Light-fermion mass hierarchy and grand unification

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(Received 18 October 1979)

We show that the existence of a hierarchy of masses among light fermions, with typical mass ratios (α/π) (where α is the unified gauge coupling at the unification mass scale), is a natural feature of a certain class of grand unification schemes that have recently been discussed.

I. INTRODUCTION

One of the outstanding puzzles that confronts theorists at the present time is the enormous range spanned by the masses of the known elementary fermions. Between the mass of the lightest (excepting the neutrinos)—the electron—and the mass of the heaviest presently known—the b quark—lie four orders of magnitude.

This impressive fact has suggested to many theorists[1] that the masses of the lighter fermions may be higher-order radiative effects. For example, it is an old speculation[2] that the empirical relation $m_e = O(\alpha m_\mu)$ may be explained by the hypothesis that the electron mass arises from one-loop diagrams like the one shown in Fig. 1 (where the wavy line in that diagram is the propagator of some "horizontal" gauge boson). More generally one might suppose that the large ratios between the masses of fermions of successive generations may be due to a similar mechanism, i.e., there could be a hierarchy in which the fermion masses of one generation are of order α times those of the next higher generation since they arise as higher-order effects in some horizontal interaction.

Recently Georgi[3] and others have suggested that by embedding the SU(5) unified gauge group of Georgi and Glashow[4] in yet higher unified groups such as SU(N) ($N>5$) one might avoid the replication of families of fermion representations. [In the SU(5) model a "family" consists of a left-handed $\overline{5}$ and a left-handed 10 of fermions.] These higher gauge groups provide us with just the kind of "horizontal" interactions which could produce the hierarchical pattern of fermion masses we have noted above. (Note that by "hierarchy" here we are referring to the hierarchy of masses among the known light fermions, *not* that between known light fermions and superheavy fermions.) In the next section we outline a somewhat general scenario in which such a hierarchy could arise in a grand unification scheme.

II. GENERAL SCENARIO

We adopt the basic philosophy of Georgi in Ref. 3. Specifically, we assume the gauge group of the world is SU(N). The fermions are all contained in a set of SU(N) representation which we will refer to henceforth as the "primitive representations." We assume that these primitive representations are left-handed, totally antisymmetric SU(N) tensors. This is plausible as no quarks which are 6's or 8's of color have yet been observed. For notational convenience a left-handed primitive representation with p antisymmetric contravariant indices will be referred to as a $[p]$; e.g., $T_L^{[\alpha\beta\gamma\delta]}$ is a $[4]$.

We assume (following Georgi) that SU(N) is broken down to SU(5) by some mechanism (probably "dynamical") which generates fermion masses which are "ultraheavy" [to distinguish from the "superheavy" SU(5) breaking]. These masses are SU(5) singlets but are SU(N) noninvariant. This breaking mechanism also gives ultraheavy masses to the gauge bosons which change SU(5) indices to non-SU(5) indices (denoted by $\tilde{W}^\alpha_A$, where $\alpha=1,\ldots,5$ and $A=6,\ldots,N$). We assume the validity of Georgi's principle, namely, all fermions which *can* acquire (ultraheavy) SU(5)-singlet masses *do* acquire them. That is, there is an imperative for a 10_L and a $\overline{10}_L$ to combine to form an SU(5)-invariant mass term (and likewise for a 5_L and a $\overline{5}_L$). After this stage of breaking, therefore, we will be left with an exact SU(5) world with some excess 10_L (or $\overline{10}_L$) and excess $\overline{5}_L$ (or 5_L) representations of fermions which are not ultraheavy (are indeed massless at this stage). We call these "light fermions." As Georgi has observed, if the primitive fermion representations are anomaly free then the number of these light-fermion $\overline{5}_L$'s is equal to the number of light-fermion 10_L's. In other words, the light fermions are arranged in a certain number of SU(5) "families" such as seem to be seen in nature.

Henceforth, we denote a light-fermion $\overline{5}_L$ which

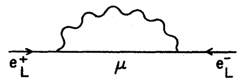

FIG. 1. Old speculation on origin of masses of light fermions.

was part of a primitive $[p]$ by $\bar{5}_L[p]$ (and similarly for $10_L[p]$, etc.). An ultraheavy $\bar{5}_L$ that is part of a primitive $[p]$ is denoted by $\bar{5}_L\{p\}$. (Actually, of course, a given $\bar{5}_L$ of light fermions may be a linear combination like $\sum_i C_i \bar{5}_L[p_i]$, i.e., it may come from *several* primitive representations. We assume otherwise in the rest of the paper only to simplify notation and give a clearer picture of the basic idea. It is a simple matter to generalize to the more realistic case. The qualitative features are the same in the general case.)

The breaking of SU(5) down to SU(3) × SU(2) × U(1) and then to $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{\mathrm{QED}}$ happens, we assume, in a manner similar to that usually considered.[4] There may be a 24 of Higgs bosons that breaks SU(5) down to SU(3) × SU(2) × U(1) and gives "superheavy" masses to the gauge bosons that connect color and flavor ($\tilde{W}^i a$, $i = 1, 2$, $a = 3, 4, 5$) but leaves the light fermions massless. We make the usual assumption that the light fermions develop masses from the same symmetry-breaking mechanism that breaks SU(2) × U(1) and makes the $W^\pm$ and Z^0 heavy. In particular, we assume that this mechanism is the Higgs mechanism, with an elementary scalar field developing an SU(2) × U(1)-noninvariant vacuum expectation value. It seems simplest to make this Higgs scalar transform under SU(N) like a fundamental representation. We denote it therefore by φ^α ($\alpha = 1, \ldots, N$; $\langle\varphi^2\rangle \neq 0$). Since φ^α is assumed to be elementary, its Yukawa couplings must respect SU(N) invariance. This is a very important point. It means that a $\bar{5}_L[p]$ and a $10_L[q]$ can have a Yukawa to φ^α of the type which, in the Georgi-Glashow SU(5) model, produces lepton and down-quark masses only if $p + q - 1 = N$. Similarly a $10_L[p]$ and a $10_L[q]$ couple as in the Georgi-Glashow model only if $p + q + 1 = N$. So, in general, not all Yukawa couplings which would give masses to the light fermions are allowed by SU(N) invariance. In fact, for such a coupling to be allowed would be somewhat fortuitous. To repeat, some (but in general not all) light fermions can obtain mass at tree level in the (broken) theory. These tree-level masses are represented pictorially in Fig. 2. Those light fermions which are forbidden by SU(N) invariance to acquire tree-level masses *will*, however, obtain *calculable* masses at one-loop (or higher-loop) levels due to the breaking of SU(N) by the ultraheavy dynamical mass terms. Let us now see how this happens. As noted above,

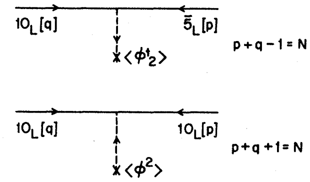

FIG. 2. Origin of "tree-level" light-fermion masses in present scenario. Fermions are **"light fermions"** such as have been observed. The Higgs field is in a fundamental representation of SU(N). The conditions $p + q \pm 1 = N$ come from SU(N) invariance.

Georgi's principle requires that $(\overline{10}_L)$'s and (10_L)'s [and (5_L)'s and (5_L)'s, etc.] combine to form SU(5)-singlet, but SU(N)-noninvariant, ultraheavy mass terms. Such a mass term typically can connect a $\overline{10}_L\{p\}$ to a $10_L\{q\}$ with $p \neq q$. We represent these ultraheavy masses pictorially in Fig. 3 by blobs. As we have already pointed put, there are also ultraheavy SU(N) gauge bosons in the theory which connect the light fermion multiplets $\bar{5}[p]$, etc. to the ultraheavy fermion multiplets $\overline{10}\{p\}$, etc. (denoted $\tilde{W}^\alpha_A$, $\alpha = 1, \ldots, 5$, $A = 6, \ldots, N$). Therefore, one expects one-loop mass diagrams for the light fermions of the sort shown in Fig. 4. The internal fermion lines in these diagrams are propagators of ultraheavy fermions of typical mass M_F. The wavy lines are propagators of ultraheavy gauge bosons of typical mass $M_{\tilde{W}}$. It is readily apparent that the one-loop mass terms shown in Fig. 4 are then of magnitude

$$M(1\ \text{loop}) \sim \frac{\alpha_u}{\pi}\left(\frac{M_F}{M_{\tilde{W}}}\right)^2 \ln\left(\frac{M_{\tilde{W}}}{M_F}\right)^2 M(\text{tree}), \tag{1}$$

where M(tree) is a typical light-fermion mass that comes from the diagrams shown in Fig. 2. α_u is the SU(N) grand unified gauge coupling evaluated at the SU(N) unification mass scale [probably $O(10^{-1})$]. We expect $M_F/M_W = O(1)$ [by analogy with the known breaking of SU(2) × U(1) where $M_b/M_{W_\pm}$ and $M_t/M_{W_\pm}$ are not very different from 1]. We end up then with

$$\frac{M(1\ \text{loop})}{M(\text{tree})} \sim \frac{\alpha_u}{\pi} \sim 10^{-1} \text{ to } 10^{-2}. \tag{2}$$

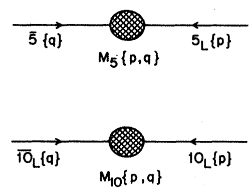

FIG. 3. Diagrammatic representation of ultraheavy masses that break SU(N) down to SU(5). They are SU(5) singlets.

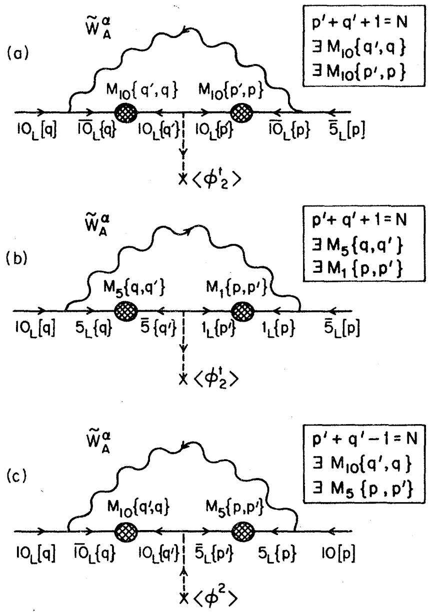

FIG. 4. One-loop mass terms for light fermions. The internal fermion line is ultraheavy. The gauge boson is also ultraheavy. In boxes are conditions for such one-loop mass terms to exist.

To recapitulate, some light fermions are allowed by SU(N) to get masses at tree level from $\langle\varphi^\alpha\rangle$. But, in general, others are forbidden to by SU(N) and receive calculable masses of $O((\alpha_u/\pi)M(\text{tree}))$ from loop diagrams due to the breaking of SU(N) by the ultraheavy masses. It is clearly possible that the pattern of ultraheavy masses will not permit some light fermions to get masses even by the diagrams shown in Fig. 4 (e.g., there may be no ultraheavy mass connecting certain primitive representations). In this case these light fermions may only get mass at the two-loop level of $O((\alpha_u/\pi)^2 M(\text{tree}))$. For example, it may well be that the t, b, and τ^- have tree-level masses, while the c, s, and μ^- have one-loop masses and the u, d, and e^- have two-loop masses.

One can make some statements about what kinds of SU(N) primitive fermion representations one would like, given a particular elementary Higgs representation. For example, suppose (as above) that the elementary Higgs is in an SU(N) fundamental representation φ^α. Then, as we have seen, tree-level masses for up-type ($Q=\frac{2}{3}$) quarks require the existence of representations $10_L[p]$ and $10_L[q]$ satisfying $p+q+1=N$. (This is a true statement even if the light-fermion families are mixtures $\sum_i c_i 10[p_i]$.) So if an up-type quark has a tree-level mass then there exist primitive representations $[p]$, $[q]$ with $p+q+1=N$. Furthermore, if this up-type quark mass term is "flavor diagonal" i.e., if the left-handed components of the quark and antiquark are in the *same* 10_L then $p=q=(N-1)/2$. More specifically, if the t and b have tree-level masses and $\binom{t}{b}_L$ and $(\bar{t})_L$ are in the same $10_L[p]$ and $\bar{b}_L$ is in a $\bar{5}_L[q]$, then we must have $p=(N-1)/2$ and $q=(N+3)/2$. (For example, in the trivial case $N=5$, which is just Georgi and Glashow's model, $p=2$ and $q=4$ which tell us that we need to have $\bar{5}$ and 10 representations—as we already knew.) It would be desirable therefore to require the existence of primitive representations $[(N-1)/2]$ and $[(N+3)/2]$.

III. EXAMPLES

To make these ideas more concrete, consider $N=9$. We want to find a set of primitive representations which are anomaly-free and give three (at least three families of light fermions. To do this we must relax one of Georgi's[3] aesthetic criteria and allow some replication of primitive representations (we still satisfy Frampton's[3] criterion, however). One solution (there are a whole family of solutions, of course) is to have eleven [8]'s, one [6], two [4]'s, and two [2]'s. This yields three families of light fermions. Further, this satisfies our criterion of having an $[(N-1)/2]$ and an $[(N+3)/2]$. It is not a trivial matter to discover *which* three $(\bar{5}_L)$'s and (10_L)'s remain light after SU(9) breaks down to SU(5). Instead of trying to solve this problem let us consider two cases to illustrate our ideas. First, suppose the light fermions are in three $\bar{5}_L[6]$ and three $10_L[4]$ representations. Then all of the quarks and leptons can acquire mass at tree-level from $10_L[4]$-$10_L[4]$-φ^α and $10_L[4]$-$\bar{5}_L[6]$-φ_α couplings. So, no hierarchy among quark and lepton masses will result. Second, let us consider what would happen if the light fermions were in three $10_L[4]$ and three $\bar{5}_L[8]$ representations. Then the up-type ($Q=\frac{2}{3}$) quarks can enjoy tree-level masses while the leptons and down-type quarks all must receive mass at the one-loop level [in fact, from a diagram of the type shown in Fig.

TABLE I. Anomalies of the $[m]$ representations of SU(N) normalized so that the anomaly of [1] is 1.

m \ N	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1
2	−1	0	1	2	3	4	5	6
3		−1	−1	0	2	5	9	14
4			−1	−2	−2	0	5	14
5				−1	−3	−5	−5	0
6						−4	−9	−14
7						−1	−5	−14
8							−1	−6
9								−1

4(b)].

It should be obvious that if the $\overline{5}_L$ representations are different (from different primitive representations), then the down-type quarks and leptons can have a hierarchy of masses. Similarly, if the 10_L are different the up-type quarks can have a hierarchy. Obviously, much more investigation remains to be done to discover which (if any) groups, sets of primitive representations, and choices of elementary Higgs representations will lead to a hierarchy of light-fermion masses resembling that observed in nature, and, of these, which is simplest.

ACKNOWLEDGMENTS

I would like to thank A. Zee, G. Segre, and H. A. Weldon for helpful discussions.

APPENDIX A

We wish to mention a trivial but useful fact about the anomalies of totally antisymmetric representations of SU(N). Table I is a table of anomalies of the $[m]$ representations of SU(N) normalized so that the anomaly of $[1]$ is 1. The point to be noticed is that this table can be generated in the same way as Pascal's triangle. Each entry is the sum of the entry to its left and the entry above and to the left. This saves a certain amount of arithmetic.

There is also a Pascal triangle for the number of SU(5) families that result from a given $[m]$ representation of SU(N) under the assumption that the total set of representations is anomaly-free (see Table II). For illustration consider the example in the text. There $N=9$ and there are eleven $[8]$'s, one $[6]$, two $[4]$'s and two $[2]$'s. Then

$$\text{No. of families} = 11\times(1)+1\times(6)+2\times(-3)+2\times(-4)$$
$$=3\,,$$
$$\text{Anomaly} = 11\times(-1)+1\times(-9)+2\times(5)+2\times(5)$$
$$=0\,.$$

TABLE II. Number of SU(5) families that result from a given $[m]$ representation of SU(N) under the assumption that the total set of representations is anomaly-free.

m \ N	5	6	7	8	9	10
1	−1	−1	−1	−1	−1	−1
2	0	−1	−2	−3	−4	−5
3	0	0	−1	−3	−6	−10
4	+1	+1	+1	0	−3	−9
5		+1	+2	+3	+3	0
6			+1	+3	+6	+9
7				+1	+4	+10
8					+1	+5
9						+1

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[2]See especially the papers of Georgi and Glashow, Vinciarelli, and Barr and Zee of Ref. 1.

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