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τ radiative decay and pion structure form factors

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The decay mode of the τ lepton $\tau^- \to \nu_\tau \pi^- \gamma$ is calculated in terms of the axial-vector and vector structure-dependent form factors of the pion, which also describe the radiative π decay $\pi^- \to l^- \bar{\nu}_l \gamma$. The isospin-rotated vector form factor is related (as usual) to the decay $\pi^0 \to 2\gamma$, as well as to $e^+e^- \to \pi^0\gamma$. These reactions thus serve to probe the isospin structure of the vector form factor, as well as the momentum-transfer dependence of both form factors. The numbers expected in the τ radiative decay are calculated using vector-meson-dominance estimates for the form factors.

We calculate the weak radiative decay for the τ lepton $\tau^- \to \nu_\tau \pi^- \gamma$, assuming the τ^- is a sequential lepton of spin $\frac{1}{2}$, has a mass $M = 1782$ MeV/c^2, couples with the full strength $G_F/\sqrt{2}$ to the standard form of $V-A$ current, and that its accompanying neutrino ν_τ is massless. These assumptions are consistent with present experiments.[1] The transition $\tau^-(p) \to \nu_\tau(q) + \pi^-(K) + \gamma(k)$ is described in terms of the same structure-dependent vector and axial-vector form factors, v and a, respectively, which describe the radiative pion decay[2] $\pi^- \to e^- \nu_e \gamma$ under the foregoing assumption of lepton universality. However, $v(s)$ and $a(s)$ in the present case are functions of $s = (p-q)^2 = (K+k)^2$, which varies over the kinematic range $\mu_{\pi^-}^2 \le s \le M^2$, where μ_{π^-} = charged-pion mass. In the π^- radiative decay, the argument of the form factors varies between m_e^2 and $\mu_{\pi^-}^2$, a very small range, and they have been well approximated by $v(0)$ and $a(0)$. These structure functions of the pion are of intrinsic interest because they represent a description of the pion structure by nonhadronic probes (the weak and electromagnetic currents). Thus, if v and a are calculated in a quark model, they will depend on the effective quark masses, which in turn can depend on s. In more general terms, the s dependence of v and a will give important information on the bound-state wave function of the pion.[3]

We use the conventions of Bjorken and Drell.[4] The width for the nonradiative decay $\tau^-(p) \to \nu_\tau(q) + \pi^-(K)$ is given by[5]

$$\Gamma_{\nu\pi} = (16\pi)^{-1}(G_F \cos\theta_C f_\pi)^2 M^3 (1-\Delta)^2 ,$$

where G_F is the Fermi weak decay constant, θ_C is the Cabbibo angle, $f_\pi = 0.945\mu_{\pi^-}$ is the pion-decay constant, and $\Delta \equiv (\mu_{\pi^-}/M)^2 = 6.13 \times 10^{-3}$. The invariant amplitude for the radiative decay is

$$\begin{aligned} M_{\nu\pi\gamma} = {} & ie\frac{G_F}{\sqrt{2}}\cos\theta_C f_\pi M\bar{u}(q)(1+\gamma_5) \\ & \times\left[\frac{\epsilon\cdot p}{k\cdot p} - \frac{\epsilon\cdot K}{k\cdot K} + \frac{\not{\epsilon}\not{k}}{2k\cdot p}\right]u(p) \\ & + ie(G_F/\sqrt{2})\cos\theta_C a(s) F_{\mu\nu}K^\mu L^\nu \\ & - e(G_F/\sqrt{2})\cos\theta_C v(s)\tilde{F}_{\mu\nu}K^\mu L^\nu . \end{aligned} \tag{1}$$

Here, e is the electric charge of the proton, ϵ_μ is the polarization four-vector of the photon, $L^\nu \equiv \bar{u}(q)\gamma^\nu(1-\gamma_5)u(p)$ is the τ^- leptonic current, $F_{\mu\nu} \equiv k_\mu\epsilon_\nu - k_\nu\epsilon_\mu$ is the electromagnetic field tensor, and $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is its dual. The first term corresponds to inner bremsstrahlung (denoted below by subscript B), associated with the emission of the photon from the external τ^- or (structureless) π^- lines, and includes the contact term required by gauge invariance. The second and third terms correspond to the axial-vector (A subscript below) and vector (V subscript below) structure terms. The form factors a and v are defined by

$$\begin{aligned} \langle\pi^-(K)\gamma(k,\epsilon)|A_\mu(0)|0\rangle &= +iea(s)F_{\mu\nu}K^\nu , \\ \langle\pi^-(K)\gamma(k,\epsilon)|V_\mu(0)|0\rangle &= +ev(s)\tilde{F}_{\mu\nu}K^\nu , \end{aligned} \tag{2}$$

where A_μ and V_μ are the hadronic axial-vector and vector currents, respectively, and a and v are real by time-reversal arguments.

It is convenient to work with dimensionless variables. $x \equiv 2E_k/M$ and $y = 2E_K/M$, where E_k and E_K are the photon and pion energies. We work in the τ rest frame where the kinematically allowed region is given by

$$\begin{aligned} & 2\Delta^{1/2} \le y \le 1+\Delta , \\ & 1-\tfrac{1}{2}y-(y^2/4-\Delta)^{1/2} \le x \le 1-\tfrac{1}{2}y+(y^2/4-\Delta)^{1/2} . \end{aligned} \tag{3}$$

The neutrino variables are integrated over, and the differential spectrum with respect to x and y obtained. This is expressed as a dimensionless branching ratio with respect to the nonradiative

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mode $\tau^- \to \pi^- \nu_\tau$.

$$R \equiv \frac{1}{\Gamma_{\nu\pi}} \frac{d^2\Gamma_{\nu\pi\gamma}}{dx dy} \equiv \sum_i R_i , \tag{4}$$

where $i = BB, VV, AA, BV, BA, VA$; and BB is pure bremsstrahlung, BV is the bremsstrahlung-vector interference term, etc. The result is

$$\begin{aligned}
&R_{VV} = C\left(\frac{v(s)}{v(o)}\right)^2 F_{VV}, \quad R_{BB} = \frac{\alpha}{2\pi}(1-\Delta)^{-2} F_{BB},\\
&R_{AA} = C\left(\frac{a(s)}{v(o)}\right)^2 F_{AA}, \quad R_{AB} = D\frac{2a(s)}{v(o)} F_{AB},\\
&R_{AV} = C\frac{2a(s)v(s)}{[v(o)]^2} F_{AV}, \quad R_{VB} = -D\frac{2v(s)}{v(o)} F_{VB},\\
&F_{AA} = F_{VV} = [x+y-1-\Delta][x(1-\Delta-x)+y(1+\Delta-y)]\\
&\qquad -2\Delta x(1+\Delta-y),\\
&F_{AV} = [x+y-1-\Delta][x(1-\Delta-x)-y(1+\Delta-y)],\\
&F_{VB} = x^{-1}(x+y-1-\Delta)(1+\Delta-y),\\
&F_{AB} = x^{-1}y(1+\Delta-y) - x^{-1}(x+y-1-\Delta)(1-\Delta-x)\\
&\qquad +(1-\Delta-x) - 2\Delta(1+\Delta-y)(x+y-1-\Delta)^{-1},\\
&F_{BB} = -2\Delta(2-x-y)(x+y-1-\Delta)^{-2} - 2x^{-2}(1-x-\Delta)\\
&\qquad +2x^{-1}y(2-x-y)(x+y-1-\Delta)^{-1} - x^{-1}(1-\Delta-x)\\
&\qquad -x^{-1}y(1+\Delta-y)(x+y-1-\Delta)^{-1}\\
&\qquad +(1-\Delta-x)(x+y-1-\Delta)^{-1},
\end{aligned} \tag{5}$$

$$\begin{aligned}
&C = (4\pi^2\alpha)^{-1}\Gamma_{\pi^0} M^4 \mu_{\pi^0}^{-3} f_\pi^{-2}(1-\Delta)^{-2} = 6.7\times 10^{-3},\\
&D = (\Gamma_{\pi^0}/8\pi^3\mu_{\pi^0})^{1/2} M^2 \mu_{\pi^0}^{-1} f_\pi^{-1}(1-\Delta)^{-2} = 2.8\times 10^{-3},\\
&\alpha = (137)^{-1}, \quad \mu_{\pi^0} = 135\ \mathrm{MeV}/c^2, \quad \Gamma_{\pi^0} = 7.95\ \mathrm{eV},
\end{aligned} \tag{6}$$

and in obtaining C and D, the conserved-vector-current hypothesis at $s=0$ was used to relate $v(0)$ to the neutral-pion decay. The relation is $v(0) = (2\Gamma_{\pi^0}/\pi\alpha^2\mu_{\pi^0}^3)^{1/2}$.

It is apparent that R depends on the two unknown functions $v(s)$ and $a(s)$. However, $s = M^2(x+y-1)$ so v and a are functions of $x+y$ only, and are constant on the straight lines $x+y=$ constant in the x,y plane. Consequently, two measurements at different (x,y) on the same line allow v and a to be determined for that particular s. Because of the quadratic dependence, the solution of the equations will involve ambiguities; these can be resolved if more than two points are measured on the same line. For example, five independent measurements give five linear equations for the unknowns v^2, a^2, av, v and a. v and a can be found unambiguously and the three remaining equations are redundant, and provide consistency checks. The procedure is repeated for different values of $x+y$ to obtain the s dependence.

It is interesting to compare this situation to the radiative π decay.[2] In the latter, the bremsstrahlung contribution could be suppressed by going to the electronic mode (the electron mass supplies the suppression) and hard photons. Here, there is no small lepton mass, and all six terms must be considered. For the π decay, the s dependence could be neglected and $v(0)$ was directly related to $\pi^0 \to 2\gamma$. So the π decay could be parametrized by the single unknown number $a(0)/v(0)$. Here, as seen above, the decay is described by the two functions $v(s)$ and $a(s)$. In an interesting recent publication,[6] the question of isospin breaking has been raised,[7] and the suggestion made for also measuring directly $v(0)$ as well as $a(0)$ in the radiative π decay.

For the radiative τ decay, the function $v(s)$ can also be related by an isorotation to a physically accessible reaction. This is $e^+e^- \to \pi^0\gamma$ at the same value of s, where s here is the center-of-mass energy squared. This reaction can be described by the form factor $v_0(s)$, defined by

$$\langle \pi^0(K)\gamma(k,\epsilon)\,|J_\mu^{\mathrm{em}}(0)|\,0\rangle = -e v_0(s)\tilde{F}_{\mu\nu}K^\nu ,$$

and where the 0 subscript denotes the third component of isospin. We can write $v_0(s) = \frac{1}{2}[v_0^1(s) + v_0^0(s)]$, where the superscript denotes the total isospin. $v(s)$ corresponds, of course, to isospin 1, and isospin symmetry implies[8] $v(s) = \sqrt{2}v_0^1(s)$. The spin-averaged cross section in the center of mass is ($z = \cos\theta$, $k=$photon momentum, $p = \frac{1}{2}\sqrt{s}$)

$$\frac{d\sigma}{d\Omega}(e^+e^- \to \pi^0\gamma) = \tfrac{1}{2}\pi\left(\frac{\alpha k}{p}\right)^3 v_0^2(1+z^2) \tag{7}$$

and allows v_0 to be measured directly, except for sign. If isospin symmetry is assumed and $v_0^1(s)$ is obtained from $v(s)/\sqrt{2}$, then $v_0^0(s)$ can be extracted. These might then be compared with theoretical models, for example vector-meson dominance (VMD), where ρ, ρ' are expected to contribute to v_0^1, and ω, ϕ to v_0^0. We note that $v_0^1 = v_0^0$ is expected only at $s=0$.[6]

To estimate the numbers expected in $\tau^- \to \nu_\tau \pi^- \gamma$, we have estimated $v(s)$ and $a(s)$ using VMD, with $v(s)$ ρ,ρ'-dominated and $a(s)$ A_1-dominated.

$$v(s) = v(0)\left\{\frac{f_{\rho\pi\pi}}{f_\rho}\left[\left(1-\frac{s}{m_\rho^2}\right)^2 + \frac{\Gamma_\rho^2}{m_\rho^2}\right]^{-1/2} + \frac{f_{\rho'\pi\pi}}{f_{\rho'}}\left[\left(1-\frac{s}{m_{\rho'}^2}\right)^2 + \frac{\Gamma_{\rho'}^2}{m_{\rho'}^2}\right]^{-1/2}\right\}, \tag{8a}$$

$$a(s) = \frac{2f_\pi}{m_{A_1}^2}\left[\left(1-\frac{s}{m_{A_1}^2}\right)^2 + \frac{\Gamma_{A_1}^2}{m_{A_1}^2}\right]^{-1/2}, \tag{8b}$$

where $f_{\rho\pi\pi}/f_\rho$ is 1.20 from experiment[9] and $f_{\rho'\pi\pi}/f_{\rho'}$ is determined to be -0.18 from the normalization at $s=0$. For $a(0)/v(0)$, however, the experimental and theoretical situation is far from clear[10,11] and we have chosen the form of Eq. (8b) for $a(s)$ [i.e., $a(0)/v(0) \simeq 1$]. It turns out that though the bremsstrahlung contribution favors small x and therefore hard photons are desired in radiative π decay to accentuate the structure terms, for the τ decay this is not necessarily the case.

$[v(s)/v(0)]^2 \simeq 25$ near the ρ corresponding to $x+y=1.19$, so R_{VV} can be considerably larger than R_{BB} in the ρ band for x not too small. For example, at $x=0.80$, $y=0.39$, $R_{VV} \simeq 11\times10^{-3}$, $R_{BB} \simeq 3\times10^{-3}$, $R_{BV} \simeq -4\times10^{-3}$, $R_{AB} \simeq 4\times10^{-3}$, $R_{AA} \simeq 2\times10^{-3}$, and $R_{VA} \simeq -2\times10^{-3}$. If we integrate over the ρ band $[s\pm\Delta s=(m_\rho \pm \Gamma_\rho)^2$ or $1.12 \lesssim x+y \lesssim 1.27$, say] we estimate $\int_\rho R_{VV} \simeq 9.2\times10^{-4}$, $\int_\rho R_{BB} \simeq 8.4\times10^{-4}$, and $\int_\rho R \simeq 2.4\times10^{-3}$.

With $\Gamma(\tau^-\to\nu_\tau\pi^-)/\Gamma(\tau^-\to\text{all})=0.083$, the total branching ratio (B) in the ρ band is 2.0×10^{-4}. Assuming the accompanying τ^+ would be detected by a signal of e^+ or μ^+ with $B \simeq 2\times0.18=0.36$, we estimate an effective detectable B of $\simeq 7\times10^{-5}$ or about one detectable radiative event in the ρ band for every $\simeq 1.4\times10^4$ τ pairs produced. If we consider only $\int_\rho R_{VV}=9.2\times10^{-4}$ in this band we obtain

$$(B)_{VV}=\frac{\Gamma(\tau^-\to\nu_\tau\rho\to\nu_\tau\pi^-\gamma)}{\Gamma(\tau^-\to\text{all})}\simeq 7.6\times10^{-5}\,.$$

The experimental data[1,8] give

$$\frac{\Gamma(\tau^-\to\nu_\tau\rho^-)}{\Gamma(\tau^-\to\text{all})}\times\frac{\Gamma(\rho^-\to\pi^-\gamma)}{\Gamma(\rho^-\to\text{all})} \simeq [0.24]\times[2.4\times10^{-4}]=5.8\times10^{-5}$$

which is in good agreement with our estimate. (The ρ' contribution is small.)

Similarly, since $[a(s)/v(0)]^2 \simeq 16$ near the A_1, R_{AA} may be significantly larger than R_{BB} in the A_1 band $[s\pm\Delta s=(m_{A_1}\pm\Gamma_{A_1})^2$ or $1.20 \lesssim x+y \lesssim 1.62]$. Integrating over the A_1 band gives us $\int_{A_1} R_{AA} \simeq 1.9\times10^{-3}$, $\int_{A_1} R_{BB} \simeq 4.8\times10^{-4}$, and $\int_{A_1} R \simeq 4.1\times10^{-3}$ and we estimate one detectable radiative event for every $\simeq 8.3\times10^3$ τ pairs produced. Since

$$\Gamma(\tau^-\to\nu_\tau A_1)/\Gamma(\tau^-\to\text{all})=0.104$$

has been seen, we can also predict the branching ratio

$$\Gamma(A_1\to\pi^-\gamma)/\Gamma(A_1\to\text{all})=1.5\times10^{-3}\,.$$

At $s=15$ GeV^2 (just above τ threshold), $\sigma(e^+e^-\to\tau^+\tau^-) \simeq 3.3\times10^{-33}$ cm^2. A luminosity of 10^{31} cm^{-2} sec^{-1} and 15 days running time will produce 4.3×10^4 τ pairs or three detectable radiative events in the ρ band and five events in the A_1 band. For the overlapping ρ-A_1 band ($1.12 \lesssim x+y \lesssim 1.62$) seven detectable events can be expected.

When we now integrate over all the phase space ($1.0 \lesssim x+y \lesssim 2.0$) with a photon-energy cut of 100 MeV, we get $\int R \simeq 1.4\times10^{-2}$. This implies that we need about 20 hours running time with a luminosity of 10^{31} cm^{-2} sec^{-1} to produce one detectable radiative event in the $\tau^-\to\nu_\tau\pi^-\gamma$ process.

To measure separately v and a as functions of s would still require correspondingly greater luminosity or running time. (We do not enter here into the question of identification and background suppression.) By considering the facts that $\int R$ is large and that W. Bacino *et al.*[12] have recently observed a clear signal of the decay $\tau^-\to\nu_\tau\pi$ based on a sample of 41 events of the type, $e^+e^-\to e^\pm x^\mp (x\neq e)$ and no observed photons, it should be clear that a first generation experiment, in which $\tau^-\to\nu_\tau\pi^-\gamma$ is simply detected, should be feasible now with existing beams and detectors. A second generation experiment, in which $v(s)$ and $a(s)$ are measured near the ρ or A_1 bands might just be feasible if present facilities are pushed to the limit, and if VMD estimates are valid. Since the higher region $1.62 < x+y < 2$ contributes only $<2\%$ to the total branching ratio, the measurement of $v(s)$ and $a(s)$ for larger s (approaching the kinematic limit $M^2 \simeq 3.2$ GeV^2) is probably not possible with present luminosities, although there is probably an enhancement at $s \simeq 2.6$ GeV^2 due to $\rho'(1600)$.

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[3]This is being investigated further. After this paper was submitted for publication, we were informed (private communication) that R. Montemayor and M. Moreno are also investigating the $\pi\nu\gamma$ decay mode of the τ lepton, with the pion structure form factors calculated in a quark-loop model.

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