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Symmetry breaking and the πK amplitudes in the unphysical region

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We apply two different methods of analytic continuation (fixed-t and hyperbolic dispersion relations with discrepancy) to determine the expansion parameters of the πK amplitudes in the unphysical region near the symmetry point. Part of the available phase-shift data shows inconsistencies with crossing symmetry. The values we obtain for the parameters and for the so-called σ term are then compared with predictions that we derive from different models of chiral-symmetry breaking.

I. INTRODUCTION

It has been well known for many years that the algebra of hadronic currents provides a valuable tool for predicting low-energy properties of scattering amplitudes. These predictions are strictly valid only in regions that are experimentally not accessible by definition. In the last years, however, the understanding and practicing of analytic continuation procedures has been enormously improved (cf. the review of Ciulli *et al.*[1]) so that it is just a technical problem whether the point at which one wants to know the amplitude is physical or not. The condition that a continuation gives sensible results with reasonable errors is of course a good experimental knowledge of the physical-region amplitude.

It has taken some time from putting the theoretical question (How good are the current-algebra predictions for amplitudes?) until experiments improved sufficiently to allow for the phenomenological answer. This is especially true for meson-meson scattering data, which cannot be measured directly but have to be determined by analytic continuation. For pion-pion scattering one has obtained a very precise knowledge and theoretical understanding of the low- and medium-energy region.[2] For pion-nucleon scattering the large amount of data allows very accurate determination of low-energy parameters.[3] For the simplest process involving mesons of unequal masses and strangeness—pion-kaon scattering—experiments have improved in the last few years and now allow analytic-continuation procedures with results of reasonable accuracy.[4,5] In this paper we determine the values of the πK scattering amplitude in the unphysical region by different methods of analytic continuation from the physical-region scattering data: fixed-t and hyperbolic dispersion relations. We obtain the so-called πK σ term, which allows us to decide on the type of symmetry breaking and the coefficients of the expansion of the amplitude in the unphysical region; we find a very good agreement with the current-algebra predictions.

In Sec. II we introduce our notation and discuss the current-algebra results for different models of symmetry breaking. The parametrization of the amplitudes and the data that we use is presented in Sec. III. In Sec. IV we discuss briefly the different methods of analytic continuation applied in our case before we finally compare results with the different predictions in Sec. IV.

II. NOTATION AND CURRENT-ALGEBRA PREDICTIONS

For a more detailed discussion of the notation and singularity structure of the πK scattering amplitude we refer to Lang.[6] In the u and s channel, one has πK scattering with isospin $I=\frac{1}{2}$ and $\frac{3}{2}$. The s-channel amplitude for definite isospin is projected into partial waves:

$$T_s^{I_s}(s,t)/16\pi \equiv A_s^{I_s}(s,t) = \sum_l (2l+1) q_s^{2l} a_l^{I_s}(s) P_l(z_s)\,, \qquad (2.1)$$

with the momentum in the c.m. frame

$$q_s = \{[s-(M+\mu)^2][s-(M-\mu)^2]/4s\}^{1/2} \qquad (2.2)$$

and the cosine of the scattering angle

$$z_s = 1 + t/2q_s^2\,. \qquad (2.3)$$

In the t channel the amplitude for $\pi\pi \to K\bar{K}$ may have isospin values $I_t=0$ and 1 and it is represented through partial waves as

$$T_t^{I_t}(s,t)/16\pi \equiv A_t^{I_t}(s,t) = \sqrt{2} \sum_{\substack{\text{even}\\ \text{odd}}\, l\, \substack{\text{for even}\\ \text{for odd}}\, I_t} (2l+1)(p_t q_t)^l \times a_l^{I_t}(t) P_l(z_t)\,, \qquad (2.4)$$

where $p_t = (t/4-\mu^2)^{1/2}$ and $q_t = (t/4 - M^2)^{1/2}$ are the pion and kaon momenta in the c.m. system; the cosine of the scattering angle is

$$z_t = (s-u)/4p_t q_t\,. \tag{2.5}$$

Throughout the paper we abbreviate $m_K \equiv M$, $m_\pi \equiv \mu$. Where the units are omitted we use $\mu^2 \equiv 1$ and $M^2 \equiv 12.5$. The s- and t-channel isospin amplitudes are dependent through the crossing relations

$$\begin{aligned} A^+ &\equiv \frac{1}{\sqrt{6}} A_t^{I_t=0} = \tfrac{1}{3}(A_s^{I_s=1/2} + 2A_s^{I_s=3/2})\,, \\ A^- &\equiv \tfrac{1}{2} A_t^{I_t=1} = \tfrac{1}{3}(A_s^{I_s=1/2} - A_s^{I_s=3/2})\,, \end{aligned} \tag{2.6}$$

where we have introduced the usual definition of the crossing-even and -odd amplitudes. In the unphysical region an expansion of the amplitudes[3] at $t=0$, $\nu \equiv (s-u)/4M = 0$ has the form

$$\begin{aligned} 16\pi A^+(s,t) &= T^+(s,t) = \sum_{i,j=0}^{\infty} c_{ij}^+ t^i \nu^{2j}\,, \\ 16\pi A^-(s,t) &= T^-(s,t) = \nu \sum_{i,j=0}^{\infty} c_{ij}^- t^i \nu^{2j}\,. \end{aligned} \tag{2.7}$$

In the framework of current algebra (cf. Ref. 7) it is possible to predict values of the scattering amplitudes for one or two of the external particles in the soft meson limit, i.e., with vanishing four-momentum. Following Weinberg's[8] approach, originally applied to $\pi\pi$ and πN scattering, Griffith[8] investigates πK amplitudes with $I_s = \frac{3}{2}$ that can be easily decomposed into crossing-even $I_t = 0$ and crossing-odd $I_t = 1$ contributions. The amplitude is extended off the mass shell of a pion or a kaon according to the Lehmann-Symanzik-Zimmermann reduction formalism. The interpolating pion or kaon fields are defined through the divergences of the axial-vector currents with corresponding flavor [partial conservation of axial-vector current (PCAC)]. One finds that the amplitude vanishes if the momentum of the off-shell particle vanishes (Adler[9] zero, cf. Fig. 1). Taking two pions off shell one finds

$$T_s^{3/2} \xrightarrow[q_1,q_2\to 0]{} O(q^2) - 2\frac{p\cdot q}{f_\pi^2} - \frac{2}{f_\pi^2}\langle K^+(p)|[F_{\pi^-}^5,[F_{\pi^+}^5,\epsilon H']]|K^+(p)\rangle\,, \tag{2.8}$$

where $q_1 = q_2 = q$ are the momenta of the pions and $p_1 = p_2 = p$ are the momenta of the kaons. The second term stems from a commutator of the two axial-vector currents that is known from the $SU(3)_L \times SU(3)_R$ current algebra; it is the antisymmetric contribution ($I_t = 1$) to the off-shell amplitude $T_s^{3/2}$ and proportional to $(s-u)/f_\pi^2$—the Adler-Weisberger condition (f_π is the pion decay constant). At the off-shell point $q_1 = q_2 = 0$ ($s = u = M^2$, $t = 0$) we have

$$T_s^{3/2}(s=u=M^2, t=0) = -\frac{2}{f_\pi^2}\langle K^+(p)|[F_{\pi^-}^5,[F_{\pi^+}^5,\epsilon H']]|K^+(p)\rangle \equiv -\frac{2}{f_\pi^2}\sigma_{KK}\,, \tag{2.9}$$

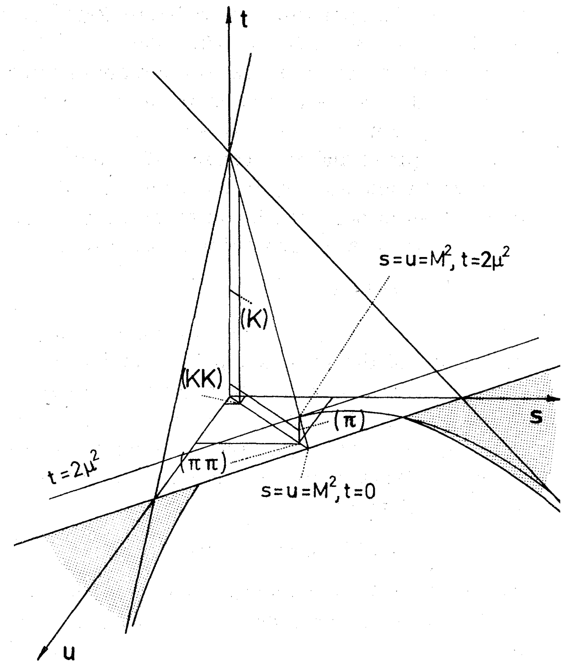

FIG. 1. This figure illustrates the kinematical situation in the case of πK scattering; on-shell scattering is characterized by $s+t+u = 2M^2 + 2\mu^2$, the physical s- and u-channel regions are shaded. The soft particles at the corresponding off-shell points are given in parentheses.

where we have implicitly defined the so-called σ term; in this notation the index denotes the on-shell particles. In the above expression F_α^5 are the chiral charge operators with flavor α and $\epsilon H'$ is the symmetry-violating contribution to the total Hamiltonian. Analogously, one may define $\sigma_{\pi\pi}$ for the case of two off-shell kaons by interchanging $\pi \leftrightarrow K$, $\mu^2 \leftrightarrow M^2$, $p \leftrightarrow q$, $f_\pi \leftrightarrow f_K$, and $F_\pi^5 \leftrightarrow F_K^5$ in the equations. The $\sigma_{\pi\pi}$ term, however, is much more unphysical than σ_{KK} and therefore we do not try to obtain its value by extrapolation from experiments.

One may determine the values of σ_{KK} and $\sigma_{\pi\pi}$ that result from different models for the nature of the symmetry breaking. Performing another low-energy limit, one finds

$$\langle K^+(p')|[F_{\pi^-}^5,[F_{\pi^+}^5,\epsilon H']]|K^+(p)\rangle \xrightarrow[p'\to 0]{} -i\frac{\sqrt{2}}{f_K}\langle 0|[F_{K^-}^5,[F_{\pi^-}^5,[F_{\pi^+}^5,\epsilon H']]]|K^+(p)\rangle \tag{2.10}$$

and correspondingly for soft kaons.

Three representations for $SU(3)_L \times SU(3)_R$ symmetry breaking have been under discussion in this context: $(3,3^*)+(3^*,3)$ (cf. Ref. 10), $(8,8)$ (Ref. 11), and $(6,6^*)+(6^*,6)$ (Ref. 12). For the transformation properties of the corresponding set of

18, 64, or 72 operators we refer to Reya[7] or the original papers (Refs. 10–12). The first of the three models corresponds to the standard quark model and it is the one compatible with quantum chromodynamics.[13] The other two representations have isospin-2 and hypercharge-2 components. This will show up only in the transformation properties in our case because we discuss only singlet and octet contributions, namely,

$$\epsilon H' = \epsilon_0 t_0 + \epsilon_8 t_8 , \tag{2.11}$$

where t_0 and t_8 are the general combinations of operators of the representation that conserve isospin, hypercharge, and parity. These terms guarantee the Gell-Mann–Okubo mass formulas and are therefore sufficient to describe lowest-order chiral-symmetry breaking. For the numerical values we use

$$f_\pi = f_K = f = 131 \text{ MeV} \equiv 0.936\mu . \tag{2.12}$$

By PCAC we have a relation between mass and decay constant of a meson α and a chiral commutator

$$\frac{m_\alpha{}^2 f_\alpha}{\sqrt{2}} = \frac{1}{i}\langle 0 | [F^5_\alpha, \epsilon H'] | \alpha\rangle , \tag{2.13}$$

which leads to a value for ϵ_8/ϵ_0 for each type of symmetry breaking. We find the following:

$(3, 3^*) + (3^*, 3)$:

$$\epsilon_8/\epsilon_0 = -2\sqrt{2}\,\frac{M^2 f_K - \mu^2 f_\pi}{2M^2 f_K + \mu^2 f_\pi} = -1.25 , \tag{2.14a}$$

soft $\pi\pi$:

$$T_s^{3/2}(s = u = M^2, t = 0) = -\frac{2}{f_\pi^2}\sigma_{KK} = -\frac{\mu^2}{f_\pi f_K} = -1.14 , \tag{2.14b}$$

soft KK:

$$T_s^{3/2}(s = u = \mu^2, t = 0) = -\frac{2}{f_K^2}\sigma_{\pi\pi} = -\frac{M^2}{f_\pi f_K} = -14.27 , \tag{2.14c}$$

$(8, 8)$:

$$\epsilon_8/\epsilon_0 = -\sqrt{10}\,\frac{M^2 f_K - \mu^2 f_\pi}{2M^2 f_K + \mu^2 f_\pi} = -1.40 , \tag{2.15a}$$

soft $\pi\pi$:

$$T_s^{3/2}(s = u = M^2, t = 0) = -\frac{2}{f_\pi^2}\sigma_{KK} = -\frac{7\mu^2}{3 f_\pi f_K}\,\frac{1 + (\sqrt{10}/7)(\epsilon_8/\epsilon_0)}{1 + (2/5)^{1/2}(\epsilon_8/\epsilon_0)} = -8.50 , \tag{2.15b}$$

soft KK:

$$T_s^{3/2}(s = u = \mu^2, t = 0) = -\frac{2}{f_K^2}\sigma_{\pi\pi} = -\frac{7M^2}{3 f_\pi f_K}\,\frac{1 + \frac{1}{14}(2/5)^{1/2}(\epsilon_8/\epsilon_0)}{1 - (1/\sqrt{10})(\epsilon_8/\epsilon_0)} = -21.62 , \tag{2.15c}$$

$(6, 6^*) + (6^*, 6)$:

$$\epsilon_8/\epsilon_0 = \frac{25}{7}\,\frac{M^2 f_K - \mu^2 f_\pi}{2M^2 f_K + \mu^2 f_\pi} = 1.58 , \tag{2.16a}$$

soft $\pi\pi$:

$$T_s^{3/2}(s = u = M^2, t = 0) = -\frac{2}{f_\pi^2}\sigma_{KK} = -\frac{13\mu^2}{5 f_\pi f_K}\,\frac{1 - \frac{22}{65}(\epsilon_8/\epsilon_0)}{1 - \frac{14}{25}(\epsilon_8/\epsilon_0)} = -11.97 , \tag{2.16b}$$

soft KK:

$$T_s^{3/2}(s = u = \mu^2, t = 0) = -\frac{2}{f_K^2}\sigma_{\pi\pi} = -\frac{13M^2}{5 f_\pi f_K}\,\frac{1 - \frac{1}{65}(\epsilon_8/\epsilon_0)}{1 + \frac{7}{25}(\epsilon_8/\epsilon_0)} = -25.10 . \tag{2.16c}$$

All multiple commutators have been checked with help of the REDUCE[14] system for algebraic manipulations.

There are two possibilities to compare these predictions with scattering data. One is to adopt a linear ansatz for the scattering amplitude[8] and to use the current-algebra values to determine the parameters. This gives values for the s- and p-wave scattering lengths in Table I; in Fig. 2 we compare these values with an average over the experimental values (cf. Ref. 6). Linearity, however, is too stringent a restriction because the physical amplitude deviates substantially from a linear approximation due to unitarity and resonance structure. This is true even in the unphysical region, as can be demonstrated by dispersion relations. Therefore it is difficult to decide from Fig. 2 on the type of symmetry breaking and on the quality of other soft-meson predictions.

TABLE I. The nonvanishing scattering lengths α_l^I due to the linear soft-meson model (in units μ^2).

	$(3, 3^*) + (3^*, 3)$	$(6, 6^*) + (6^*, 6)$	$(8, 8)$
$\alpha_0^{1/2}$	0.142	0.237	0.206
$\alpha_0^{3/2}$	−0.071	0.024	−0.006
$\alpha_1^{1/2}$	0.010	0.010	0.010

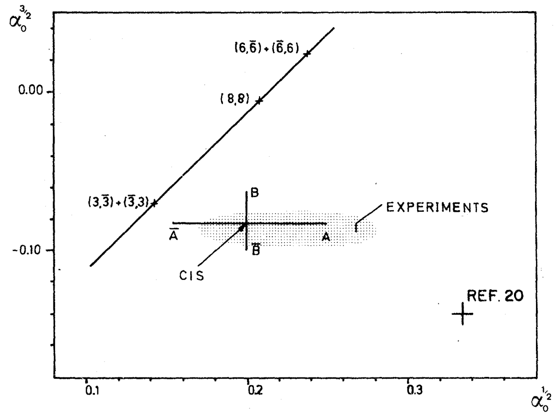

FIG. 2. The predictions of the soft-meson theory for $\alpha_0^{1/2}$ and $\alpha_0^{3/2}$ are compared with the experimental average values. The values corresponding to our parametrization (CIS) are also shown, as well as the results of Ref. 20.

The increase in the amount and quality of πK data in the last few years now allows us to select a more promising method, namely, to continue analytically from the physically accessible to the unphysical region near $s=u$, $t=0$. There we expand the amplitude into a polynomial in ν and t [compare Eq. (2.7)] and compare it with the predictions. Linear extrapolation from the soft-meson points—where the predictions ought to be exact—to the neighbored points in the Mandelstam plane (Fig. 1) is certainly much better justified. This linearity over a distance of $2\mu^2$ gives in the Cheng-Dashen[15] prescription the connection between the off-shell and the on-shell σ term

$$T^+(s=u=M^2, t=2\mu^2)=-T^+(s=u=M^2, t=0)$$
$$=\frac{2}{f_\pi^2}\sigma_{KK}\,. \tag{2.17}$$

A corresponding relation for the soft-kaon σ term involves linearity over the much larger distance $2M^2$ and is therefore hardly reliable.

III. PARAMETRIZATION OF THE DATA

For the analytic continuation we need values of the total crossing-even or -odd amplitudes in the physical region. The data is given for the partial waves and show local instabilities. This may be due to systematic errors in the process of analytic continuation from $KN\to K\pi N$ data to $K\pi$ on-mass-shell scattering data. We therefore parametrize the partial waves $s^{1/2}$, $s^{3/2}$, $p^{1/2}$, and $d^{1/2}$ in an unitary and analytic way. The $p^{3/2}$ and $d^{3/2}$ as well as higher partial waves are found to be negligible in this energy region in the experiments.

In the energy range below 1.4 GeV the partial waves are practically elastic:

$$a_l^{I_s}(s)q_s^{\,2l}=\frac{\sqrt{s}}{2q_s}\,\frac{1}{2i}\left[\exp(2i\delta_l^{I_s})-1\right], \tag{3.1}$$

and we may solve a k-matrix equation to obtain

$$[a_l^I(s)]^{-1}=\Big(k_l^I(s)+\sum_R \alpha_R^{\,2}d_R(s)\Big)^{-1}$$
$$-\,d_{\pi K,l}(s)\,. \tag{3.2}$$

Here $d_{\pi K,l}$ is the Chew-Mandelstam function for two particles of unequal mass; it has only a right-hand cut with the imaginary part $2p_s^{2l}/\sqrt{s}$. The functions

$$d_R=16\pi s/(2l+1)s_R(s_R-s)$$

are the propagators for the bare resonance R coupled to the partial wave. For a discussion of the special type of k-matrix equation and the explicit form of the d's cf. Lang.[16] The parameter α_R gives the strength of the coupling to the πK system. Then k_l^I has no right-hand cut from πK elastic unitarity and no poles due to resonances R. It does have left-hand singularities such as a_l^I and cuts starting at higher-lying inelastic thresholds. In the elastic-unitarity region it is therefore singularity free and can be approximated by smooth functions; this was supported by the results of model calculations.[16] We find from the data that it is sufficient to approximate k for the s waves by straight lines ($k_0^I=\gamma_0^I+\beta_0^I s$) and for the p and d waves by a constant ($k_1^{1/2}=\gamma_1^{1/2}$, $k_2^{1/2}=\gamma_2^{1/2}$).

As our central input set of data we use for the $s^{1/2}$ wave the data by Matison *et al.*[17] below 1 GeV

TABLE II. The input parameters for our unitary parametrization of the πK phase-shift data; in the upper half we give the values due to the data in Refs. 17–19 as discussed in the text; the lower half contains the values due to the data of Ref. 20.

l	I	α_R	s_R	β	γ
0	$\frac{1}{2}$	1.042	83.7	0.0089	-0.035 ± 0.069
0	$\frac{3}{2}$	···	···	-0.0039	-0.128 ± 0.051
1	$\frac{1}{2}$	0.225	41.8	···	0.016 ± 0.003
2	$\frac{1}{2}$	0.048	102.8	···	···
0	$\frac{1}{2}$	1.082	93.0	0.0033	0.365 ± 0.047
0	$\frac{3}{2}$	···	···	-0.0007	-0.465 ± 0.022
1	$\frac{1}{2}$	0.227	42.0	···	0.032 ± 0.004
2	$\frac{1}{2}$	0.048	102.9	···	···

and of Firestone *et al.*[17] above; this corresponds to a κ resonance with mass 1.31 GeV and width 0.91 GeV. The $s^{3/2}$ data come from Ref. 18, and for the $p^{1/2}$ wave we determine α_R and s_R from the data of Mercer *et al.*[19] Finally, for the $d^{1/2}$ wave α_R and s_R are determined from the $K^*_{I=2}$ parameters $m_{K^*}=1.421$ GeV, $\Gamma_{K^*}=0.108$ GeV, and since there are no adequate data, $\gamma_2^{1/2}=0$. We check that our results are only weakly dependent on this last assumption by also running our programs with a different value for $\gamma_2^{1/2}$ which corresponds to a different d-wave scattering length and a zero of the d wave at infinity. We give the values of the parameters[5] in Table II. The standard deviations of the parameters γ are obtained from the fits to the experimental data and define the error bands as can be seen in Fig. 3.

The estimation of the errors of the analytic continuation, due to the parametrization of the input data, is done in the following way. Each extrapolation is repeated for different sets of the input parameters. The central values define the central input set (CIS); sets A and $\bar{A}$ are equal to CIS except for $\gamma_0^{1/2}$, which differs by plus or minus 1 standard deviation. The same holds for B and $\bar{B}$ with regard to $\gamma_0^{3/2}$ and for C and $\bar{C}$ with regard to the p-wave parameter $\gamma_1^{1/2}$. The systematic errors of the numerical analytic continuation of the discrepancy function (Sec. IV) and of the integrals involved are comparably small, and therefore the error of the result can be expressed through the uncertainty of the input values in the form

$$f(\mathrm{CIS})\pm\tfrac{1}{2}\{[f(A)-f(\bar{A})]^2+[f(B)-f(\bar{B})]^2 +[f(C)-f(\bar{C})]^2\}^{1/2}\,. \tag{3.3}$$

This is an approximation derived from the law for propagation of errors under the assumption that the input parameters are uncorrelated.

Recently, there has been a new high-statistics experiment for the determination of πK scattering partial waves.[20] Especially the low-energy data deviate from the other determinations (cf. Figs. 2 and 3); we therefore parametrize this data set separately in the same way as discussed above. The values obtained for the parameters are also given in Table II, and the error bands are shown in Fig. 3. Owing to the very small errors claimed in Ref. 20, the error band for $s^{1/2}$ is smaller than for the other set of data. In both cases, however, the oscillations of the phase shift below 1 GeV indicate an underestimation of the systematic errors of the analytic method of πK data extraction.

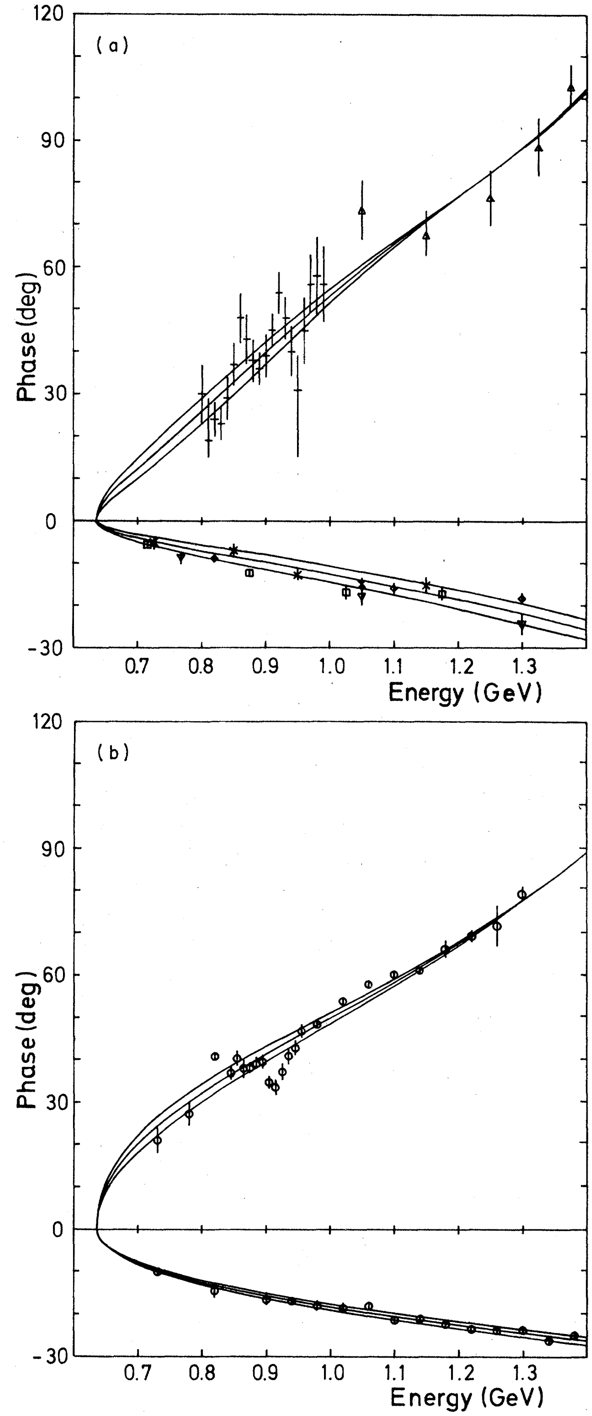

FIG. 3. (a) The $\delta_0^{1/2}$ (upper part) and $\delta_0^{3/2}$ (lower part) data from Refs. 17 and 18 [+ Matison *et al.* (Ref. 17), ⩚ Firestone *et al.* (Ref. 17), ⁎ Bakker *et al.* (Ref. 18), ⌽ Cho *et al.* (Ref. 18), ▼ Linglin *et al.* (Ref. 18), and ⌷ Jongejans *et al.* (Ref. 18)] compared to our parametrizations CIS, A, $\bar{A}$, B, $\bar{B}$ as discussed in the text. (b) The $\delta_0^{1/2}$ (upper part) and $\delta_0^{3/2}$ (lower part) data from Ref. 20 compared to our parametrizations as discussed in the text.

IV. DISPERSION TECHNIQUES APPLIED

For the determination of the expansion parameters in Eq. (2.7) we have to determine the amplitudes in the unphysical region near $s=u, t=0$. We do this with the help of finite-contour dispersion relations[21] at different values of fixed $t=-1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0\mu^2$. The σ term $(s=u, t=2\mu^2)$ is also determined with a dispersion relation (DR) on a hyperbola.

The fixed-t DR for crossing-even or -odd amplitudes can be written

$$T^{\pm}(s, t\text{ fixed}) = \frac{1}{\pi} P \int_{s_{\text{thr}}}^{s_{\max}} \operatorname{Im} T^{\pm}(s', t) \times \left(\frac{1}{s'-s} \pm \frac{1}{u'-u}\right) ds' + \Delta^{\pm}(s). \tag{4.1}$$

We know $\operatorname{Im} a_l^I$ to good accuracy between threshold and $100\mu^2$; between 100 and $150\mu^2$ our parametrization holds approximately—we therefore choose $s_{\max}=150\mu^2$. We expect that the sum of partial waves converges quickly to $\operatorname{Im}T^{\pm}$ even at $t=2\mu^2$. Then the discrepancy function $\Delta^{\pm}(s)$ has singularities for $s>100\mu^2$ and $u>100\mu^2$ $(s<2M^2+2\mu^2-t-100\mu^2)$; presumably the cut discontinuity is small for $100\mu^2<s<s_{\max}$ and $100\mu^2<u<s_{\max}$, respectively. $\Delta^{\pm}(s)$ is real and free of singularities in the region $2M^2+2\mu^2-t-100\mu^2<s<100\mu^2$.

The points $s_{\text{thr}}<s<100\mu^2$, $-1\leq t\leq 2\mu^2$ are inside the smaller Lehmann ellipse of convergence for the partial wave sum for the real part of T; we take, however, values of $\operatorname{Re}T^{\pm}$ only in the region $28\mu^2<s<60\mu^2$ where we have reliable experimental data and parametrization. One, therefore, can compute the real values of $\Delta^{\pm}(s)$ at these points; since the discrepancy function has no nearby singularity, it can be easily extrapolated to $s=u=M^2+\mu^2-t/2$ by a low-order polynomial.

This introduces the inevitable model dependence of the analytic continuation. In a more sophisticated approach one allows for more (in principle, infinitely many) coefficients and suppresses the higher coefficients, e.g., through the introduction of a smoothness norm (Pietarinen[1]). In our case this is practically equivalent with our choice of a low-order polynomial, as we have checked. A polynomial of degree 2 in ν^2 for Δ^+ and Δ^-/ν fits the numerical values better than two decimals (cf. Fig. 4). The only exception is the antisymmetric amplitude constructed from the input data of Estabrooks *et al.*[20] One clearly sees that the discrepancy is hardly consistent with crossing antisymmetry; remember that the discrepancy function has no singularities in the energy region around the symmetry point and should show no

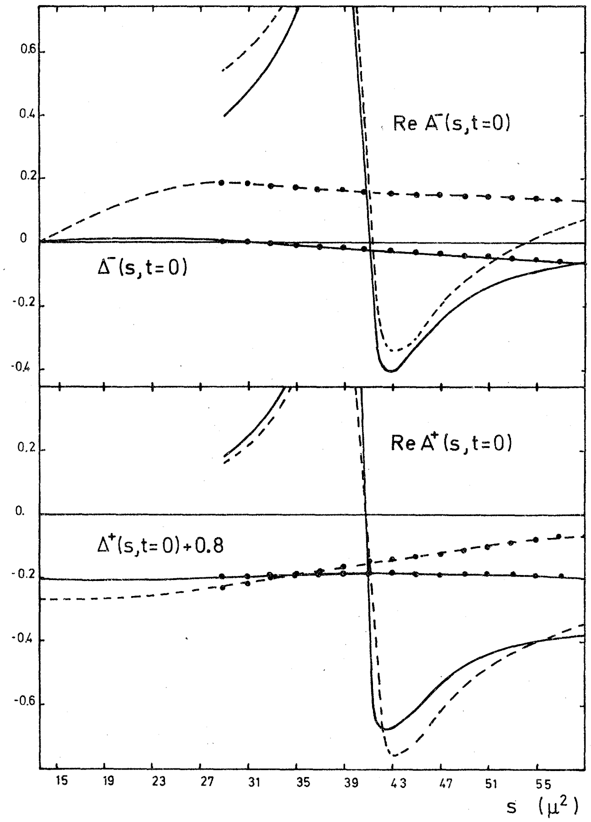

FIG. 4. Here we compare the values we obtain at $t=0$ for the discrepancy functions Δ^- and Δ^+ (full circles) and the polynomial fit (lines through the circles) for the two sets of data due to Refs. 17–19 on one hand (full lines) and Ref. 20 on the other (dashed lines). We also give the values for Re A^- and Re A^+ to allow a comparison of the order of magnitude. The crossing-odd or -even functions are antisymmetric or symmetric in $s-13.5\mu^2$ (for $t=0$).

strong structure there. If one forces a good fit of Δ^-/ν one needs a fourth-order polynomial in ν^2 as shown in Fig. 4. We discuss this problem in Sec. V. Having determined the discrepancy, the integral can be easily computed near $s=u$ and so we find the values for the amplitude at $t=2\mu^2$.

The expansion coefficients c_{00}^+ and c_{01}^+ are determined by fitting the values of the amplitudes T^+ at different values of ν for $t=0$ by a low-order polynomial in ν. The coefficients c_{10}^+ and c_{11}^+ are found by repeating the determination of $c_{00}^+(t)$ and $c_{01}^+(t)$ for different values of t and fitting these values by a straight line in t—its derivatives then give c_{10}^+ and c_{11}^+. Owing to systematic and experimental uncertainties, it is not possible to give reliable values for higher coefficients. A similar procedure is applied to obtain the c_{ij}^-.

We also use a DR on a hyperbola to continue the amplitude to $t=2\mu^2$, $s=u$. The advantage of such curves[4-6,22] is that they lie asymptotically inside the physical regions of different channels. The hyperbola of the type

$$(s-\alpha)(u-\alpha)=\beta \tag{4.2}$$

approaches asymptotically fixed-s or -u lines. For a specific choice of β, namely,

$$\beta=\alpha^2-\alpha(2M^2+2\mu^2)+(M^2-\mu^2)^2\,, \tag{4.3}$$

the hyperbola is always inside the physical region above threshold. The curves are related to the kinematic Kibble function[23] $\phi=4tp_t^2q_t^2\sin^2\theta_t$, i.e., they correspond to $\phi/t^2=-\alpha$ with $\alpha<0$. With $\alpha=-12\mu^2$ the hyperbola runs through $s=u, t=2\mu^2$.

In the DR the analytic continuation is now performed in t, which means that there is a cut starting at $t=4\mu^2$ due to $\pi\pi\to K\bar{K}$. The phase of the s-wave amplitude is that of elastic $\pi\pi$ scattering because of extended unitarity. We introduce the Omnes function

$$\Omega[\delta_{\pi\pi}(t)]=\exp\left[\frac{t}{\pi}\int_{4\mu^2}^{4M^2}\frac{\delta_{\pi\pi}(t')}{t'(t'-t)}\,dt'\right]\,, \tag{4.4}$$

which is calculated from the Roy-equation solution[24] of the experimental $\pi\pi$ phase shift δ_0^0 with scattering length $a_0^0(\pi\pi)=0.3\mu^{-1}$. We then apply a DR for T^+/Ω which does not have the strong s-wave $\pi\pi$ cut above $4\mu^2$:

$$T^+/\Omega=\frac{1}{\pi}\int_{t_{\rm min}}^{0}\mathrm{Im}(T^+/\Omega)\,\frac{dt'}{t'-t}+\Delta(t)\,. \tag{4.5}$$

The discrepancy function has cuts for $t<t_{\rm min}=-100\mu^2$ and for $t>4\mu^2$; the discontinuity of the latter is small because it comes only from d waves and higher partial waves as well as from deviations of the exact $\delta_{\pi\pi}$ from our ansatz.[24] Again, the real values of Δ for $-30\mu^2<t<-1\mu^2$ can be extrapolated to $t=2\mu^2$. Evaluation of the integral gives then a further value of the amplitude $T^+(s=u, t=2\mu^2)$.

V. RESULTS AND DISCUSSION

In Table III we give the results for the σ term for our standard input as determined from fixed-$t=2\mu^2$ and hyperbolic dispersion relations. We compare these values with the current-algebra results due to the three different models for symmetry breaking in Sec. II. We find that only the $(3,3^*)+(3^*,3)$ model is within 1 standard deviation compatible with our results, the other two types of symmetry breaking are excluded with more than 4 standard deviations. The results of both types of analytic continuation agree within the errors. We think that the errors we obtain are realistic with regard to the variation of the experimental input.

In Table IV we compare the expansion parameters with the results of other authors and from current algebra (linear model). The first coefficient is the only one that depends on the type of symmetry breaking, and again only $(3,3^*)+(3^*,3)$ is compatible with our result for the central input set in the last line of the table. The coefficient c_{00}^- is in the linear model given through the Adler-Weisberger condition and our result is in good agreement with it. This number is, in Fig. 2, responsible for the linear relation between the s-wave scattering lengths; there (on-shell and in the physical region) the comparison with the experiments exhibits no agreement. The other coefficient that does not vanish in the linear soft-meson theory (SMT) model is c_{10}^+; this coefficient is also independent of the type of symmetry breaking. This follows from the mass relations due to PCAC [Eq. (2.13)]. Our result is in excellent agreement with this value too.

The other coefficients correspond to nonlinear terms in ν and t and are therefore vanishing in the linear SMT model by assumption, whereas our results show that they certainly do not vanish in the true amplitudes. This offers an explanation for the failure of the SMT predictions at threshold (cf. Fig. 2). Our results agree within 1 standard deviation with the values for c_{10}^+, c_{00}^-, and c_{10}^- obtained by Nielsen and Oades[4] and Hedegaard-Jensen[4]; we also agree with the values for c_{00}^- and c_{10}^- given by Blatnik *et al.*,[5] who use the same input data but a different method of continuation on different hyperbolas. We do not agree with the values

TABLE III. We compare the theoretical predictions for different models of symmetry breaking (Sec. II) with the results of the analytic continuation of our central input set of data for $t=2\mu^2$ and on a hyperbola; we also give the arithmetic mean between the two results. Linearity in t over the distance $2\mu^2$ implies $T^+(s=u=M^2, t=2\mu^2)=-T^+(s=u=M^2, t=0)$ (Ref. 15).

$-T^+(s=u=M^2,t=0)$	$T^+(s=u=M^2,t=2\mu^2)$ Fixed-t	Hyperbola	Mean
$(3,3^*)+(3^*,3)$: 1.14			
$(6,6^*)+(6^*,6)$: 11.98	0.54 ± 2.03	-1.15 ± 2.06	-0.31 ± 2.05
$(8,8)$: 8.50			

TABLE IV. We compare our results for the expansion coefficients in Eq. (2.7) with the results from soft-meson theory and from other methods of extrapolation from data. In the last line we show the results due to our central input set of parameters (due to the data of Refs. 17–19). In the second line from below, the results for the parametrization corresponding to the data of Ref. 20 are given; for this data set we find inconsistencies with crossing symmetry as discussed in the text.

	c_{00}^{+}	c_{10}^{+}	c_{01}^{+}	c_{11}^{+}	c_{00}^{-}	c_{10}^{-}	c_{01}^{-}	c_{11}^{-}
Linear soft-meson theory	$(3,3^*)+(3^*,3)$: 0 $(6,6^*)+(6^*,6)$: 10.83 $(8,8)$: 7.63	0.57	0	0	8.08	0	0	0
Nielsen and Oades (Ref. 4)	-0.8 ± 0.7	0.5 ± 0.1	⋯	⋯	6.7 ± 1.5	0.2 ± 0.2	⋯	⋯
Fox and Griss (Ref. 25)	3.4 ± 0.7	⋯	5.0	⋯	12.0 ± 1.5	0.2 ± 0.2	⋯	⋯
Hedegaard-Jensen (Ref. 4)	⋯	0.4 ± 0.1	2.0 ± 0.5	⋯	⋯	⋯	⋯	⋯
Blatnik *et al.* (Ref. 5)	⋯	⋯	⋯	⋯	8.65 ± 1.00	0.28 ± 0.06	⋯	⋯
Results for the data of Estabrooks *et al.* (Ref. 20)	-1.31 ± 1.26	0.75 ± 0.09	3.53 ± 0.26	-0.18 ± 0.03	14.30 ± 0.93	0.31 ± 0.07	0.31 ± 0.11	-0.11 ± 0.02
Results for the central input set of the data (Refs. 17–19)	-0.52 ± 2.03	0.55 ± 0.07	2.06 ± 0.22	-0.04 ± 0.02	7.31 ± 0.90	0.21 ± 0.04	0.51 ± 0.10	-0.04 ± 0.02

for c_{00}^{+}, c_{01}^{+}, and c_{00}^{-} obtained by Fox and Griss.[25]

We also determine the expansion parameters in the unphysical region by continuing analytically the parametrization of the data by Estabrooks *et al.*[20] (Table IV). As one may already expect from the larger scattering lengths, the coefficients c_{00}^{+} and c_{10}^{+} deviate clearly from our main results; they are compatible with the linear model values within 2 standard deviations. For the antisymmetric combinations we find a discrepancy function that is hardly consistent with analyticity and crossing (cf. the discussion in Sec. IV). We think that the reason for this is that the s-wave scattering lengths are too large in absolute magnitude. This effect is canceled in the symmetric combination but is amplified in the antisymmetric combination [Eq. (2.6)]. We conclude that the low-energy s-wave phase shifts due to the work in Ref. 20 are hardly consistent with s-u crossing.[26] If we force the antisymmetry by allowing more structure in the discrepancy, we find an Adler-Weisberger coefficient c_{00}^{-} too large by a factor of 2 (6 standard deviations). These problems may be due to an underestimation of the errors of the data[20] in the low-energy region.

As a conclusion we want to say that the different techniques for analytic continuation seem to be well suited to enlarge the region where the amplitudes can be determined, i.e., the region of "observability." It is preferable to continue the data by these methods to those unphysical points, where theoretical predictions can be tested easily, than to continue the theoretical model rather crudely to the physical region. For πK scattering we find a very good agreement with the assumption of $(3, 3^*)+(3^*, 3)$ symmetry breaking of $SU(3)_L \times SU(3)_R$.

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