Gauge invariance and fermion mass dimensions

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Renormalization-group-equation fermion mass dimensions are shown to be gauge dependent in gauge theories possessing nonvector couplings of gauge bosons to fermions. However, the *ratios* of running fermion masses are explicitly shown to be gauge invariant in the SU(5) and SU(2) × U(1) examples of such theories.

One of the outstanding successes of the SU(5) model[1] has been its ability to relate differing fermion masses to a hierarchy of symmetry breaking.[2,3] Specifically, d, s, and b quarks have been shown to be heavier than their corresponding leptons (e, μ, τ) because leptons do not communicate with the quantum-chromodynamic (QCD) subgroup of the unified theory.[4] However, inconsistencies between the formulas for fermion mass dimensions ($\gamma_{m_f}^{(i)}$) appearing in Refs. 2 and 3 have led me to reexamine the calculation of these quantities in an arbitrary covariant gauge. These inconsistencies are shown below to be a consequence of gauge dependence of $\gamma_{m_f}^{(i)}$ in theories where nonvector couplings occur between fermions and gauge bosons.

To gain an understanding of this loss of gauge invariance, consider first the renormalized fermion propagator in QED. The photon-electron coupling is, of course, vector and is given by $\mathcal{L}_{\rm QED} = -e\bar{\psi}\gamma_\mu\psi A^\mu$. The renormalized electron propagator $G(p)$ is found from the electron self-energy $\Sigma(p)$:

$$G(p) = \frac{a(p^2)\not{p} + b(p^2)m}{m^2 - p^2} = \frac{1}{m - \not{p}}\left[1 + \Sigma(p)\frac{1}{\not{p} - m}\right]. \quad (1)$$

In the limit of large spacelike momenta ($|p^2| \gg m^2$), $G(p)$ and $\Sigma(p)$ can be expressed as follows:

$$\Sigma(p) = r\not{p} + sm\,,$$
$$a(p^2) = 1 + r\,, \quad (2)$$
$$b(p^2) = 1 + 2r + s\,,$$

where[5]

$$r = \frac{e^2}{16\pi^2} d^0 \ln\frac{|p^2|}{\mu^2}\,,$$
$$s = -\frac{e^2}{16\pi^2}(3 + d^0)\ln\frac{|p^2|}{\mu^2}\,, \quad (3)$$

μ is an (arbitrarily) chosen renormalization point, and d^0, the gauge parameter, is the coefficient of the longitudinal component of the photon propagator

$$D_{\mu\nu} = [g_{\mu\nu} - (1 - d^0)k_\mu k_\nu/k^2]/k^2\,.$$

Correspondence to a gauge theory of vector fermion currents is obtained by adding fermion self-energy contributions from each gauge boson of the theory. For example, the currents coupling to a red quark in QCD are

$$\Delta\mathcal{L}_I = g_3\{[\bar{r}\gamma^\mu r(V_\mu^3/2 + V_\mu^8/\sqrt{12})] + [\bar{r}\gamma^\mu y(V_\mu^1 - iV_\mu^2)/2 + \text{H.c.}] + [\bar{r}\gamma^\mu b(V_\mu^4 - iV_\mu^5)/2 + \text{H.c.}]\}\,, \quad (4)$$

so that self-energy bubbles from V^{1-5} and V^8 must be added to find $\Sigma(p)$. This sum can be obtained from the QED expressions for r and s (corresponding to the exchange of a single vector-coupled gauge boson) by replacing e^2 [Eq. (3)] with

$$g_3^2[\tfrac{1}{4} + \tfrac{1}{12} + 4(\tfrac{1}{4})] = 4g_3^2/3\,,$$

where the V^3, V^8, and the four equal V^1, V^2, V^4, V^5 contributions are respectively indicated. In general, if fermions transform under a fundamental representation of SU(N) that is vector-coupled to the gauge bosons of the group, then $\Sigma(p)$ is given by

$$r = \frac{g^2 d^0}{16\pi^2} C_2(N) \ln\frac{|p^2|}{\mu^2}\,,$$
$$s = -\frac{g^2(3 + d^0)}{16\pi^2} C_2(N)\ln\frac{|p^2|}{\mu^2}\,, \quad (5)$$

where

$$\lambda_{\alpha\beta}^I \lambda_{\beta\gamma}^I/4 = C_2(N)\delta_{\alpha\gamma} = [(N^2 - 1)/2N]\delta_{\alpha\gamma}\,.$$

The fermion mass dimension $\gamma_m^{(N)}$ is found by taking the renormalized fermion inverse propagator

$$\Gamma^{(2)}(p) = [1 - 2r(p^2)][a(p^2)\not{p} - b(p^2)m + O(m^2/p^2)]$$

through the renormalization-group equation[6]

$$0=\mu\frac{d}{d\mu}\Gamma^{(2)}(p)$$
$$=\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+m\gamma_m^{(N)}\frac{\partial}{\partial m}+\gamma_\psi^{(N)}+\delta(g,d)\frac{\partial}{\partial d}\right]\Gamma^{(2)}(p). \tag{6}$$

We substitute r and s [Eq. (5)] into the inverse-propagator coefficients $a(p^2)$ and $b(p^2)$ [Eq. (2)] to obtain mass ($\gamma_m^{(N)}$) and wave-function ($\gamma_\psi^{(N)}$) fermion dimensions to order g^2 [$\beta(\partial/\partial g)$ and $\delta(\partial/\partial d)$ are $O(g^4)$]:

$$\gamma_m^{(N)}=-3g^2C_2(N)/8\pi^2,$$
$$\gamma_\psi^{(N)}=-g^2C_2(N)d^0/8\pi^2. \tag{7}$$

$\gamma_m^{(N)}$ is independent of d^0 and is therefore gauge invariant.[7]

Now consider the self-energy contribution $\Delta\Sigma(p)$ from the interaction of fermions to a single non-vector-coupled gauge boson B:

$$\Delta\mathcal{L}=g\bar{f}\gamma^\mu[a_+(1+\gamma_5)/2+a_-(1-\gamma_5)/2]fB_\mu.$$

This contribution is given by

$$\Delta r=\frac{g^2}{16\pi^2}d^0\frac{(a_+^2+a_-^2)}{2}\ln\frac{|p^2|}{\mu^2},$$
$$\Delta s=-\frac{g^2}{16\pi^2}(3+d^0)a_+a_-\ln\frac{|p^2|}{\mu^2} \tag{8}$$

[for vector-coupled theories $a_+=a_-=1$, and Eq. (8) becomes the same as Eq. (3)]. These values for Δr and Δs can be used to determine the corrections to the inverse propagator $\Gamma^{(2)}=(1-r)\not{p}-(1+s)m$. Subsequent substitution of this new inverse propagator into the renormalization-group equation leads to the following contributions to fermion dimensions:

$$\Delta\gamma_\psi=\frac{-g^2}{16\pi^2}d^0(a_+^2+a_-^2),$$
$$\Delta\gamma_m=\frac{g^2}{16\pi^2}[-(6+2d^0)a_+a_-+d^0(a_+^2+a_-^2)]. \tag{9}$$

The above expression $\Delta\gamma_m$ for the mass dimension from a single gauge boson is the same as that given in Ref. 3 [Eq. (2.22)], *provided one chooses the Feynman gauge* ($d^0=1$). Similarly, Eq. (9) yields the same values as Ref. 2 for U(1) fermion-mass dimensions from the Weinberg-Salam U(1) current,[8]

$$\mathcal{L}_I=(g_1B^\mu/\sqrt{60})\{\bar{u}\gamma_\mu[(1+\gamma_5)/2+4(1-\gamma_5)/2]u+\bar{d}\gamma_\mu[(1+\gamma_5)/2-2(1-\gamma_5)/2]d+\bar{e}\gamma_\mu[-3(1+\gamma_5)/2-6(1-\gamma_5)/2]e+\bar{\nu}\gamma_\mu[-3(1+\gamma_5)/2]\nu\}, \tag{10}$$

provided one chooses the Landau gauge ($d^0=0$).[9] In any case, Eq. (9) shows that gauge invariance of γ_m is lost unless $a_+=a_-$, in which case the coupling is vectorial.

What is the meaning of this failure of gauge invariance? The u, d, and e mass dimensions in the SU(2)× U(1) (Weinberg-Salam[8]) and the SU(5) (Georgi-Glashow[1]) theories are easily calculated by using the recipe of Eq. (9) to obtain the contribution of each gauge boson in SU(2), U(1), and SU(5) to $\gamma_m^{(N)}$. Adding each contribution to $\gamma^{(2)}$, $\gamma^{(1)}$, and $\gamma^{(5)}$, one obtains

$$\gamma_{e,d,u}^{(2)}=\frac{g_2^2}{16\pi^2}(\tfrac{3}{4}d^0),\quad \gamma_e^{(1)}=\frac{g_1^2}{16\pi^2}(-\tfrac{9}{5}+\tfrac{3}{20}d^0),$$
$$\gamma_d^{(1)}=\frac{g_1^2}{16\pi^2}(\tfrac{1}{5}+\tfrac{3}{20}d^0),\quad \gamma_u^{(1)}=\frac{g_1^2}{16\pi^2}(-\tfrac{2}{5}+\tfrac{3}{20}d^0),$$
$$\gamma_{e,d}^{(5)}=\frac{g_5^2}{16\pi^2}(-\tfrac{54}{5}+\tfrac{12}{5}d^0),\quad \gamma_u^{(5)}=\frac{g_5^2}{16\pi^2}(-\tfrac{72}{5}+\tfrac{12}{5}d^0). \tag{11}$$

Note that the difference between $\gamma^{(i)}$'s obtained for different fermion identities is always independent of d^0. The ratios of fermion masses depend only on these differences. For example, if two fermions f and g have mass dimensions $\gamma_f^{(i)}$, $\gamma_g^{(i)}$ under the subgroup G_i, then

$$\ln\frac{m_f(\mu)}{m_g(\mu)}=\ln\frac{m_f(M)}{m_g(M)}+\sum_i\int_M^\mu[\gamma_f^{(i)}(\mu')-\gamma_g^{(i)}(\mu')]d\mu'/\mu'. \tag{12}$$

Thus the calculations of fermion mass ratios given in Refs. 2 and 3 remain gauge invariant, even though the individual fermion-mass dimensions are not gauge invariant.[10,11]

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[7]$\gamma_\psi^{(N)}$ [Eq. (7)] corresponds to $-2\gamma_f$ in Eq. (4.19) of D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973), as can be seen by comparing Eq. (6) with their parametrization of the renormalization-group equation [Eq. (5.18)].

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[9]The authors of Ref. 2 erred by assigning a nonzero value to $\gamma_f^{(2)}$. Their value corresponds to what $\gamma^{(2)}$ would be if SU(2) were coupled vectorially rather than left-handedly. However, erroneous but equal factors for $\gamma_e^{(2)}$, $\gamma_d^{(2)}$, $\gamma_u^{(2)}$ canceled in their mass-ratio calculations, which are unaffected.

[10]Note also that $|\gamma_u^5| > |\gamma_d^5|$. The SU(5) model currents (Ref. 1) are entirely contained within SO(10). (I am grateful to S. Rajpoot for demonstrating this explicitly.) Thus if u and d fermions have the same mass in an SO(10) theory until breakdown into an SU(5) intermediate step occurs, $m_{u,c,t}$ will become greater than $m_{d,s,b}$, respectively, until SU(5) symmetry is broken further to SU(3) ×SU(2) ×U(1). Therefore, an SO(10) → SU(5) → SU(3) ×SU(2) ×U(1) hierarchy admits the possibility of u, d, and e (or c, s, and μ; t, b, and τ) evolving from a single mass. However, my estimate of the ultimate unifying mass scale for such a theory is much greater than the Planck mass and, therefore, unrealistic.

[11]This behavior is to be contrasted with that of the running coupling constants $g(Q^2)$, which are manifestly independent of the choice of gauge (D. J. Gross and F. Wilczek, Ref. 7). Indeed, one of the reasons for believing in QCD is the calculable decrease of $g_3(Q^2)$ with increasing Q^2, as manifested in the scaling of leptoproduction cross sections. Note, however, that running coupling constants and "running mass ratios" are intrinsically dimensionless, whereas the running mass is not.