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Determination of the pion-nucleon scattering lengths from partial-wave analyses

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A study of the factors which govern a precise determination of the S- and P-wave pion-nucleon scattering lengths using phenomenological energy-dependent partial-wave analyses is made. The nuclear scattering lengths obtained result from rapidly convergent effective-range expansions and are shown to be insensitive to the energy cutoff. These phenomenological values are in general agreement with theoretical treatments.

The determination of pion-nucleon threshold parameters using an energy-dependent partial-wave data analysis provides a useful test of strong-interaction dynamics. Tests of such theories usually require a knowledge of the scattering lengths when the Coulomb interaction is turned off. Such purely hadronic quantities depend for their determination on assumptions concerning the low-energy dynamics itself in order to implement the separation of hadronic and electromagnetic effects, assumptions which themselves may well bias the results obtained. Thus the hadronic scattering lengths provide a test of strong-interaction theories only within a definite constellation of assumptions concerning these theories, e.g., assumptions about the applicability of dispersion relations (their high-energy behavior, number of subtraction constants, error propagation, etc.), of the form of fundamental interaction Lagrangians, and so on. As a result it is most desirable to determine the nuclear rather than the hadronic threshold parameters because they depend on fewer dynamical assumptions. Corrections can then be made by those favoring the use of purely hadronic quantities.

The nuclear scattering lengths $a_{2J}^{(2I)}$ are defined by the effective-range expansion of the nuclear phase shift $\delta_{2J}^{(2I)}$ in the usual manner,

$$C_l^2(\eta)\rho_l(q)[\cot\delta_{2J}^{(2I)}+2\eta h(\eta^2)/C_0^2] = \frac{1}{a_{2J}^{(2I)}}+\tfrac{1}{2}r_{2J}^{(2I)}q^2+\cdots, \quad (1)$$

where $\rho_l(q)=q^{2l+1}$, q is the barycentric three-momentum, I labels the isospin, and $J=l\pm\frac{1}{2}$ is the total angular momentum of the state. The nuclear phase shift is obtained from a partial-wave analysis of the scattering data after corrections for the Coulomb amplitude and Coulomb-nuclear interference have been made. The other terms in Eq. (1) arise from low-energy Coulomb effects. The quantity

$$C_0^2=2\pi\eta/(e^{2\pi\eta}-1) \quad (2)$$

is the Coulomb S-wave barrier-penetration factor,

$$\eta=1/(qa_0) \quad (3)$$

is the Coulomb parameter, a_0 is the Bohr radius,

$$C_l^2=C_{l-1}^2[1+(\eta/l)^2], \quad (4)$$

and

$$h(\eta^2)=\eta^2\sum_{j=1}^{\infty}\frac{1}{j(j^2+\eta^2)}-\ln\eta-0.5772. \quad (5)$$

The effective-range formula is a direct consequence of potential theory and as such is strictly valid only in the nonrelativistic limit for finite-range, velocity-independent potential scattering. Explicit introduction of a potential can be avoided by utilizing some ansatz for the wave function, but such an ansatz is ultimately justified only in the context of potential theory. For sufficiently low energies only the first two terms of Eq. (1) are required for a good description of the data; the shape of the potential itself plays no role, so the parametrization is virtually model independent.

In a phase-shift analysis the method of extracting values for the scattering lengths and effective ranges necessarily involves some ambiguity. Ideally we would prefer to fit only the data near the elastic threshold (below 30 MeV, for example), but in reality, measurements are conducted somewhat farther away. Owing to the virtual absence of very-low-energy πN scattering data, in order to phenomenologically determine the πN scattering lengths with any reliability, it is necessary to either extrapolate the phase shifts to threshold by generating continuous energy representations of the single-energy scattering phases, or better yet, to perform an energy-dependent analysis of all the available data up to some maximum energy E_{max}. It then becomes a question of deciding at which maximum energy to truncate the fit. If E_{max} is too small, essential data may be excluded and the low-energy fit may not extrapolate reasonably to higher energies. If E_{max} is too large, the fit at low energies may be sacrificed to fit the higher-energy data.

In addition to these considerations there may be

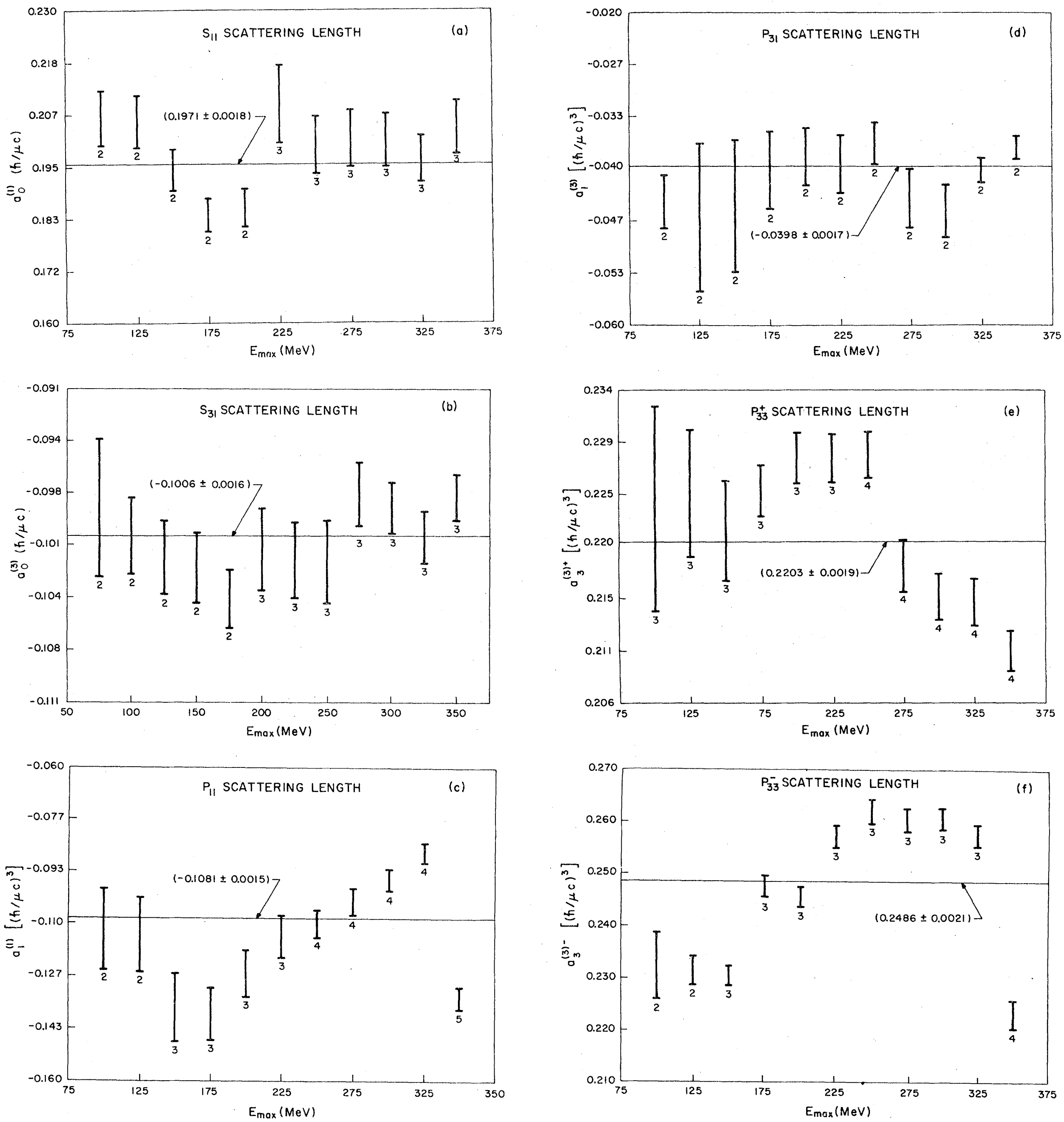

FIG. 1. Scattering lengths as determined by our 0–75 MeV to 0–350 MeV fits. The horizontal line is the result of fitting a constant to all energy-range fits. The numbers under the error bars denote the number of nonzero parameters in the effective-range expansion for a given partial wave.

doubts concerning the validity of a parametric expansion of the phase shifts in q^2. These are of two kinds. First the expansion itself is a low-energy result and may not be applicable at high energies due to the poor convergence of the series. The earliest attempts to determine the scattering lengths from πN partial-wave analyses were, indeed, frustrated by the lack of convergence of the effective-range expansion at energies greater than 50 MeV (Ref. 1). A good determination of the scattering lengths would require that their values remain stable as more parameters are included in the series for a given partial wave.

Equation (1) accounts for the essential singularity at $q^2 = 0$ due to the Coulomb interaction and can be used to analytically describe the scattering process "near threshold." There is no *a priori*, foolproof way of deciding what this means, however. The Yukawa hypothesis requires that one-particle exchange introduces a branch point in the amplitude at $q = \pm i(m/2)$. For πN scattering, the exchange of a scalar meson of mass 600 MeV means that the effective-range expansion defined by Eq. (1) diverges for pion laboratory kinetic energies above 300 MeV. The exact details of pion-nucleon dynamics are still in dispute, however, and the difficulties in assessing the range of validity of Eq. (1) require that it be adopted as a simple phenomenology in order to extract values for the scattering lengths.

Secondly, the Coulomb corrections contained in Eq. (1) are good only for nonrelativistic speeds. It may therefore be judicious to neglect them if E_{max} is greater than, say, 100 MeV. Ultimately, however, this decision is not one for theory but for phenomenology, as the large errors on the experimental data may mask the uncertainties arising from these other sources.

Finally, the low-energy pion-nucleon data cannot be adequately represented by pure S-wave scattering. As a result we may anticipate correlations between S and P waves, a feature that must be accounted for in any realistic error determination.

Our approach is to find a minimal parametrization of the partial waves to obtain an optimum fit to all the data between 0 and E_{max} and then to plot the scattering lengths thus obtained as functions of E_{max} in the hope of observing the required behavior. Thus, once an initial fit is found, we investigate the sensitivity of the fit to the addition of extra parameters in a given phase in order to test the convergence of the effective-range expansion. When higher coefficients in a given partial wave prove to be smaller than their errors, they are neglected and the fit "optimized" in that partial wave.

Our study is based on the pion-nucleon phase shifts obtained from our recent partial-wave analysis.[2] These phase shifts resulted from a minimal fit to all the $\pi^{\pm}p$ elastic and $\pi^{-}p$ charge-exchange scattering data from 0 to 350 MeV (pion lab kinetic energy) which incorporated charge splitting in both the P_{33} and S_{31} partial waves. In the present study we permit charge splitting only in the P_{33} phase shift. The data base was pruned by eliminating all high-χ^2 data sets until the resultant solution, after minimization, gave a χ^2/datum of 1. This reduced data base was then used in our determination of the πN scattering lengths in the hope of eliminating as much instability as possible due to inconsistent data. The parametrization given by Eq. (1) was used in all partial waves except P_{11}. For this

TABLE I. Scattering lengths from πN data analyses.

Scattering length $[(\hbar/\mu c)^{2l+1}]$						
S_{31}	S_{11}	P_{31}	P_{11}	P_{33}^{+}	P_{33}^{-}	Notes
−0.1006 ± 0.0016	0.1971 ± 0.0018	−0.0398 ± 0.0017	−0.1081 ± 0.0015	0.2203 ± 0.0019	0.2486 ± 0.0021	Constant fit Figs. 1(a)–1(f)
−0.0981 ± 0.0015	0.2035 ± 0.0060	−0.0374 ± 0.0015	−0.1334 ± 0.0036	0.2106 ± 0.0018	0.2231 ± 0.0028	Best fit, 0–350 MeV
−0.1029 ± 0.0017	0.2030 ± 0.0071	−0.0445 ± 0.0017	−0.1081 ± 0.0039	0.2185 ± 0.0022	0.1814 ± 0.0038	Coulomb-corrected phases for best fit
$S_{11}-S_{31}$	$S_{11}+2S_{31}$	$P_{11}-P_{31}$	$P_{11}+2P_{31}$			
0.2975 ± 0.0018	−0.0054 ± 0.0021	−0.0689 ± 0.0018	−0.1904 ± 0.0024			Constant fit Figs. 2(a)–2(d)
0.3016 ± 0.0062	0.0073 ± 0.0067	−0.0960 ± 0.0039	−0.2082 ± 0.0047			Best fit, 0–350 MeV
0.3059 ± 0.0073	−0.0028 ± 0.0079	−0.0636 ± 0.0043	−0.1971 ± 0.0052			Coulomb-corrected phases for best fit

TABLE II. Compilation of pion-nucleon S-wave scattering lengths.

Scattering length ($\hbar/\mu c$) S_{31}	S_{11}	$S_{11}-S_{31}$	$S_{11}+2S_{31}$	Reference
−0.0981 ± 0.0015	0.2035 ± 0.0060	0.3016 ± 0.0062	0.0073 ± 0.0067	0–350 MeV fit of this work
−0.098 ± 0.003	0.185 ± 0.008	0.283 ± 0.009	−0.011 ± 0.008	4
−0.092 ± 0.002	0.170 ± 0.004	0.262 ± 0.004	−0.014 ± 0.005	5
−0.092 ± 0.010	0.149 ± 0.019	0.241 ± 0.021	−0.035 ± 0.028	6[a]
−0.113 ± 0.005	0.135 ± 0.005	0.248 ± 0.007	−0.091 ± 0.011	7
−0.0891 ± 0.0045	0.1815 ± 0.0122	0.2706 ± 0.0113	0.0031 ± 0.0080	8
−0.0892 ± 0.0054	0.1866 ± 0.0089	0.2724 ± 0.0123	0.0151 ± 0.0104	9
		0.276 to 3.0	−0.30 to 0.60	10[b]

[a] The author presents three different sets of scattering lengths. We use his preferred solution II.

[b] We use the small error band results preferred by the author.

phase shift we parametrized the right-hand side by an inverse polynomial in s in order to adapt the representation to the fact that the P_{11} passes through zero at about 170 MeV.

Our results for the S- and P-wave scattering lengths are presented in Figs. 1(a)–1(f) for fits to the data from $E_{max}=75$ to 350 MeV in 25-MeV steps. We found that the P_{13} phase shift was too small to permit a determination of its scattering length by our methods, and consequently it was fixed throughout our analysis at the value determined by our previous best fit.[2] Owing to the occurrence of charge splitting we must distinguish between the P_{33} scattering length determined by the π^+p analysis, $a_3^{(3)+}$, and the P_{33} scattering length determined by the π^-p analysis, $a_3^{(3)-}$.

We observe that both the S_{31} and S_{11} scattering lengths are remarkably stable when regarded as functions of E_{max}. Furthermore, both are very stable as additional parameters are included in the partial wave. Among the P-wave scattering lengths, only the P_{31} manifests a similar behavior. Both the P_{11} and P_{33} scattering lengths rise slowly then drop suddenly near the end of the energy range. The cause of this behavior can only be partly ascribed to inconsistencies in the data. In general, we observe that the low-energy data are too sparce to permit a precise determination of the threshold parameters.

Comparison of our results with other phenomenologies is not straightforward because many authors prefer to report Coulomb-corrected scattering

TABLE III. Compilation of pion-nucleon P-wave scattering lengths.

Scattering length [$(\hbar/\mu c)^3$] P_{31}	P_{33}	P_{13}	P_{11}	Reference
−0.0374 ± 0.0015	0.2106 ± 0.0018 0.2231 ± 0.0028	−0.0419 ± 0.0143	−0.1334 ± 0.0036	Δ^{++} / Δ^0 0–350 MeV fit of this work
−0.029 ± 0.002	0.205 ± 0.05	−0.013 ± 0.002	−0.047 ± 0.004	4
−0.043 ± 0.011	0.214 ± 0.009	−0.039 ± 0.016	−0.089 ± 0.02	6
−0.040 ± 0.0021	0.2022 ± 0.0001	−0.0243 ± 0.0049	−0.0575 ± 0.0033	7
−0.038 ± 0.005	0.220 ± 0.005	−0.023 ± 0.006	−0.078 ± 0.007	11
−0.0429 ± 0.0071	0.2041 ± 0.0045	−0.0266 ± 0.0063	−0.0845 ± 0.0102	12
$P_{11}-P_{31}$	$P_{13}-P_{33}$	$P_{11}+2P_{31}$	$P_{13}+2P_{33}$	
−0.096 ± 0.0039	−0.2525 ± 0.0144 −0.2650 ± 0.0146	−0.2082 ± 0.0039	0.3793 ± 0.0147 0.4043 ± 0.054	Δ^{++} / Δ^0 0–350 MeV fit of this work
−0.034 ± 0.0045	−0.218 ± 0.500	−0.105 ± 0.0057	0.397 ± 0.100	4
−0.046 ± 0.023	−0.253 ± 0.018	−0.175 ± 0.041	0.389 ± 0.024	6
−0.0175 ± 0.0039	−0.2265 ± 0.0049	−0.1375 ± 0.0053	0.3801 ± 0.0049	7
−0.0475 to −0.038	−0.255 to −0.0243	−0.174 to −0.159	0.399 to 0.42	10
−0.040 ± 0.009	−0.243 ± 0.008	−0.154 ± 0.012	0.418 ± 0.012	11
−0.0416 ± 0.0124	−0.2307 ± 0.0077	−0.1703 ± 0.0175	0.3816 ± 0.0110	12

lengths. In Table I we present our values for the S-, P-, and D-wave scattering lengths. Table I includes the values determined by (1) the best fit to the reduced data base from 0 to 350 MeV, (2) a constant fit to the scattering-length "data" of Figs. 1(a)–1(f), and (3) the hadronic scattering lengths as determined with the aid of the Coulomb corrections of Tromberg *et al.*[3] (corrections to P_{13} and P_{11} are negligible) to the best fit.

In Tables II–IV we present our values for the nuclear scattering lengths along with those of other authors[4-12] all of which, with the exception of Rowe *et al.*[4] and Brunet,[7] were obtained using dispersion relations to close the energy gap from threshold to regions where reliable data are available. Here we include only those authors who make use of the new total-cross-section data of Carter *et al.*[13] or the phase shifts derived from them. We have also included values for the isospin-odd and -even combinations defined by

$$a_{2J}^{(+)} = \tfrac{1}{3}(a_{2J}^{(1)} + 2a_{2J}^{(3)}) ,$$
$$a_{2J}^{(-)} = \tfrac{1}{3}(a_{2J}^{(1)} - a_{2J}^{(3)}) , \qquad (6)$$

and exhibited graphically in Figs. 2(a)–2(d). In spite of several new analyses, substantial disagreements still exist. Even so, the S- and P-wave scattering lengths obtained from our study are in good agreement with the general trend of the dispersion relation methodologies.

Measurements of low-energy charge-exchange differential cross sections permit a direct determination of $a_0^{(1)}$ if the P-wave phase shifts are known. Duclos *et al.*[14] quote

$$a_0^{(-)} = (0.270 \pm 0.014)\hbar/\mu c .$$

In addition, threshold pion photoproduction data supplemented by an experimental value for the Panofsky ratio may also be used to determine $a_0^{(-)}$:

Donnachie *et al.*[15]: $a_0^{(-)} = (0.245 \pm 0.013)\hbar/\mu c$,

Spuller *et al.*[16]: $a_0^{(-)} = (0.263 \pm 0.005)\hbar/\mu c$.

Our values are seen to be somewhat larger than these results.

The basic objective of this paper has been to reevaluate the possibility of using experimental data via partial-wave analysis to extract values for the pion-nucleon scattering lengths. Our results suggest there is some reason for optimism in this regard if careful attention is given to the details of the parametrization. It must be kept in mind that our values for the scattering lengths have been adjusted for a best fit to all the data and hence their values depend on both the form of our phenomenological representation and on the completeness of the data base. More experimental data below 100 MeV is required to improve our results as well as higher-precision data over the whole energy range. The S- and P-wave scattering lengths may be shown to be largely independent of the way the right-hand side of Eq. (1) is written (resonance plus background, inverse polynomial, etc.). This is not the case for the D waves; they are quite sensitive to the form of the parametrization as well as to the number of parameters included per partial wave.

In keeping with our goal of reporting only the nuclear scattering lengths, the effective-range formula itself neglects mass-difference effects and radiative capture, both of which contribute an imaginary part to the scattering amplitude and alter the threshold phase-space factor. The consequences of vacuum polarization and the extended

TABLE IV. Compilation of pion-nucleon D-wave scattering lengths.

Scattering length $[(\hbar/\mu c)^5]$				
D_{33}	D_{35}	D_{13}	D_{15}	Reference
$(1.03 \pm 0.56)\times10^{-4}$	$(-1.05 \pm 0.21)\times10^{-4}$	$(3.00 \pm 0.22)\times10^{-4}$	$(1.08 \pm 0.18)\times10^{-4}$	0–350 MeV fit of this work
		$(1.3 \pm 0.5)\times10^{-3}$	$(1.2 \pm 0.5)\times10^{-3}$	4
$(-2.0 \pm 7.0)\times10^{-3}$	$(-8.0 \pm 5.0)\times10^{-3}$	$(-4.0 \pm 12.0)\times10^{-3}$	$(-2.0 \pm 9.0)\times10^{-3}$	6
$(2.1 \pm 0.4)\times10^{-3}$	$(-4.8 \pm 0.4)\times10^{-3}$	$(1.2 \pm 0.5)\times10^{-3}$	$(3.8 \pm 0.5)\times10^{-3}$	11
2.7×10^{-3}	-4.2×10^{-3}	1.5×10^{-3}	5.0×10^{-3}	12
$D_{13}-D_{33}$	$D_{13}+2D_{33}$	$D_{15}-D_{35}$	$D_{15}+2D_{35}$	
$(1.97 \pm 0.60)\times10^{-4}$	$(5.06 \pm 1.14)\times10^{-4}$	$(2.13 \pm 0.28)\times10^{-4}$	$(-1.02 \pm 0.46)\times10^{-4}$	0–350 MeV fit of this work
$(-2.0 \pm 18)\times10^{-3}$	$(-8.0 \pm 8.0)\times10^{-3}$	$(6.0 \pm 13)\times10^{-3}$	$(-18 \pm 6)\times10^{-3}$	6
$(-0.72$ to $-0.54)\times10^{-3}$	$(0.51$ to $0.57)\times10^{-2}$	$(0.96$ to $1.05)\times10^{-2}$	$(-0.66$ to $-0.57)\times10^{-2}$	10
$(-9 \pm 6)\times10^{-4}$	$(54 \pm 9)\times10^{-4}$	$(86 \pm 5)\times10^{-4}$	$(-59 \pm 9)\times10^{-4}$	11
-1.2×10^{-3}	6.9×10^{-3}	9.2×10^{-3}	-3.4×10^{-3}	12

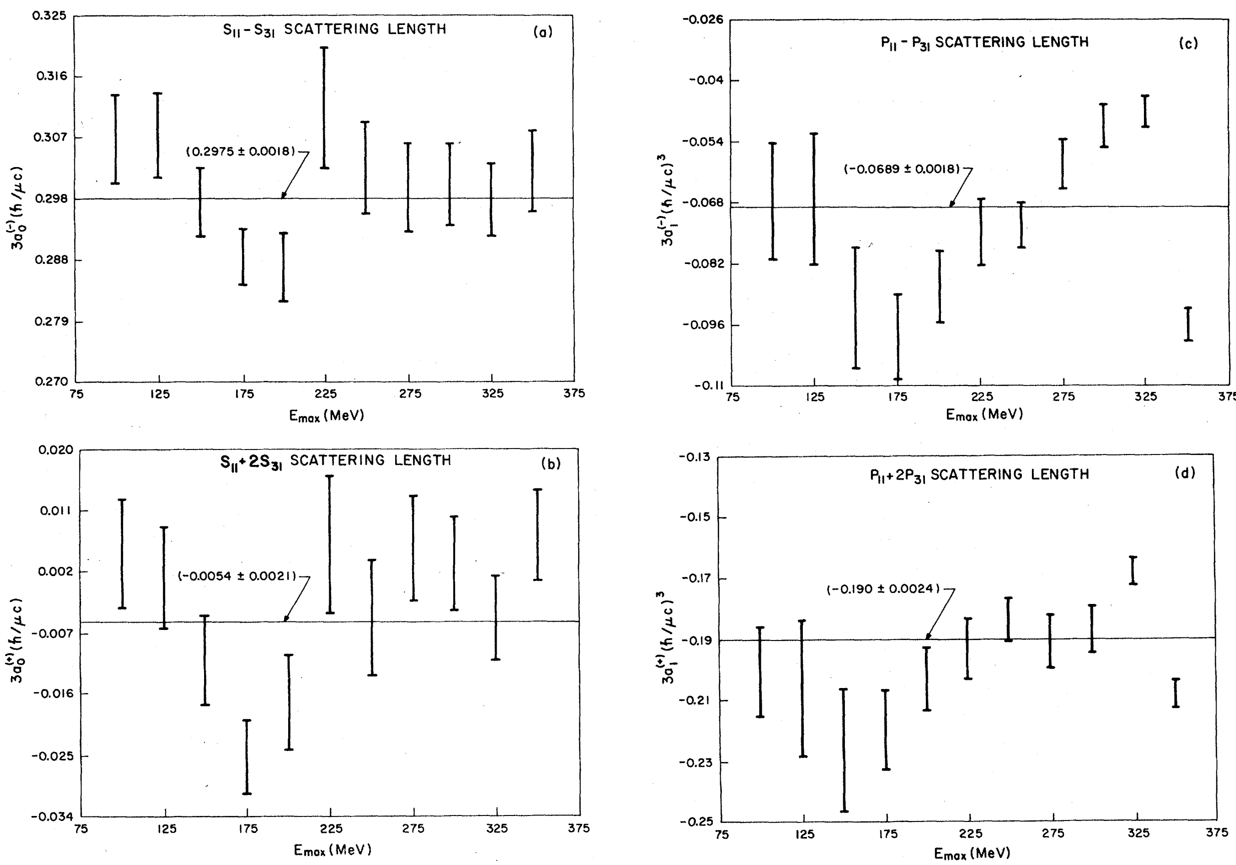

FIG. 2. Isospin-odd and -even combinations of scattering lengths.

charge-current structure of the pion and nucleon are also neglected. Structure effects are expected to be dominant only in the *S* wave, being a short-range influence. The exchange of an electron-position pair, on the other hand, leads to a long-range force affecting a large number of angular momentum states. It would be very useful for the data analyst to have access to an extended effective-range formula that included all these effects and permitted the direct determination of the hadronic scattering lengths.

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