η_{ab}) which have compact support. Elements of $V_{(r,s)}$ will be called "test fields." Recall that a *tensor distribution* of rank (r,s) is a map from $V_{(r,s)}$ to the real numbers which is linear and (in a suitable sense) continuous. For example, any vector field V_a on M gives rise to a vector distribution, whose action on the test field μ^a is the real number obtained by integrating $V_a\mu^a$ over the spacetime. (The convergence of the integral is guaranteed by the compact support of μ^a.) Thus every tensor field gives rise to a tensor distribution. Of course, there exist distributions which do not arise from tensor fields in this manner. For example, fix a point $p\in M$ and consider the scalar distribution (the "Dirac δ distribution at p") defined by $\delta_p(f)=f(p)$ for all test functions f.

Many operations on tensor fields can be extended to distributions. The method for doing so is the following: One requires that when a given operation is applied to a distribution arising from a tensor field, the operation reduce to the familiar one on the tensor field. Several examples of operations on distributions are given below. Although a particular rank distribution is used in each example, the generalization to other distributions is immediate.

(1) *Addition.* If R_a and S_a are two vector distributions, their sum is the vector distribution defined by

$$(R_a+S_a)(\mu^a)\equiv R_a(\mu^a)+S_a(\mu^a)\,, \tag{A1}$$

for all μ^a in $V_{(1,0)}$.

(2) *Outer product by tensor fields.* If S_a is a vector distribution and V_a a smooth vector field, then their outer product is the rank-(0, 2) distribution defined by

$$V_aS_b(\mu^{ab})\equiv S_b(\mu^{ab}V_a)\,, \tag{A2}$$

for all μ^{ab} in $V_{(2,0)}$. Notice that $\mu^{ab}V_a$ will always have compact support, and thus the right-hand side is well defined.

(3) *Contraction.* If $S^a{}_b$ is a rank-(1, 1) distribution, then $S^m{}_m$ is the scalar distribution defined by

$$S^m{}_m(f)\equiv S^a{}_b(f\delta^b{}_a) \tag{A3}$$

for all f in $V_{(0,0)}$, where $\delta^b{}_a$ is the identity tensor.

(4) *Derivation.* If S_b is a vector distribution, ∇_aS_b is the rank-(0, 2) distribution defined by

$$\nabla_aS_b(\mu^{ab})\equiv -S_b(\nabla_a\mu^{ab})\,, \tag{A4}$$

for all μ^{ab} in $V_{(2,0)}$.

(5) *Convolution.* Fix an origin in Minkowski spacetime. Recall that the convolution of two tensor fields of compact support, e.g., V_a, μ^{ab} is again a tensor field of compact support:

$$V_a*\mu^{ab}(x)\equiv\int V_a(x-x')\mu^{ab}(x')d^4x'\,. \tag{A5}$$

If S_b is a vector distribution and V_a is a vector field of compact support, then we define

$$V_a*S_b(\mu^{ab})\equiv S_b(\tilde V_a*\mu^{ab})\,, \tag{A6}$$

for all μ^{ab} in $V_{(2,0)}$ where $\tilde V_a(x)=V_a(-x)$. That is, the convolution of a vector field V_a with a vector distribution S_b is the second-rank tensor distribution whose action on μ^{ab} is the real number obtained by acting with S_b on the convolution of μ^{ab} and the reflection of V_a. One can easily verify that for a distribution S^a arising from a vector field, this definition reduces to the usual definition of convolution on vector fields.

(6) *Action under diffeomorphisms*: Let ϕ: $M\to M$ be a diffeomorphism, and S_a be a vector distribution. We define

$$\phi S_a(\mu^a)=S_a(\phi^{-1}\mu^a)\,, \tag{A7}$$

for all μ^a in $V_{(1,0)}$. A distribution is said to be invariant under a diffeomorphism ϕ if $\phi S_a=S_a$.

There are at least two additional operations on tensor fields which do not appear in the above list: the product of two distributions and the Fourier transform of a distribution. Each of these operations can only be defined on a restricted class of distributions. Roughly speaking, the product of two distributions S_a and R_a can only be defined when "S_a is more regular than R_a is irregular." For example, the product of two distributions arising from smooth tensor fields is clearly the distribution arising from the product of the tensor fields, while the square of the Dirac δ distribution is not defined. A more precise statement of the class of distributions on which multiplication can be defined is given in Ref. 37. Now consider the Fourier transform of a distribution. One would like to simply define $\hat S(f)=S(\hat f)$ for all test functions f where a caret denotes the Fourier transform (since this is the formula for distributions arising from functions). However, if f is a real-valued function of compact support, then $\hat f$ may be neither real valued nor have compact support. Therefore, we enlarge the class of test fields as follows. Let $S_{(r,s)}$ denote the class of all complex tensor fields of rank (r,s) whose components (in the usual Minkowski coordinates) go to zero asymptotically faster than any polynomial in the coordinates. One can easily verify that if a tensor field is in this class, then its Fourier transform is also in this class. We now define a *tempered distribution* of rank (r,s) to be a continuous linear map from $S_{(r,s)}$ to the complex numbers. The Fourier transform of a tempered distribution, e.g., T_a is now defined to be the tempered distribution

$$\hat{T}_a(\mu^a) = T_a(\hat{\mu}^a)\,, \tag{A8}$$

for all μ^a in $S_{(1,0)}$. (Note that this equation now makes sense.) Since tempered distributions take values in the complex numbers, we can define the complex conjugate of a tempered distribution to be

$$\bar{T}(f) \equiv \overline{T(\bar{f})} \tag{A9}$$

for all f in $S_{(0,0)}$ where a bar denotes complex conjugation.

One can prove several useful properties of distributions by combining the above operations. We mention two examples. First, the product of a tensor field and a distribution satisfies the Liebnitz rule under differentiation, e.g., $\nabla_a(V_b S_c) = (\nabla_a V_b)S_c + V_b(\nabla_a S_c)$. Second, the Fourier transform of the convolution of a tempered distribution with a tensor field in $S_{(r,s)}$ is the product of the Fourier transforms, e.g., $(V_a * S_b)\hat{} = \hat{V}_a\hat{S}_b$.

The notion of the support of a tensor field can also be generalized to distributions. The support of a distribution S_a is defined to be the complement of the largest open set $\mathcal{O} \subset M$ such that $S_a(\mu^a) = 0$ for all test fields μ^a with support in $\mathcal{O}$. As examples, the distribution arising from the vector field V^a has the same support as the vector field, and the Dirac δ distribution has support at a single point.

We now consider the assignment of dimensions to distributions. Recall that a tensor field $T^a{}_{bc}$ on M is said to have dimension $(\mathrm{cm})^d$ if $\tilde{\eta}_{ab} = \Omega^2\eta_{ab}$ (for constant Ω) implies that $\tilde{T}^a{}_{bc} = \Omega^p T^a{}_{bc}$ where $d = p -$ (number of covariant indices) + (number of contravariant indices). (Thus the dimension of a tensor field is independent of the location of its indices.) Similarly, a tensor distribution $S^a{}_{bc}$ is said to have dimensions $(\mathrm{cm})^d$ if $\tilde{\eta}_{ab} = \Omega^2\eta_{ab}$ implies that

$$\tilde{S}^a{}_{bc}(\tilde{\mu}_a{}^{bc}) = \Omega^{d+4} S^a{}_{bc}(\mu_a{}^{bc}) \tag{A10}$$

for all test fields $\mu_a{}^{bc}$ with dimension $(\mathrm{cm})^0$. For example, consider the Dirac δ distribution at the origin δ_0. Since δ_0 is independent of the metric, $\tilde{\delta}_0(\tilde{f}) = \delta_0(f) = f(0)$ for any test function f of dimension $(\mathrm{cm})^0$. Thus δ_0 has dimension $(\mathrm{cm})^{-4}$. [It would perhaps be more natural to omit the 4 in Eq. (A10) and, e.g., assign δ_0 the dimension $(\mathrm{cm})^0$. The 4 is included, however, to be compatible with the frequent integral notation for distributions, e.g., $\int \delta_0(x) f(x)\, d^4x = f(0)$.] As a second example, we have

$$(\nabla_a\tilde{\delta}_0)(\widetilde{\xi^a}) = (\nabla_a\tilde{\delta}_0)(\Omega^{-1}\xi^a) = \Omega^{-1}(\nabla_a\delta_0)(\xi^a)\,.$$

[Thus the derivative of the Dirac δ distribution has dimension $(\mathrm{cm})^{-5}$ as one might expect. As our final example, consider the Green's function G to the massless wave equation (the "integrate over the light cone" distribution). It is not hard to show that $\tilde{G}(\tilde{f}) = \Omega^2 G(\tilde{f}) = \Omega^2 G(f)$, and therefore G has dimensions cm^{-2}.

Finally, we remark on one useful generalization of tensor distributions. A two-point tensor distribution is a map $V_{(r,s)} \times V_{(r,s)} \to \mathrm{R}$ which is continuous and linear in each factor. For example, if S_a and T_{bc} are two- (one-point) distributions, then

$$U_{ab'c'}(\mu^a, \xi^{b'c'}) \equiv S_a(\mu^a) \times T_{bc}(\xi^{bc}) \tag{A11}$$

defines a two-point distribution where the right-hand side is just the product of two real numbers. All properties and operations discussed above can easily be generalized to two-point distributions. For example, the action of a diffeomorphism ϕ on a two-point tensor distribution $U_{aa'}$ is defined to be

$$\phi U_{aa'}(\xi^a, \mu^{a'}) = U_{aa'}(\phi^{-1}\xi^a, \phi^{-1}\mu^{a'})\,. \tag{A12}$$

2. Poincaré-invariant distributions

In this section we discuss a characterization of two-point tensor distributions which are invariant under the action of the Poincaré group. We proceed in three steps. First, Poincaré-invariant two-point distributions are shown to be equivalent (in a well-defined sense) to Lorentz-invariant one-point distributions. Next, Lorentz-invariant tensor distributions are expanded in terms of smooth tensor fields and Lorentz-invariant scalar distributions. Finally, Lorentz-invariant scalar distributions are shown to be essentially equivalent to distributions on the real numbers. Thus, in terms of their distributional nature, Poincaré-invariant two-point tensor distributions are no more complicated than distributions on the real numbers.

We begin by considering the translation subgroup of the Poincaré group. We claim that *if H is a translation-invariant two-point scalar distribution, then there exists a one-point distribution S such that*

$$H(\ \ ,g) = g * S\,, \tag{A13}$$

for all test functions g. Roughly speaking, this says that if H is translation invariant $\int H(x,y)g(y) = \int S(x-y)g(y)$. The idea of the proof is the following. One defines a distribution T on the manifold $M \times M$ [with coordinates $(x^0,\ldots,x^3, y^0,\ldots,y^3) \equiv (x,y)$] by first setting $T(f(x)g(y)) = H(f,g)$ for each test function of $M\times M$ that can be written as a product of a function of x times a function of y, and then extending to all test functions using continuity and linearity. Now fix an origin in M and define a smooth map ψ: $M\times M\to M$ by $\psi(x,y)=x-y$. Define a distribution S on M by $S(f)=T(c\overleftarrow{\psi}f)$ where f is a test function on M, $\overleftarrow{\psi}f$ is the pullback of f to $M\times M$, and c is a suitable cutoff function on $M\times M$ which ensures that the argument of T has compact support. The last step of the proof is to use the fact that H is translation invariant to show that S is in fact independent of the choice of c, and satisfies Eq. (A13). Notice that the distribution S will in general depend upon the choice of origin in M. Similarly, the operation of convolution depends upon the choice of origin. However, one finds that the distribution $g*S$ (for any test function g) is in fact independent of the choice of origin [as it must be to satisfy Eq. (A13)].

Equation (A13) can be generalized from scalar distributions to tensor distributions. For example, if $H_{aa'}$ is a translation-invariant two-point tensor distribution then there exists a one-point tensor distribution S_{ab} such that

$$H_{aa'}(\ \ ,\mu^a)=\mu^b * S_{ab} \tag{A14}$$

for all test fields μ^b. The proof consists essentially of taking components and using the result for scalar distributions, as we now indicate. Introduce four constant orthonormal vector fields e^a_i, $i=1,\ldots,4$ on M. Fix i and j and consider the two-point scalar distribution $H_{ij}\equiv e^a_i e^{a'}_j H_{aa'}$. (Recall that the multiplication of a tensor distribution by a smooth tensor field is well defined.) Let S_{ij} be the one-point scalar distribution satisfying Eq. (A13) with $H=H_{ij}$. Repeat for all i and j (choosing the same origin each time) and set

$$S_{ab}=\sum_{i,j=1}^{4} e^i_a e^j_b S_{ij}\,, \tag{A15}$$

where e^i_a is the dual basis to e^a_i at each point. One now easily verifies that the distribution S_{ab} so obtained satisfies Eq. (A14).

Now let $H_{aa'}$ be a Poincaré-invariant two-point distribution. Since, in particular, $H_{aa'}$ is translation invariant, there exists a one-point distribution S_{ab} satisfying Eq. (A14). However, in addition, it is easy to show that S_{ab} must now be Lorentz invariant (i.e., invariant under Lorentz transformations defined with respect to the origin chosen in the definition of S_{ab}). Thus, Poincaré-invariant two-point distributions are equivalent [in the sense of Eq. (A14)] to Lorentz-invariant one-point distributions. We now examine Lorentz-invariant distributions. We will only consider the subgroup G of the full Lorentz group that preserves a time orientation (but possibly reverses spatial parity). Thus G has two connected components.

Recall that every Lorentz-invariant tensor field on Minkowski space time can be expanded in a sum of terms, each consisting of a Lorentz-invariant function times a tensor product of the spacetime metric η_{ab} and position vector field x^a, e.g., $(x^m x_m)\eta_{ab}+x_a x_b$. We now claim that an analogous statement is true for distributions: *Every Lorentz-invariant tensor distribution can be expanded in a sum of terms, each consisting of a Lorentz-invariant scalar distribution times a product of* η_{ab} *and* x^a. The idea of the proof (say, for vector distributions) is the following. Let L_a be a Lorentz-invariant vector distribution. Let $\{V^i_a\}$ be a sequence of smooth vector fields such that the distributions defined by these vector fields converge to L_a. (The existence of such vector fields can be shown by taking components and using the well-known fact that every scalar distribution can be approximated by smooth functions—see, e.g., Ref. 35, p. 47). One now suitably Lorentz averages the vector fields $\{V^i_a\}$ to obtain a sequence of Lorentz-invariant vector fields which define distributions converging to L_a. Finally, one uses the fact that every Lorentz-invariant vector field is of the form $x_a f$ for some Lorentz-invariant function f, to show that $L_a=x_a L$ for some Lorentz-invariant scalar distribution L. Notice that no exception is needed for Lorentz-invariant distributions that have support at a point, e.g. $\nabla_a\delta_0=x_a(-\frac{1}{2}\nabla^2\delta_0)$ where δ_0 is the Dirac δ distribution at the origin.

We now complete our characterization of Poincaré-invariant two-point distributions by showing that every Lorentz-invariant scalar distribution (e.g., the coefficients in the expansion of a Lorentz-invariant tensor distribution) comes from a pair of distributions on the reals. To begin, let ψ: $M\to R$ be defined by $\psi(x)=\frac{1}{2}x^a x_a$. In order to distinguish the future time direction from the past, we define ψ_+ to be the restriction of ψ to the exterior of the past of the origin [i.e., $M-\overline{I^-(0)}$] and similarly define ψ_- to be the restriction of ψ to the exterior of the future of the origin [i.e., $M-\overline{I^+(0)}$]. Given a distribution s on R we obtain a Lorentz-invariant distribution on $M-\overline{I^-(0)}$ by setting

$$(\overleftarrow{\psi}_+ s)(f)=s(g)\,, \tag{A16}$$

where f is a test function with support in $M-\overline{I^-(0)}$ and g is the test function on R whose value at the point α is the integral of f over the hyperbola $\psi_+^{-1}(\alpha)$. We similarly define the distribution $\overleftarrow{\psi}_- s$. Methée[27] has proved the following important theorem: *For each Lorentz-invariant distribution* S *on* M, *there exists a unique pair of distributions* (s_+,s_-) *on* R (*which agree on all test functions with support in* R^+) *such that* $S=\overleftarrow{\psi}_+ s_+$ *on* $M-\overline{I^-(0)}$ *and* $S=\overleftarrow{\psi}_- s_-$ *on* $M-\overline{I^+(0)}$. The idea of the proof is similar to the one involving translation-invariant two-point distributions. Given S, one defines two distributions on R by $s_+(g) = S(c_+\overleftarrow{\psi}_+ g)$ and $s_-(g) = S(c_-\overleftarrow{\psi}_- g)$ where c_+ and c_- are appropriate cutoff functions to ensure compact support. One then verifies that since S is Lorentz invariant, s_+ and s_- are independent of the cutoff functions c_+ and c_-, which completes the proof. Notice that the distributions on R (s_+ and s_-) only determine the distribution S on $M - \{0\}$. In general, given a distribution defined only on $M - \{0\}$, there is no obvious way of extending it to a distribution on all of M.

3. The distribution H_λ

Let S be any Lorentz-invariant distribution which has support on the past light cone and dimensions cm^{-4}. By Methée's theorem, there exist two distributions (s_+, s_-) on R which give rise to S on $M - \{0\}$. Since S has support on the past light cone, s_+ must be the zero distribution and s_- must have support at the origin. But the only distributions on R with support at the origin are finite linear combinations of the Dirac δ distribution and its derivatives, i.e., δ_0, δ_0', etc. (Ref. 34, p. 100). Since S has dimensions cm^{-4}, s_- must be some multiple of δ_0'. Therefore on $M - \{0\}$, S must be some multiple of $\overleftarrow{\psi}_-\delta_0'$. That is, the only Lorentz-invariant distributions on $M - \{0\}$ which have support on the past light cone and dimensions cm^{-4} are multiples of $\overleftarrow{\psi}_-\delta_0'$.

We now consider extensions of $\overleftarrow{\psi}_-\delta_0'$ to distributions defined on all of M. Methée[27] has given one such extension:

$$H(f) = \lim_{\alpha \to 0^-} [g'(\alpha) + 2\pi \ln(-\alpha) f(0)] , \qquad \text{(A17)}$$

where $g(\alpha)$ is the integral of f over the past hyperbola $\psi_-^{-1}(\alpha)$ [cf. Eq. (15)]. However, one can check that H does not have the correct dimensions (due to the presense of the logarithm). This feature can be corrected by introducing a length scale λ (i.e., a parameter with dimensions of length) and defining a new extension H_λ of $\overleftarrow{\psi}_-\delta_0'$ by Eq. (A17) with $\ln(-\alpha)$ replaced by $\ln(-\alpha/\lambda^2)$. Thus the distribution H_λ is Lorentz invariant, has support on the past light cone, and dimensions cm^{-4}.

Notice that the distribution $\psi_-\delta_0$ is the distribution on $M - \{0\}$ that integrates a test function over the past light cone. This distribution has a natural extension to all of M which is simply the retarded Green's function G_{ret} for the massless wave equation. Therefore the Green's function H_λ entering the formula for the linearized quantum stress energy can be viewed as the derivative of the retarded Green's function G_{ret} for the massless wave equation. We now compare these two Green's functions.

We first consider the equations that these distributions satisfy. Of course, G_{ret} satisfies the wave equation

$$\nabla^2 G_{\text{ret}} = 4\pi\delta_0 . \qquad \text{(A18)}$$

Is there an analogous equation for H_λ? It turns out that there is:

$$\nabla_a(x^a H_\lambda) = 4\pi\delta_0 . \qquad \text{(A19)}$$

This equation can be verified by directly applying both sides to any test function f and using the explicit formula for H_λ given in Eq. (20). Can we use Eq. (A19) to obtain a relation between the linearized stress energy $\langle \dot{T}_{ab} \rangle$ and its source $\dot{A}_{ab}$, which is independent of the Green's function H_λ? Recall that if ρ is a test function and $\phi(x) = \int_M G_{\text{ret}}(x - x')\rho(x')$, then by virtue of Eq. (A18) alone, we conclude that $\nabla^2\phi = 4\pi\rho$. Now let $\dot{A}_{ab}$ be a test field and set

$$\langle \dot{T}_{ab}(x) \rangle = \hbar \int_M H_\lambda(x - x')\dot{A}_{ab}(x') . \qquad \text{(A20)}$$

Applying (A19) to (A20) we find

$$x^m \nabla_m \langle \dot{T}_{ab}(x) \rangle - 4\pi\hbar \dot{A}_{ab}(x) = \hbar \int_M H_\lambda(x - x') \times [x'^m \nabla'_m \dot{A}_{ab}(x')] . \qquad \text{(A21)}$$

The reason we have failed to obtain an expression which is independent of the Green's function H_λ can be traced to the appearance of the position vector field x^a in Eq. (A19).

We next consider the Fourier transform of these distributions. As is well known, the Fourier transform of G_{ret} is

$$\hat{G}_{\text{ret}} = -4\pi[\text{P}(1/k^2) + \tfrac{1}{2} i\pi(\delta^- - \delta^+)] , \qquad \text{(A22)}$$

where P denotes the principal value distribution and δ^- (δ^+) denotes the distribution which integrates a test function over the past (future) light cone in momentum space. Since H_λ is the "derivative" of G_{ret}, it may not be surprising that H turns out to be the "integral" of $\hat{G}_{\text{ret}}$:

$$\hat{H}_\lambda = -2\pi[\ln\lambda^2 |k^2| + 2\gamma - 1 + i\pi\theta_-(k)] , \qquad \text{(A23)}$$

where γ is Euler's constant and $\theta_-(k)$ is the step function that takes the value -1 inside the future light cone, $+1$ inside the past light cone, and zero elsewhere. Equation (A23) can be obtained by first noticing that $\hat{H}_\lambda$ must satisfy the Fourier transform of Eq. (A19) [which yields the general form of Eq. (A23)] and then actually computing a specific example (to fix the numerical coefficients).

Finally, we remark on the existence of related Green's functions. Of course, there exists an advanced Green's function to the wave equation as well as a retarded Green's function. Similarly, one can define an advanced Green's function analog of H_λ. Consider the distribution obtained by extending $\overleftarrow{\psi}_+\delta_0'$ (rather than $\overleftarrow{\psi}_-\delta_0'$, which was used to obtain H_λ) to a distribution on M. This distribution is Lorentz invariant, has support on the *future* light cone, and dimensions cm^{-4}. It is the advanced Green's function counterpart to H_λ, and would appear in the formula for the *out-vacuum* expectation value of the linearized quantum stress energy.

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[17]That is, a conserved tensor (other than a multiple of the spacetime metric) whose value at a point p depends only on the metric and its derivatives in an arbitrarily small neighborhood of p.

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[19]There exist many conserved local curvature tensors which even satisfy this additional condition. However, we will soon find that in the linearized case, the situation simplifies considerably.

[20]The proof of this statement will be given in the next section.

[21]We adopt the sign conventions and geometrical units (i.e., $G=c=1$, but, e.g., $\hbar\neq 1$) of C. W. Misner, Kip S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[22]Henceforth, we drop the vacuum state $|0\rangle$, and let $\langle \dot{T}_{ab}\rangle$ denote the in-vacuum expectation value of the quantum stress energy.

[23]The requirement that γ_{ab} have compact support is imposed for mathematical convenience. It can be weakened to allow perturbations which are suitably well behaved asymptotically.

[24]This argument (with only slight modifications) can be made into a proof of our earlier statement that the only linearized conserved local curvature tensors with less than six derivatives of the metric are $\dot{A}_{ab}$, $\dot{B}_{ab}$, and $\dot{G}_{ab}$.

[25]The meaning of Eq. (14) is the following: Let ξ^a and η^a be two vectors in the tangent space at x. Equation (14) states that the real number $\langle \dot{T}_{ab}(x)\rangle\, \xi^a\eta^b$ is equal to the real number obtained by first extending ξ^a and η^a to constant vector fields on Minkowski spacetime, next applying the distribution G_1 to $\dot{A}_{ab}\xi^a\eta^b$ and G_2 to $\dot{B}_{ab}\xi^a\eta^b$, and finally multiplying by $\hbar$ and taking the sum.

[26]The volume element to be used in the integral is $(-\alpha)^{-1/2}$ times the natural induced volume element on the hyperbola. The factor $(-\alpha)^{-1/2}$ is inserted so that the limit $\alpha\to 0$ yields a nonzero volume element on the light cone.

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[29]Since we now have a length scale λ in the theory, we can add certain tensors to the quantity in brackets in Eq. (22) without violating properties 1–5. These tensors consist of any linearized conserved local curvature tensor multiplied by appropriate powers of λ so that it has dimensions cm^{-4}, e.g., $\lambda^{-2}\dot{G}_{ab}$, $\lambda^2\nabla^4\dot{G}_{ab}$, etc. Indeed, in the context of the nonlinearized theory, this was the reason we could not invoke dimensional considerations to rule out higher derivative terms. However, we now impose our additional condition (see Sec. II) that the stress energy not depend on sixth-order derivatives (or higher) of the metric. Thus the only remaining freedom is to add multiples of $\lambda^{-2}\dot{G}_{ab}$, which only trivially affects solutions to the linearized semi-

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