

Hadron masses in quantum chromodynamics on the transverse lattice

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Calculational methods are formulated for the transverse lattice version of quantum chromodynamics. These methods are used to study the low-lying spectrum of gluon bound states in the pure Yang-Mills theory.

I. INTRODUCTION

It has been proposed that the strong-interaction aspects of particle dynamics can be described by a Lagrangian field theory, quantum chromodynamics (QCD). The theory consists of colored quarks interacting via colored gluons. The original applications of the theory to short-distance and light-cone-dominated processes have been extended to encompass various inclusive processes¹ ($e^+e^- \rightarrow n$ jets, $pp \rightarrow \gamma^* \gamma X$, and others) and maybe even certain exclusive processes² (elastic form factors of hadrons). Renormalization-group equations and general factorization properties were used to analyze those phenomena perturbatively.

The applications of the same methods to study the large-distance structure of the theory have not resulted in the emergence of a spectrum consisting of only color-singlet bound states.³ Various attempts have been made to incorporate nonperturbative effects in QCD. Semiclassical methods utilizing instantons and other configurations have important consequences for the chiral structure of QCD and may even result in an effective MIT-like bag theory.⁴ The strong-coupling aspects are emphasized by reformulating QCD in terms of a lattice gauge theory. Wilson⁵ has proposed a four-dimensional Euclidean lattice version of QCD while Kogut and Susskind⁶ have studied a Hamiltonian formulation of QCD. Within those lattice theories, bound-state spectra have been calculated in a strong-coupling expansion. This was done for pure Yang-Mills theory⁷ and for colored quarks interacting^{5,8} with fermions. These calculations are qualitatively successful for aspects not involving chiral symmetry and seem to result in a stringlike picture for excited bound states. Finally, 't Hooft⁹ has proposed the $1/N_c$ expansion, where the color group is taken to be $SU(N_c)$,

which leads to a pictorial simplification of the theory. The task of summing the surviving planar diagrams still seems formidable. However, if QCD confines, then for large N_c a valence-quark picture emerges for the meson sector which will consist of an infinite number of stable hadrons. The pure gluon-bound-state sector will also contain stable hadrons decoupled from mesons.¹⁰ Another approach to QCD was discussed by Bardeen and Pearson¹¹ (to be referred to as I). It will be reviewed in Sec. II. Its structure is nontrivial in both the weak- and strong-coupling regimes. In this paper we discuss the calculation of hadronic gluon-bound-state masses in the strong-coupling regime. In Sec. III we discuss the nonperturbative longitudinal dynamics of the model. Bare hadrons are constructed from gauge potentials and "real" color degrees of freedom. In Sec. IV, a perturbative strong-coupling analysis of an effective field theory for bare hadrons is performed resulting in hadrons with transverse motion. The calculations are discussed in Sec. V.

II. TRANSVERSE-LATTICE VERSION OF QCD

In this section we will review the transverse-lattice version of QCD given in I. First we will present the lattice action functional in terms of the link variables introduced by Wilson.⁵ Then we will discuss the transformation to linearized degrees of freedom. Finally, we will quantize the resulting linear theory. Since this paper is only concerned with the pure Yang-Mills sector of the theory we will not discuss here any of the problems associated with describing fermions on the lattice.

If we use the matrix form of the gauge field

$$A_\mu = igA_\mu^a T^a, \quad (2.1)$$

where the group generators are normalized by the conditions

$$\text{tr}(T^a T^b) = \delta^{ab}/2, \quad [T^a, T^b] = if^{abc} T^c, \quad (2.2)$$

then the QCD action has the form

$$A = \int d^4x \frac{1}{2g^2} \text{tr}(G^{\mu\nu} G_{\mu\nu}), \quad (2.3)$$

where the Yang-Mills field strength is given by

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \quad (2.4)$$

Since there are only two dynamical gauge fields we choose a way of putting the theory on the lattice, with a gauge and quantization procedure which exploits this fact. First, we choose to leave the longitudinal coordinates t and z and the longitudinal gauge fields A_t and A_z intact while we make the transverse coordinates discrete. Thus,

$$\vec{x}_\perp = a(n_x, n_y), \quad (2.5)$$

where a is the lattice spacing and n_x and n_y are integers. The transverse gauge fields A_x and A_y are replaced by link variables $U_{\vec{x}_\perp, \alpha}(t, z)$ which are associated with the link between the lattice sites \vec{x}_\perp and $\vec{x}_\perp + \hat{\alpha}$, where $\alpha = 1, 2$ and $\hat{\alpha} = (a, 0)$ or $(0, a)$. The U 's also depend on the longitudinal coordinates t and z . The U 's may be interpreted as the phase factors

$$\exp\left(ig \int_{\vec{x}_\perp}^{\vec{x}_\perp + \hat{\alpha}} A_\beta dx^\beta\right). \quad (2.6)$$

We may now write down a discrete version of the QCD action which reproduces (2.3) in the naive continuum limit ($ga \rightarrow 0$) with the expression (2.6) substituted for the U 's and the exponential expanded out in powers of a , viz.,

$$A = \int dz dt \left(\sum_{\vec{x}_\perp, \nu} \frac{a^2}{2g^2} \text{tr}(G^{\mu\nu} G_{\mu\nu}) + \sum_{\vec{x}_\perp, \alpha, \mu} \frac{1}{g^2} \text{tr}(D_\mu U_{\vec{x}_\perp, \alpha} D^\mu U_{\vec{x}_\perp, \alpha}^\dagger) + \sum_{\vec{x}_\perp, \alpha, \beta} \frac{1}{g^4 a^2} \text{tr}(U_{\vec{x}_\perp, \alpha} U_{\vec{x}_\perp + \hat{\alpha}, \beta} U_{\vec{x}_\perp + \hat{\beta}, \alpha}^\dagger U_{\vec{x}_\perp, \beta}^\dagger) \right), \quad (2.7)$$

where $\mu, \nu = 0, 3$ and $\alpha, \beta = 1, 2$. The (longitudinal) covariant derivative of the link variables is given by

$$D_\mu U_{\vec{x}_\perp, \alpha} = [\partial_\mu + A_\mu(\vec{x}_\perp)] U_{\vec{x}_\perp, \alpha} - U_{\vec{x}_\perp, \alpha} A_\mu(\vec{x}_\perp + \hat{\alpha}). \quad (2.8)$$

The action for the longitudinal gauge fields is unchanged from the continuum except to replace integrals by sums. The action for the purely transverse gauge fields is the familiar "plaquette" action used before in lattice gauge theories. The mixed term involving $D_\mu U$ is the simplest local interaction which reproduces the corresponding mixed term in the continuum. Most importantly, the lattice action (2.7) remains gauge invariant under the restriction of the original gauge group to the lattice. We can use this gauge invariance to completely eliminate the longitudinal gauge fields A_μ from the theory. This may be accomplished by using light-cone gauge [$A_- = (A_0 - A_3)/\sqrt{2} = 0$] with light-cone quantization [$\tau = x^+ = (x^0 + x^3)/\sqrt{2}$] so that A_+ becomes a parametric field which may be eliminated by its equations of constraint in favor of a nonlocal Coulomb interaction. In this respect the treatment of each transverse-lattice site is the same as the discussion of two-

dimensional QCD given by several authors.¹² Explicitly setting A_- to zero, the terms in (2.7) which depend on A_+ become

$$A = \int dx^+ dx^- \sum_{\vec{x}} \left(\frac{a^2}{2} (\partial_- \vec{A}_+)^2 + g \vec{A}_+ \cdot \vec{J}_- + \sum_{\alpha} \frac{1}{g^2} \text{tr}(\partial_\mu U_{\vec{x}_\perp, \alpha} \partial^\mu U_{\vec{x}_\perp, \alpha}^\dagger) + \dots \right), \quad (2.9)$$

where we are using the Hermitian form of \vec{A}_+ in the vector representation and the current \vec{J}_- is given by

$$\vec{J}_- = \frac{1}{g^2} \sum_{\alpha} \text{tr} \left[\vec{T} (U_{\vec{x}_\perp, \alpha} i \vec{\partial}_- U_{\vec{x}_\perp, \alpha}^\dagger + U_{\vec{x}_\perp - \hat{\alpha}, \alpha}^\dagger i \vec{\partial}_- U_{\vec{x}_\perp - \hat{\alpha}, \alpha}) \right]. \quad (2.10)$$

The Euler-Lagrange equations for \vec{A}_+ are

$$\partial_-^2 \vec{A}_+ = \frac{g}{a^2} \vec{J}_- \quad (2.11)$$

which contain no "time" derivatives $\partial_+ = \partial/\partial\tau$ and so can be solved for \vec{A}_+ in terms of \vec{J}_- without upsetting our subsequent quantization. Eliminating \vec{A}_+ by (2.11) gives an effective action which only depends on the U 's,

$$A = \int dx^+ dx^- \sum_{\vec{x}_1} \left(\frac{1}{g^2} \sum_{\alpha} \text{tr}(\partial_{\mu} U_{\vec{x}_1, \alpha} \partial^{\mu} U_{\vec{x}_1, \alpha}^{\dagger}) + \frac{1}{g^2 a^2} \sum_{\alpha\beta} \text{tr}(U_{\vec{x}_1, \alpha} U_{\vec{x}_1 + \hat{\alpha}, \beta} U_{\vec{x}_1 + \hat{\beta}, \alpha}^{\dagger} U_{\vec{x}_1, \beta}^{\dagger}) + \int dx'^- \frac{g^2}{4a^2} |x^- - x'^-| \bar{J}_{-}(x^-) \cdot \bar{J}_{-}(x'^-) \right). \quad (2.12)$$

In order for this action to give the usual QCD action in the continuum it was necessary to assume that the matrices U have the form

$$U = e^{i\alpha \vec{T} \cdot \vec{A}} \quad (2.13)$$

so that over small regions of space, U can be expanded in a Taylor series. (This is the same as the "spin-wave" expansion used at low temperature in statistical mechanics. In I it was incorrectly stated that in order for the expansion to be valid it was necessary for U to develop a vacuum expectation value. It is actually only necessary that over any small region of transverse space that the differences between U 's be small.) Thus in particular, we are assuming a functional measure for U which restricts it to the space of unitary matrices. There are several reasons why this is not the correct choice if we wish to describe physics correctly. The first and most mundane is that there is no compelling reason for finite values of the lattice spacing a to use this measure. Any measure which preserves the remaining gauge invariance under gauge transformations which are global in the longitudinal coordinates t and z [i. e., a global $SU(n)$ associated with each vertex of the transverse lattice] is *a priori* a suitable candidate. The correct measure can only be determined by a real-space renormalization-group analysis but might take the form

$$dU^{\dagger} dU \exp \left(\int dx^+ dx^- V(a, U, U^{\dagger}) \right) \rightarrow \int_{a=0} dU^{\dagger} dU \delta(U^{\dagger} U - 1) \delta(\det(U) - 1), \quad (2.14)$$

where U is allowed to be a complex general matrix, and dU is an unrestricted measure. This potential V may depend on the lattice spacing a and the invariants $U^{\dagger} U$ and $\det(U)$. Put simply, the variables of a lattice theory can never describe the underlying continuum theory exactly but are supposed to represent, in an aggregate way, the nearby degrees of freedom of the continuum theory. If the microscopic variables satisfy a constraint the new aggregate may not and we may wish to choose a linear variable to describe the average behavior of a collection of nonlinear (i. e., unitary) variables. A more rigorous argument for the above statement comes from the, now exactly known,¹³ behavior of the nonlinear $[O(n)] \sigma$ model in two dimensions. The local de-

grees of freedom are nonlinear, i. e., satisfy a constraint, and in greater than two dimensions the field would develop a vacuum expectation value and the spectrum would consist of $(n-1)$ massless Goldstone bosons. In two dimensions the particle structure of the theory contains (n) massive scalars transforming under a linear realization of the symmetry group. Particle structure is inherently a large-distance property and thus depends on the correct aggregate variables and not directly on the microscopic variables. Any physical description of the theory in terms of the underlying nonlinear degrees of freedom must be highly nonperturbative, but it is possible to achieve a much simpler description of the physics with an effective action in terms of explicitly linear variables and their interactions. For large n the form of the effective action for the $O(n)$ nonlinear σ model can be constructed explicitly.¹⁴ In the action (2.12) the kinetic energy terms for U are identical to the action for an $SU(n) \times SU(n)$ nonlinear σ model defined at each link of the transverse lattice. If the other two terms of the lattice action which couple these σ -model systems together do not modify the above conclusions, then the physically correct variables would be some linear variables to replace the U 's as in (2.14). If we knew the exact behavior of the $SU(n) \times SU(n)$ σ model, we could rewrite the action (2.12) in terms of linear variables without approximation. Since we do not, we adopt the ansatz (2.14) with a minimal set of operators included in V . The coefficients will have to remain as free parameters to be fit to the desired spectrum and be eventually determined through the continuum limit analysis. It is useful to introduce the scalar field

$$M_{\vec{x}_1, \alpha} = \frac{1}{g} U_{\vec{x}_1, \alpha}. \quad (2.15)$$

The effective potential takes the form

$$V = \mu_0^2 \text{tr}(M^{\dagger} M) + \lambda_1 \text{tr}[(M^{\dagger} M)^2] + \lambda_2 \text{tr}(M^{\dagger} M)^2 + \lambda_3 [\det(M) + \text{H. c.}] + \dots \quad (2.16)$$

Since in later chapters we will make use of the large- N expansion, only the first two terms in (2.16) will play any role in our quantitative analysis. Thus, we can write the action in terms of M as

$$A = \int dx^+ dx^- \sum_{\bar{x}_1} \left(\sum_{\alpha} \text{tr}(\partial_{\mu} M_{\bar{x}_1, \alpha} \partial^{\mu} M_{\bar{x}_1, \alpha}) + \mu^2 \sum_{\alpha} \text{tr}(M_{\bar{x}_1, \alpha}^{\dagger} M_{\bar{x}_1, \alpha}) + \lambda_2 \sum_{\alpha} \text{tr}[(M_{\bar{x}_1, \alpha} M_{\bar{x}_1, \alpha}^{\dagger})^2] \right. \\ \left. + \frac{g^2}{a^2} \sum_{\alpha\beta} \text{tr}(M_{\bar{x}_1, \alpha} M_{\bar{x}_1 + \hat{\alpha}, \beta} M_{\bar{x}_1 + \hat{\beta}, \alpha}^{\dagger} M_{\bar{x}_1, \beta}^{\dagger}) + \int dx'^- \frac{g^2}{4a^2} |x^- - x'^-| \bar{J}_{-}(x^-) \cdot \bar{J}_{-}(x'^-) \right). \quad (2.17)$$

In this version of the action, M is a full $n \times n$ complex matrix of fields.

We have discussed in some length the reasons why we have chosen the form (2.17) for the action of the theory. In return for a more physical starting point for the description of this system, we have given up exact knowledge of some of the parameters which must be fit to the desired spectrum. We feel that there should exist an exact transformation to an effective linear theory derivable by renormalization-group analysis, but we have not carried this out. The remaining task of this section is to quantize the action (2.17) in the light-cone variables. This is standard¹⁵ and presents no new difficulties. The Hamiltonian corresponding to (2.17) is given by

$$P^+ = H = \int dx^- \sum_{\bar{x}_1} \left(\mu^2 \sum_{\alpha} \text{tr}(M_{\bar{x}_1, \alpha} M_{\bar{x}_1, \alpha}^{\dagger}) - \frac{g^2}{a^2} \sum_{\alpha\beta} \text{tr}(M_{\bar{x}_1, \alpha} M_{\bar{x}_1 + \hat{\alpha}, \beta} M_{\bar{x}_1 + \hat{\beta}, \alpha}^{\dagger} M_{\bar{x}_1, \beta}^{\dagger}) \right. \\ \left. - \lambda \sum_{\alpha} \text{tr}[(M_{\bar{x}_1, \alpha} M_{\bar{x}_1, \alpha}^{\dagger})^2] - \int dx'^- \frac{g^2}{4a^2} |x^- - x'^-| \bar{J}_{-}(x^-) \cdot \bar{J}_{-}(x'^-) \right). \quad (2.18)$$

The link fields may be decomposed into creation and annihilation operators in momentum space as

$$M_{\bar{x}_1, \alpha} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{2k} [A_{k, \bar{x}_1, \alpha} e^{-ikhx^-} + B_{k, \bar{x}_1, \alpha}^{\dagger} e^{+ikhx^-}], \quad (2.19)$$

where the A 's and B 's obey the equal-light-front commutation relations

$$[A_k, A_{k'}^{\dagger}] = [B_k, B_{k'}^{\dagger}] = 2k\delta(k - k'). \quad (2.20)$$

In the subsequent sections of this paper we will study the spectrum of the mass operator $M^2 = 2P^+P^-$ by diagonalizing it in the space generated by the decomposition (2.19) when limited to states containing few particles.

III. LONGITUDINAL DYNAMICS

In this paper we study the pure Yang-Mills sector of QCD based on the effective action of Sec. II. Since the full theory cannot be solved we seek a practical and physically motivated approximation scheme. The principal ingredients of the scheme we propose are (1) a perturbative expansion in the terms in H responsible for transverse motion on the lattice, (2) a perturbative expansion in the nonlinearities introduced by the effective potential for the link mesons, and (3) a topological expansion in the nonplanarities of link-meson interactions. The longitudinal free-link-meson dynamics and the longitudinal Coulomb potential will be treated exactly. Although we performed a nonperturbative step by changing from the original gluonic degrees of freedom to the link mesons we have not yet constructed hadrons. In this section we shall study the transformation from the link-meson gluons to intermediate hadronic degrees of freedom, the bare hadrons. This transformation is characterized by a single parameter and

may have accordingly weak-, strong-, or intermediate-coupling features.

We start by showing how confinement arises in this approximation as a consequence of both symmetry and energetics: Using the transverse lattice and the light-cone gauge, the gluon dynamics have been separated into link mesons and Coulomb potentials. The link mesons connect (x_+, x_-) sheets and interact in each sheet via Coulomb potentials which are instantaneous and confined to the given sheet. All states of finite energy are singlets with respect to color rotations at each transverse site. It is this feature of local color confinement which leads to binding of the link mesons. The bare Hamiltonian which governs the longitudinal dynamics of the link mesons may be written as

$$H_0 = \sum_{\bar{n}, \alpha} \int dx \mu_0^2 \text{tr}(M_{\bar{n}\alpha} M_{\bar{n}\alpha}^{\dagger}) \\ - \frac{g^2}{4a^2} \sum_{\bar{n}} \int dx dy |x - y| [\bar{J}_{\bar{n}}(x) \cdot \bar{J}_{\bar{n}}(y)]^{\text{Coul}}, \quad (3.1)$$

where

$$\bar{J}_{\bar{n}} = \sum_{\alpha} (\text{tr} \bar{T} M_{\bar{n}\alpha} i \bar{\partial} M_{\bar{n}\alpha}^{\dagger} + \bar{T} M_{\bar{n}-\hat{\alpha}, \alpha}^{\dagger} i \bar{\partial} M_{\bar{n}-\hat{\alpha}, \alpha}). \quad (3.2)$$

The notation Coul signifies that we are to keep only the direct Coulomb potential part of the interaction and not those parts which produce or annihilate pairs. Using the expansion of M into plane-wave creation and annihilation operators, H_0 may be rewritten as

$$H_0 = \sum_{\bar{n}, \alpha} \int_0^{\infty} \frac{dk}{2k} \frac{\mu_0^2}{2k} \text{tr}(A_{k, \bar{n}, \alpha}^{\dagger} A_{k, \bar{n}, \alpha} + B_{k, \bar{n}, \alpha}^{\dagger} B_{k, \bar{n}, \alpha}) \\ - \frac{g^2}{4\pi a^2} \sum_{\bar{n}} \int_{-\infty}^{\infty} \frac{dq}{q^2} \bar{J}_{\bar{n}}(q) \cdot \bar{J}_{\bar{n}}(-q), \quad (3.3)$$

where

$$\bar{J}_{\bar{i}}(q) = i \sum_{\alpha} \int_0^{\infty} \frac{dk}{4k(k+q)} \theta(k+q)(2k+q) \text{tr} \{ \bar{T} : [(A_k A_{k+q}^{\dagger})_{\bar{i}, \alpha} - (B_{k+q}^{\dagger} B_k)_{\bar{i}, \alpha} - (A_{k+q}^{\dagger} A_k)_{\bar{i}-\bar{\alpha}, \alpha} + (B_k B_{k+q}^{\dagger})_{\bar{i}-\bar{\alpha}, \alpha}] : \}. \quad (3.4)$$

In $J(q)$ pair operators have been dropped. The Coulomb interaction represents a momentum transfer q from one link to another with a propagator $1/q^2$. The integral is infrared divergent when $q=0$ and this divergence is regulated by a principal-value prescription. It may be shown that the properties of color-singlet states are independent of the way in which these infrared divergences are regulated.

By construction H_0 conserves the number of links or antilinks. The eigenstates of H_0 are thus all color singlets that can be initially classified according to the number of link mesons in each state. In *this gauge* the linked mesons may be ascribed the role of "valence gluons." The mass spectrum of each "number sector" is infinite and discrete due to the linear Coulomb potential. H_0 also conserves the total longitudinal momentum and the group representation at each vertex, although we only consider states which are color singlets at each vertex. The simplest nontrivial

sector of H_0 consists of one link and one antilink meson between the same pair of vertices. Such states are created by the action of $A_{\bar{i}\alpha}^{\dagger} B_{\bar{i}\alpha}^{\dagger}$ on the ground state. The only way to make a group singlet at both ends from this operator is to take the trace of the matrix product. Thus, we consider the state

$$|P\rangle = \frac{1}{N} \int_0^1 dx \phi(x) [2x(1-x)]^{-1/2} \times \text{tr}(A_{xP}^{\dagger} B_{(1-x)P}^{\dagger} |0\rangle). \quad (3.5)$$

The extra factors have been introduced so that the norm of this state is

$$\langle P|Q\rangle = 2P\delta(P-Q) \quad (3.6)$$

if ϕ is normalized to

$$\int_0^1 dx |\phi(x)|^2 = 1. \quad (3.7)$$

Applying H_0 to this state gives

$$\begin{aligned} H_0|P\rangle &= \frac{\mu_0^2}{2P} \frac{1}{N} \int_0^1 dx \phi(x) [2x(1-x)]^{-1/2} [1/x + 1/(1-x)] \text{tr}(A_{xP}^{\dagger} B_{(1-x)P}^{\dagger} |0\rangle) \\ &+ \frac{1}{2P} \frac{2g^2 C_N}{a^2} \left(\int_0^{\infty} \frac{dq(q+P)^2}{2q(q-P)^2} \right) \frac{1}{N} \int_0^1 dx \phi(x) [2x(1-x)]^{-1/2} [1/x + 1/(1-x)] \text{tr}(A_{xP}^{\dagger} B_{(1-x)P}^{\dagger} |0\rangle) \\ &- \frac{1}{2P} \frac{2g^2 C_N}{\pi a^2} \int_0^1 \frac{dx}{x(1-x)} \frac{1}{N} \int_0^1 dy \phi(y) [2y(1-y)]^{-1/2} \\ &\quad \times (x+y) \{ [(1-x) + (1-y)] / (4|x-y|^2) \} \text{tr}(A_{xP}^{\dagger} B_{(1-x)P}^{\dagger} |0\rangle). \end{aligned} \quad (3.8)$$

To be an eigenstate $|P\rangle$ must satisfy the equation

$$H_0|P\rangle = \frac{M^2}{2P} |P\rangle. \quad (3.9)$$

Projecting out the momentum components gives the integral equation for ϕ

$$M^2 \phi(x) = \left(\mu_0^2 + \frac{2g^2 C_N}{\pi a^2} \int_0^{\infty} \frac{dq(q+P)^2}{q(q-P)^2} \right) \phi(x) \left(\frac{1}{x} + \frac{1}{1-x} \right) - \frac{2g^2 C_N}{\pi a^2} \int_0^1 \frac{dx'}{|x-x'|^2} \phi(x') \frac{(x+x')[(1-x) + (1-x')]}{4[xx'(1-x)(1-x')]^{1/2}}. \quad (3.10)$$

In the first term on the right-hand side the quantity in brackets is the renormalized mass. This equation differs from the bound-state equation in the 't Hooft model¹² in two respects. First, the current vertex of a boson introduces the additional spin factor

$$\frac{(x+x')[(1-x) + (1-x')]}{4[xx'(1-x)(1-x')]^{1/2}}$$

and second, there is an overall factor of 2 since the links are bound by two Coulomb potentials, one at each end.

We do not know an explicit form of the solution to this equation; nevertheless one can recognize some qualitative properties of the solutions. Since the equation is invariant under $x \leftrightarrow (1-x)$ with $\phi(x) \rightarrow \pm \phi(1-x)$, we may classify states as being even or odd with respect to this symmetry. $\phi(x)$ will have power behavior near $x \rightarrow 0$, or 1. If we suppose $\phi(x) \rightarrow x^\beta$ near $x \rightarrow 0$, we obtain the consistency condition

$$\mu^2 - \frac{g^2 C_N}{2\pi a^2} \int_0^{1/x} dz \frac{z^{\beta-1/2}(1+z)}{(1-z)^2} \xrightarrow{x \rightarrow 0} 0 \quad (3.11)$$

from the requirement that the leading singularity in x vanishes. For the integral to converge at both ends, we require $-\frac{1}{2} < \beta < \frac{1}{2}$. If $\beta > \frac{1}{2}$, the integral approaches a finite limit independent of x as $x \rightarrow 0$, so there is no solution. Performing the integration gives the condition on β

$$\mu^2 = \frac{2g^2 C_N}{\pi a^2} \pi \beta \tan \pi \beta. \quad (3.12)$$

There are two solutions in the range $-\frac{1}{2} < \beta < \frac{1}{2}$ for any positive value of μ^2 . There are no solutions for negative μ^2 which ultimately will imply that the lightest bare hadron is massive. For the limiting case $\mu^2 = \beta = 0$, $\phi(x) = 1$ fails to be a solution and the bound-state mass remains finite. Thus at the bare-hadron approximation, gluon-bound states are inherently heavier than usual $\bar{q}q$ mesons. One could also note that the same feature holds in the strong-coupling limit of the Wilson lattice and is a general property of the theory. An analysis of the self-adjointness of H_0 rules out the negative solution for β . For highly excited states where the end-point behavior is unimportant, this equation is effectively identical to the 't Hooft equation except for the previously mentioned factor of 2; thus if one relates these bare hadrons to the spinless daughters of the Pomeron trajectory, the Pomeron trajectory would have half the slope of the "usual" particle trajectories. The parameter β determines the features of bare hadrons. For small β they are strongly bound states while for β near $\frac{1}{2}$ they are weakly bound systems well described by two link mesons. The physical hadrons will eventually be labeled by a definite β .

In order to proceed with our program, we need to know the eigenvalues and eigenfunctions for the bound states in the two-link-meson sector. A numerical procedure allows us to compute accurate values for the eigenvalues and eigenfunctions $\phi(x)$. In particular, we consider as a basis set of functions

$$\phi_n(x) = [x(1-x)]^\beta P_n(x), \quad (3.13)$$

where $P_n(x)$ is an n th-order polynomial in x which

is either even or odd under $x \leftrightarrow (1-x)$. If the P_n are chosen as the appropriately normalized and scaled Jacobi polynomials, the ϕ_n 's provide an orthonormal basis. One may then diagonalize the finite-dimensional matrix $H_{m,n} = \langle m | H_0 | n \rangle$. The eigenvalues provide estimates to the true eigenvalues which converge rapidly as the order of the matrix is increased. The eigenvectors, which are the expansion coefficients of $\phi(x)$ in this basis, also provide converging estimates to the wave functions. The evaluation of the matrix elements can be performed analytically. First the polynomials are expanded as power series in $[x(1-x)]^n$ or $(1-2x)[x(1-x)]^n$ and then the integrals are performed using the identity

$$\int_0^1 dx dy \frac{[x(1-x)]^\alpha [y(1-y)]^\beta}{|x-y|^2} = -\frac{2\pi\alpha\beta\Gamma(\alpha)\Gamma(\beta)}{2^{2\alpha+2\beta}(\alpha+\beta)\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})}. \quad (3.14)$$

The convergence of $\phi(x)$ at the end points is only algebraic due to the presence of subdominant singularities. As these singularities are weak [$\sim x^{\beta+1} \ln(x)$], the convergence is nevertheless rapid. However, if we require higher derivatives of $\phi(x)$, the method would fail to converge. In all subsequent applications the matrix elements of ϕ which are needed are not highly sensitive to the end-point behavior so this is not a restriction.

In Fig. 1 we plot the values of M^2 versus n , the principal quantum number for $\beta = 0.1$. For fixed β , M^2 increases with n approaching linear dependence on n , for large n .

Returning to the whole lattice theory, the spectrum of H_0 in the two-body sector consists of a set of Coulomb bound states which may be labeled by a transverse coordinate \vec{n} , a polarization α

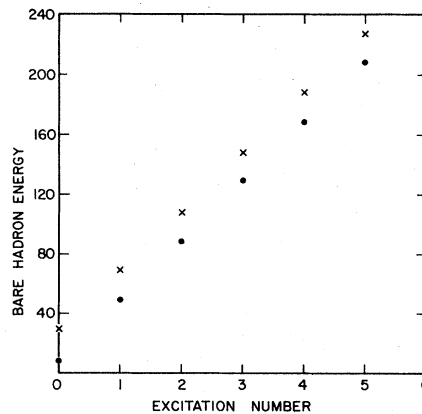


FIG. 1. The masses of two-body even (2E) [marked (•)] and odd (2O) [marked (x)] bare hadrons as a function of the radial excitation index, n . Plots are for $\beta = 0.1$.

$=\hat{x}, \hat{y}$ indicating the orientation of the bound state with respect to \vec{n} , a quantum number n indicating the excitation of the state, and a total longitudinal momentum P . We may construct from this basis states which are eigenstates of transverse momentum, but there would be no dependence of the energy on transverse momentum because of the local nature of H_0 . This unphysical restriction we will remove by a perturbative treatment of the remaining terms in H . To do so we must introduce the four-body bare-hadron states.

There are several distinct ways to place four links on the lattice so that group singlets may be formed at every vertex. These are indicated in Fig. 2. The fact that all the states should be color singlet at each transverse site leads to difficulties in defining the N_c -gluon state at each transverse site leads to difficulties in defining the N_c -

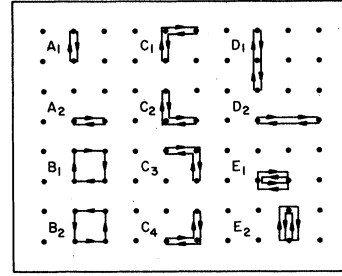


FIG. 2. Two- and four-body lattice configurations.

gluon state in the limit $N_c \rightarrow \infty$; this is like the baryon problem for quarks and thus we do not discuss in this work three-link-meson states, etc. For the open boxes $B_{1,2}$ there is only one way of forming a group singlet which is to trace the group indices around the box. Proceeding as in the two-body case we define a general state

$$|P\rangle = \frac{\sqrt{2}}{N^2} \int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(1-x_1-x_2-x_3-x_4) \phi(x_1, x_2, x_3, x_4) (2x_1 2x_2 2x_3 2x_4)^{-1/2} \text{tr}(A_{x_1 P}^\dagger A_{x_2 P}^\dagger B_{x_3 P}^\dagger B_{x_4 P}^\dagger) |0\rangle \quad (3.15)$$

with the normalization condition

$$\int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(1-x_1-x_2-x_3-x_4) |\phi(x_1, x_2, x_3, x_4)|^2 = 1. \quad (3.16)$$

Applying H_0 and projecting out momentum components leads to the bound-state equation

$$M^2 \phi = \mu^2 \phi \sum_i \frac{1}{x_i} + \left(N - \frac{1}{N}\right) (C_{12} + C_{23} + C_{34} + C_{41}) \phi, \quad (3.17)$$

where

$$C_{ij} \phi = -\frac{g^2}{\pi a^2} \int dx'_i dx'_j \delta(x_i + x_j - x'_i - x'_j) \frac{1}{|x_i - x'_i|^2} \phi(\dots, x'_i, \dots, x'_j, \dots) (x_i + x'_i) / 4 (x_i x_j x'_i x'_j)^{1/2}. \quad (3.18)$$

When four links impinge on a vertex there are two ways to make a group singlet. For example, D_2 could be drawn in the two ways shown in Fig. 3. It turns out that for large N (the number of colors) these are the correct eigenstates, in group space, of H_0 with $|1\rangle$ a true four-body bound state and $|2\rangle$ two independent two-body bound states. For general N we must be more careful. The two orthogonal combinations are $|\pm\rangle = (|1\rangle \pm |2\rangle) / [2N^3(N \pm 1)]^{1/2}$. In terms of these one obtains the bound-state equation valid for the C and D configurations

$$M^2 \phi^\pm = \mu^2 \phi^\pm \sum_i \frac{1}{x_i} + \left(N - \frac{1}{N}\right) (C_{14} + C_{23}) \phi^\pm - \frac{1}{N} (C_{12} - C_{13} + C_{14} + C_{23} - C_{24} + C_{34}) \phi^\pm + \frac{N \pm 1}{2} (C_{12} + C_{14} + C_{23} + C_{34}) \phi^\pm + \frac{(N^2 - 1)^{1/2}}{2} (C_{12} + C_{34} - C_{14} - C_{23}) \phi^\mp. \quad (3.19)$$

In the limit $N \rightarrow \infty$ neglecting nonleading terms one obtains the two uncoupled equations

$$M^2 \phi^1 = \mu^2 \phi^1 \sum_i \frac{1}{x_i} + N(C_{12} + C_{23} + C_{34} + C_{41}) \phi^1, \quad (3.20)$$

$$M^2 \phi^2 = \mu^2 \phi^2 \sum_i \frac{1}{x_i} + 2N(C_{14} + C_{23}) \phi^2.$$

The equation for $|1\rangle$ is identical to the equation

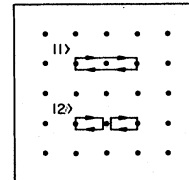


FIG. 3. Possible color structure of the D_2 four-body state.

for the B configurations. The equation for $|2\rangle$ decouples the variables (x_1, x_4) from (x_2, x_3) and will reduce to two independent two-body equations. For the E configurations, similar results are found.

If we adopt the approximation of keeping only the leading contribution in the $1/N$ expansion all of the possible configurations of four links on the lattice are described by the same bound-state equation, or reduce to two independent two-body bound states. As we shall see later in the same approximation, only the four-body bound states will contribute to the perturbative expansion of the transverse dynamics.

Many of the qualitative features of the two-body equation apply to the four-body equation as well. In particular, the consistency condition that the most singular power cancel when each of the $x_i \rightarrow 0$ requires that $\phi(\dots x_i \dots) \rightarrow x_i^\beta$ as $x_i \rightarrow 0$, where β is the same as given in Eq. (3.12). The wave functions may also be classified according to their symmetry under exchanges of various indices since the bound-state equation is invariant under cyclic permutations of the four variables or the reverse of their order. This is the symmetry of the dihedral group D_4 which has five irreducible representations, four of which are one-dimensional and one of which is two dimensional. Thus, the spectrum will be characterized by five independent trajectories, one of which will be populated by doublets. The WKB approximation to the bound-state equation is related to the spectrum of normal modes of a three-simplex. Asymptotically the density of states only depends on the volume and dimensionality of the region and behaves as $n(M^2) \propto (M^2)^3$ as $M^2 \rightarrow \infty$, or M_n^2 behaves roughly as $n^{1/3}$.

The solutions of the bound-state equation may be computed approximately just as in the two-body case by constructing a variational basis

$$\phi_{\alpha, n}(x_1, x_2, x_3, x_4) = (x_1 x_2 x_3 x_4)^\beta P_n^\alpha(x_1, x_2, x_3, x_4), \quad (3.21)$$

where $x_4 = 1 - x_1 - x_2 - x_3$, and the P_n^α are chosen as linearly independent polynomials in the four variables which transform irreducibly with representation α under the group D_4 . They are orthogonal with respect to α , but it is not particularly useful to require them to be orthogonal with respect to n since this is easily dealt with in the final numerical diagonalization. The eigenvalue equation is then cast into the form of a finite-dimensional matrix eigenvalue problem in each sector separately by working out the necessary integrals analytically. This part of the calculation was greatly facilitated by the algebraic manipulation program MACSYMA. The resulting solutions

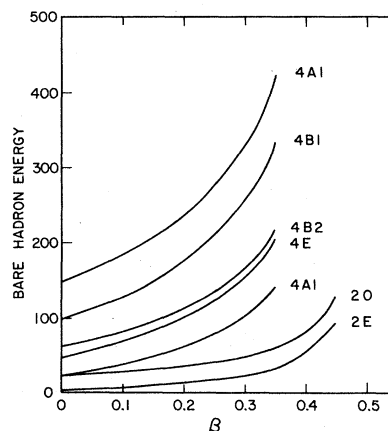


FIG. 4. Ground-state energies of two- and four-body bare hadrons is plotted as a function of β .

to the matrix problems provide estimates to the masses and wave functions as before. The details of this calculation have been relegated to Appendix A.

Patterns of the radial excitations are not presented for our numerical calculations. The ground-state energies of the various sectors (including the even and odd two-body sector) are plotted as a function of β in Fig. 4. Note that all ground-state energies tend to some finite fixed value as $\beta \rightarrow 0$. In particular, the smallest two-body eigenvalue approaches $\approx \pi^2/2$; the symmetric four-link meson A1 seems to have a lower limiting value than the odd two-link-meson state. In Fig. 5 another look at the spectroscopy of bare hadrons is given. All the results can be generalized to include hadrons with a higher link-meson number; however, in the spirit of the approximation scheme one wants to explore the consequences of truncating the link-meson sector at some small number and at a latter stage check that indeed such an expansion is convergent. By now enough

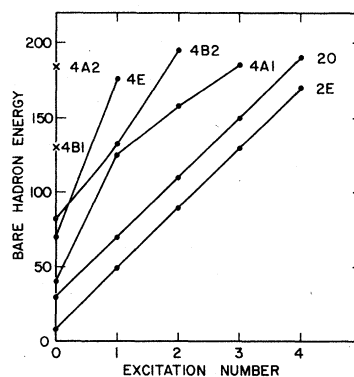


FIG. 5. Bare-hadron mass spectrum for $\beta = 0.1$.

information about the bare hadrons has been accumulated and one may proceed to evaluate the full Hamiltonian in the Hilbert state of bare hadrons.

IV. TRANSVERSE DYNAMICS

The bare hadrons constructed in Sec. III from link mesons with Coulomb binding are static on the transverse lattice. In the full Hamiltonian there are additional terms which generate nearest-neighbor residual couplings enabling the various bare-hadron states to move along the transverse lattice. They also break the conservation of link-meson number mixing various n -link-meson states. In particular, our calculation involves the mixing of two- and four-body bare-hadron

states. This effective field theory of bare hadrons is treated as a strong-coupling perturbation expansion.

The various residual couplings can be identified by examining the full Hamiltonian governing the motion of the system. These terms are generated by the Coulomb interaction, the magnetic interaction, and local potential interactions.

The Coulomb terms are identified by expanding the current-current interaction V^{cc} in terms of creation and annihilation operators. We obtain

$$V^{cc} = \sum_n \frac{g^2}{4\pi a^2} \int \frac{dk}{k^2} \vec{J}_n(k) \cdot \vec{J}_n(k), \quad (4.1)$$

where the Fourier transform of the current is given by

$$\begin{aligned} \vec{J}_n = \int_0^\infty \frac{dq}{\sqrt{2q}} \frac{dq'}{\sqrt{2q'}} & [\delta(k-q-q')(q'-q)A_{q,\vec{n},\alpha}^T B_{q',\vec{n},\alpha}^T + \delta(k-q+q')(-q'-q)A_{q,\vec{n},\alpha}^T B_{q',\vec{n},\alpha}^\dagger \\ & + \delta(k+q-q')(q+q')A_{q,\vec{n},\alpha}^\dagger B_{q',\vec{n},\alpha}^T + \delta(k+q+q')(q-q')A_{q,\vec{n},\alpha}^\dagger B_{q',\vec{n},\alpha}^\dagger]. \end{aligned} \quad (4.2)$$

In Fig. 6 we show the various vertices generated by this interaction; Figs. 6(a) and 6(b) show the Coulomb scattering vertices, already used in the longitudinal dynamics calculations, which correspond to the terms

$$\begin{aligned} \frac{1}{\sqrt{2q}} \frac{1}{\sqrt{2q'}} \delta(k-q+q')(q+q')A_q^\dagger A_{q'}, \\ \frac{1}{\sqrt{2q}} \frac{1}{\sqrt{2q'}} \delta(k-q+q')(-q-q')B_{q'}^\dagger B_q. \end{aligned} \quad (4.3)$$

The new vertices, Figs. 6(c) and 6(d), describe Coulomb production and annihilation and are given by

$$\begin{aligned} \frac{1}{\sqrt{2q}} \frac{1}{\sqrt{2q'}} \delta(k+q+q')(q-q')A_q^\dagger B_{q'}^\dagger, \\ \frac{1}{\sqrt{2q}} \frac{1}{\sqrt{2q'}} \delta(k-q-q')(q-q')A_q B_{q'}. \end{aligned} \quad (4.4)$$

The transverse-magnetic term, which we call the nonlocal box (NLB), also gives rise to scattering and production terms as shown in Figs. 7(a) and 7(b). The scattering term [Fig. 7(a)] is

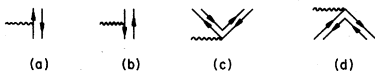


FIG. 6. The vertices generated by the Coulomb terms in the current-current interaction. (a) and (b) are the Coulomb scattering terms. (c) and (d) represent Coulomb production and annihilation, respectively.

$$-\frac{g_{NLB}}{\pi a^2} \delta(u'_1 + u'_2 - u_1 - u_2) \frac{1}{\sqrt{2u_1} \sqrt{2u_2} \sqrt{2u'_1} \sqrt{2u'_2}}$$

and the production term [Fig. 7(b)] is

$$-\frac{g_{NLB}}{\pi a^2} \delta(u'_1 - u_1 - u_2 - u_3) \frac{1}{\sqrt{2u'_1} \sqrt{2u_1} \sqrt{2u_2} \sqrt{2u_3}}.$$

As noted in Sec. II, the linearization of the link-meson degrees of freedom is only consistent if we add local potential terms to the effective Hamiltonian. One necessary term is a local four-link-meson interaction whose matrix elements are identical to those of the nonlocal box (replace g_{NLB} by g_{LB}). All the vertices are written with a normalization appropriate for direct matrix elements between the normalized wave functions $\phi(x_1, \dots, x_n)$.

Our purpose is eventually to diagonalize the full Hamiltonian in a finite basis of bare hadrons. To this end we must calculate all the matrix elements connecting the various bare mesons which are given in terms of wave functions of the form

$$\phi(x_1, \dots, x_N) = (x_1, \dots, x_N)^\beta x_1^{\eta_1} \cdots x_N^{\eta_N},$$

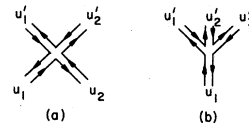


FIG. 7. Scattering (a) and production (b) terms resulting from the transverse-magnetic interaction.

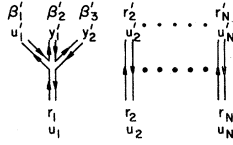


FIG. 8. The process denoted by $B(1, 1'2'3')\delta_{24}$.

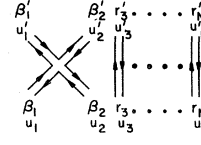


FIG. 9. The nonlocal-box scattering term.

where $\sum_{i=1}^N x_i = 1$. Thus, we wish to calculate analytically the various matrix elements of operators appearing in the Hamiltonian in this power basis.

The Coulomb scattering matrix elements have already been calculated in Eq. (3.18) in addition to the kinetic energy. These interactions do not change the link meson number. Both local- and nonlocal-box scattering will have the same structure. Let us discuss in detail the nonlocal-box case, which involves link mesons on four different sites. This interaction mixes the link-meson num-

ber two and four sectors enabling the bare hadrons to move on the lattice. We note that a bare hadron containing two link mesons can move only via a second-order transition through a four-link-meson state. The second effect of the coupling is to couple four-link-meson states with different "polarization" configurations. Thus, four-link-meson states can move on the transverse lattice on their own without having to couple to the two- or six-link-meson sector. An example of a process induced by the magnetic term is shown in Fig. 8, the process is denoted by $B(1, 1'2'3')\delta_{24}$ (for generality the number of spectators has been increased to $N - 1$), which is given by

$$B(1'2'3', 1)\delta_{4'2} = -g_{\text{NLB}}^2 \int du_1 \cdots du_N \delta\left(1 - \sum_{i=1}^N u_i\right) \int du_1' dy_1' dy_2' \delta(u_1' + y_1' + y_2' - u_1) \frac{1}{(2u_1' 2u_1 2y_1' 2y_2')^{1/2}} \times \phi_{N+2}(u_1', y_1', y_2', u_2, \dots, u_N) \phi_N(u_1, u_2, \dots, u_N), \quad (4.5)$$

where

$$\begin{aligned} \phi_N(u_1, \dots, u_N) &= u_1^{\beta_1} u_2^{\gamma_2} \cdots u_N^{\gamma_N}, \\ \phi_{N+2}(u_1', y_1', y_2', u_2, \dots, u_N) &= (u_1')^{\beta_1'} (y_1')^{\beta_2'} (y_2')^{\beta_3'} (u_2)^{\gamma_2'} \cdots (u_N)^{\gamma_N'}, \\ B(1'2'3', 1)\delta_{4'2} \delta \cdots &= \frac{1}{4} \frac{\Gamma(\beta_1' + \frac{1}{2})\Gamma(\beta_2' + \frac{1}{2})\Gamma(\beta_3' + 1)}{\Gamma(\beta_1' + \beta_2' + \beta_3' + \frac{3}{2})} g_{\text{NLB}}^2 \\ &\times \frac{\Gamma(\beta_1 + \beta_1' + \beta_2' + \beta_3' + 1)\Gamma(\gamma_2 + \gamma_2' + 1) \cdots \Gamma(\gamma_N + \gamma_N' + 1)}{\Gamma(\beta_1 + \gamma_2 + \cdots + \gamma_N + \beta_1' + \beta_2' + \beta_3' + \gamma_2' + \cdots + \gamma_N' + N)}. \end{aligned} \quad (4.6)$$

All other matrix elements are calculated in an analogous manner. The (non)local-box scattering relates two-link-meson states in the four-link-mesons number four sector. It is represented by Fig. 9 and its value is

$$\frac{(\beta_1' + \beta_2' + \beta_1 + \beta_2 + 1)\Gamma(\beta_3' + \beta_3 + 1) \cdots \Gamma(\beta_N' + \beta_N + 1)}{\Gamma(\beta_1' + \beta_2' + \cdots + \beta_N' + \beta_1 + \cdots + \beta_N + N - 1)} \frac{1}{4} \frac{\Gamma(\beta_1' + \frac{1}{2})\Gamma(\beta_2' + \frac{1}{2})\Gamma(\beta_1 + \frac{1}{2})\Gamma(\beta_2 + \frac{1}{2})}{\Gamma(\beta_1' + \beta_2' + 1)\Gamma(\beta_1 + \beta_2 + 1)}. \quad (4.7)$$

Two other terms are induced by the current-current interaction. They are, respectively, the Coulomb annihilation term which is shown in Fig. 10. It relates states in the link-meson four sector. The reason it does not appear in the link-meson two sector is that it contains a colored gluon as the intermediate state in that case. It is given by

$$\frac{1}{4} \frac{\Gamma(\beta_1' + \beta_2' + \beta_1 + \beta_2 + 1)\Gamma(\beta_3' + \beta_3 + 1) \cdots \Gamma(\beta_N' + \beta_N + 1)}{\Gamma(\beta_1' + \beta_1 + \cdots + \beta_N' + \beta_N + N - 1)} \frac{\Gamma(\beta_1 + \frac{1}{2})\Gamma(\beta_2 + \frac{1}{2})\Gamma(\beta_1' + \frac{1}{2})\Gamma(\beta_2' + \frac{1}{2})}{\Gamma(\beta_1 + \beta_2 + 2)\Gamma(\beta_1' + \beta_2' + 2)} (\beta_1' - \beta_2')(\beta_1 - \beta_2). \quad (4.8)$$

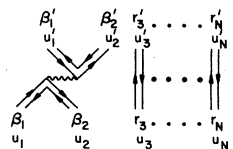


FIG. 10. Coulomb annihilation terms.

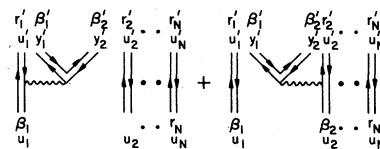


FIG. 11. Coulomb production terms.

Note that this term is nonzero only between states that have the same parity under an exchange $u_1 \leftrightarrow u_2$ ($u'_1 \leftrightarrow u'_2$). The last matrix element needed is the Coulomb production matrix element. It relates the two- and four-link-meson sectors, as shown in Fig. 11. It is given by

$$\frac{1}{4} \frac{\Gamma(\beta'_1 + \beta_1 + 1) \cdots \Gamma(\beta'_N + \beta_N + 1)}{\Gamma(\beta_1 + \cdots + \beta_N + \beta'_1 + \gamma_1 + \gamma_2 + \beta'_2 + \cdots + \beta'_N + N)} \frac{(\gamma_1 - \gamma_2)\Gamma(\gamma_1 + \frac{1}{2})\Gamma(\gamma_2 + \frac{1}{2})}{(\gamma_1 + \gamma_2)(\gamma_1 + \gamma_2 + 1)} \\ \times \left(- \frac{\Gamma(\beta'_1 + \frac{1}{2})\Gamma(\gamma_1 + \gamma_2 + \beta'_1 + \beta_1 + 1)(\gamma_1 + \gamma_2 + 2\beta'_1 + 1)}{\Gamma(\beta_1 + \beta'_1 + 1)\Gamma(\beta'_1 + \gamma_1 + \gamma_2 + \frac{3}{2})} + \frac{\Gamma(\beta'_1 + \frac{1}{2})\Gamma(\gamma_1 + \gamma_2 + \beta_2 + \beta'_2 + 1)(\gamma_1 + \gamma_2 + 2\beta'_2 + 1)}{\Gamma(\beta'_2 + \beta_2 + 1)\Gamma(\beta'_2 + \gamma_1 + \gamma_2 + \frac{3}{2})} \right). \quad (4.9)$$

All matrix elements induced by the transverse current-current interaction have not been multiplied by the Coulomb coupling constant $(g^2 C_N)/\pi a^2$ which is set to be one. This coupling actually sets the mass scale of the problem. We have accumulated by now all the necessary information to calculate the masses of the gluon-bound states. Having allowed the bare hadrons to move it would thus be natural to form a basis of transverse-momentum eigenfunctions:

$$|\psi\rangle_{p_T} = \sum e^{iP_T \cdot na} |\psi n\rangle \quad (4.10)$$

and calculate the Hamiltonian in this basis. We have constructed the matrix for a general p_T along the lines described in this section. The full matrix is given in Appendix B.

Before turning to the actual computation of the hadron masses, we note that although we have restricted our discussion to a link-meson number not larger than four it is obvious that we have by now all the machinery necessary to deal with any link-meson number. All one needs is to solve the n -link-meson bare-hadron wave-function equation by the same techniques used for the four-body wave function and then set up the Hamiltonian as done above. The number of terms increases rapidly of course, but the calculations are straightforward. This scheme is based on the proposition that the perturbation in the link-meson number is indeed reasonable and provides a convergent procedure; this will be tested by the calculations.

V. CALCULATIONS

The theory as formulated consists of three parameters (including an overall scale) and an unknown function (the potential in the linear representation). This is a consequence of the theory being treated in a noncovariant gauge and with a noncovariant cutoff. The relation between parameters fixed by Lorentz invariance can only be recovered in the continuum limit. The fact that we have modified the theory for a finite lattice spacing was discussed in detail in Sec. II. This left us

with the unknown potential function.

What we propose to check is that we have chosen relevant degrees of freedom in terms of which a tractable scheme to calculating hadron masses can be described. At our present state of knowledge this would require a parameter fitting to masses of known particles. We thus limit ourselves to describing features of this scheme.

The parameters are

(i) $(g^2 C_N)/\pi a^2$: Chosen to be one during the calculations, it sets the scale of bare-hadron masses.

(ii) γ^{*2} (or β): The mass of the link meson (the edge-point behavior of the n -link-meson wave function). In the nonlinear σ model with $O(N)$ symmetry this mass was actually generated dynamically. One would expect that in a covariant formulation there should be only one hadronic mass scale. The value of β controls the nature of the longitudinal dynamics. Large (small) β corresponds to weakly (strongly) bound link mesons.

(iii) g_{NLB} : The coupling of the nonlocal box, in the continuum theory; it is essentially the gauge field self-coupling g^2 . For a fixed transverse-lattice separation a , it should be related to g^2 . However, in this calculation we treat it as a free parameter. $g_{NLB}^2/[\Delta E(\beta)]$, where $\Delta E(\beta)$ are bare-hadron energy splittings (which are functions of β), is essentially the expansion parameter of the effective bare-hadron Hamiltonian.

(iv) There are many other parameters associated with the effective potential containing $M_{n\alpha}$, all of which are in principle determined dynamically and calculable from g^2 . We, however, do not know, at the moment, how to perform such a calculation and we thus pick one term—the local-box interaction and treat the coupling g_{LB} as another free parameter.

Fixing three parameters, one must next choose the basis of states and truncate them. In the calculations reported here we have chosen three even and three odd states in the link-meson number two sector to each such classification. There correspond two possible “polarization” states (Fig. 2). In the four-link-meson sector, one state was chosen from each one-dimensional representation. There are four such representations ($A1, A2, B1, B2$)

and to each correspond 10 different polarization states (Fig. 2). Two excitations were picked from the two-dimensional representation E . To each there correspond eight states (E does not contain e_1, e_2 states). All together the basis contained $12 + 40 + 32 = 84$ bare-hadron states. The states were chosen by their energy ordering in the mass spectrum of bare hadrons.

We first studied the hadronic mass spectrum by diagonalizing the total Hamiltonian of zero transverse momentum. In this case the same symmetries that served to classify the n -link-meson states, namely the bare hadrons, also categorize the eigenstates of the full Hamiltonian. For an illustrative example let us turn to the link-meson number two sector. In that case the states A_1 and A_2 can form only three of the five representations of the group D_4 .

Taking the symmetric and antisymmetric combinations of even two-body states one forms the one-dimensional representations of A_1 and B_2

$$|A_1\rangle = \frac{1}{\sqrt{2}} (|A_1\rangle^{\text{even}} + |A_2\rangle^{\text{even}}),$$

$$|B_2\rangle = \frac{1}{\sqrt{2}} (|A_1\rangle^{\text{even}} - |A_2\rangle^{\text{even}}).$$

The odd two-body wave function can be placed in a doublet to form the two-dimensional E representation

$$|E\rangle = \begin{bmatrix} |A_1\rangle_{\text{odd}} \\ \pm |A_2\rangle_{\text{odd}} \end{bmatrix}.$$

Both the A_2 and B_1 representations are absent from the link meson number two sector. In order to appreciate this fact let us consider a spin l particle on a transverse lattice. (These results are valid also for helicity states in the infinite-momentum frame.) The irreducible representations of dimension $2l + 1$ will be broken into one- and two-dimensional representations on the transverse lattice. By applying standard methods of group theory one concludes that the spin-zero state transforms like A_1 . The $l_z = 0$ component of the spin-one state is in A_1 while the $l_z = \pm 1$ components form two-dimensional E representations. In the spin-two case the $l_z = 0$ component transforms like A_1 . The $l_z = \pm 1$ are in an E representation and the $l_z = \pm 2$ form symmetric and antisymmetric combinations which transform according to B_1 and B_2 , respectively. The general rule is

$$\left. \begin{array}{l} A_1 \\ A_2 \end{array} \right\} l_z = \pm 4, \pm 8, \pm 12, \left\{ \begin{array}{l} m = 0 \text{ symmetric} \\ \text{antisymmetric} \end{array} \right\},$$

$$\left. \begin{array}{l} B_1 \\ B_2 \end{array} \right\} l_z = \pm 2, \pm 6, \pm 10, \left\{ \begin{array}{l} \text{antisymmetric} \\ \text{symmetric} \end{array} \right\},$$

$$E \left\{ l_z = \pm 1, \pm 3, \pm 5. \right.$$

In the continuum limit one should recover the Lorenz degeneracy. Thus, the absence of the B_1 and A_2 representations in the two-link meson sector means that these would have to come from the higher link meson number sector. This implies that the two and four (etc.), sectors have to mix in the continuum limit.

Note that one knows already from the study of the bare hadrons in the link meson number two sector that the lowest even eigenstate is lighter, for all β , than the lowest odd eigenstate. This leads to degenerate A_1, B_2 states split away from an E state, identifying the A_1 state with a spin-zero hadron, B_2 with an $m = \pm 2$ symmetric component of a spin-two component, and E with $m = \pm 1$. We get an embryonic degeneracy between a scalar and a tensor piece of a gluon-bound state. If E will also be related to the tensor then the vector gluon-bound state lies way above the scalar and some components of the tensor. Even in the case where the E will be related to a vector particle, a hierarchy scalar-tensor and a heavier vector has formed.

A similar classification can be obtained for the link-meson number four sector at $k_1 = 0$. These symmetry considerations serve also as a check of our computer program. Most of the eigenfunctions have been checked to show that indeed all eigenfunctions of the total Hamiltonian can be classified according to D_4 and that they contain only the allowed combinations in the two- and four-link-meson number scalar. The classification of all four-body states is shown in Appendix C.

In Fig. 12 we show the mass spectrum for two

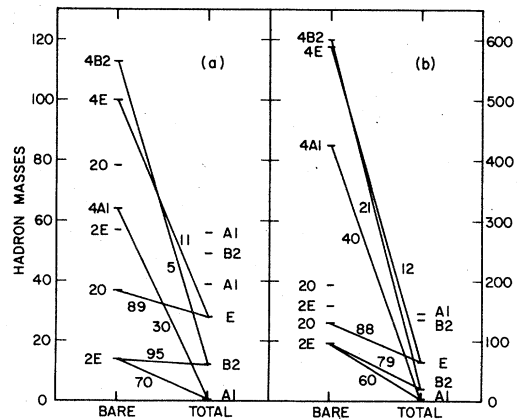


FIG. 12. The hadronic mass spectrum for (a) $\beta = 0.2$, $g_{\text{NLB}}^2 = -3.65$, $g_{\text{LB}} = 0$; (b) $\beta = 0.45$, $g_{\text{NLB}}^2 = -46$, $g_{\text{LB}} = 0$. The bare-hadron masses are also shown. The arrowed lines indicate the main bare-hadron wave-function decomposition of the hadrons. The number is the mixing probability (in %). For $\beta = 0.45$ only the important two-body states were marked.

sets of parameters:

$$(a) \beta = 0.2, \quad g_{\text{NLB}}^2 = -3.65, \quad g_{\text{LB}}^2 = 0,$$

$$(b) \beta = 0.45, \quad g_{\text{NLB}}^2 = -46, \quad g_{\text{LB}}^2 = 0.$$

These lower-lying states are classified according to the D_4 symmetry.

The next question to be addressed is the consistency of the link-number expansion. It is checked by studying the four-body content of the low-lying excitations. These results are summarized in Fig. 12. It turns out that the expansion is rather satisfactory. The four-body mixing in all possible cases ($A1$, $B2$, and E) is not too large as to upset the expansion and not too small, thus allowing the two-body states to move on the transverse lattice. [For the set of values (a) perturbation theory would give a 50% error while for the set of values (b) although g_{NLB}^2 is very large, perturbation theory gives a correct result within 10% because $g_{\text{NLB}}^2 / \Delta E(\beta)$, the expansion parameter, is small.]

In the absence of a physical gluon-bound-state spectrum qualitative considerations are used to zone the relevant range of parameters. For a general set of parameters the Hamiltonian is not positive definite. The requirement that all bound states have positive energies constitutes a nontrivial constraint. In particular, for $g_{\text{LB}}^2 = 0$, a curve forms in the β , g_{NLB}^2 plane. One defines g_{max}^2 to be the largest absolute value of g_{NLB}^2 allowed for a fixed value of β . g_{max}^2 is an increasing function of β . Its behavior is shown in Fig. 13. In the allowed region of the plane one looks for those parameters which simulate on a rather large lattice the continuum limit.

The $A1$ - $B2$ degeneracy is broken once the full Hamiltonian is diagonalized leading to a scalar-tensor-vector-like ordering. This result is stable under a large variation of all the parameters of the theory and is one of the qualitative results of our analysis. One should point out that in the cal-

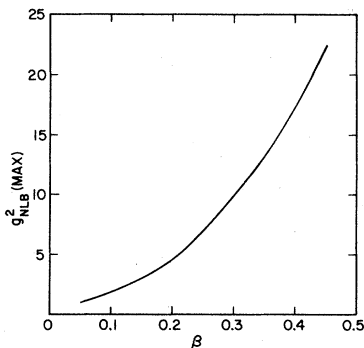


FIG. 13. The maximum g_{NLB}^2 for a given β which leads to positive masses.

ulation of Kogut and Susskind a scalar-tensor degeneracy is obtained (or more precisely, in a cubic lattice a spin two breaks into one three- and one two-dimensional representation; the statement is that the scalar is degenerate with the three-dimensional piece of the tensor) and they are both lighter than the vector (all of whose components fall in a single three-dimensional representation on the cubic lattice and thus remain degenerate). As a matter of fact, two interacting spin-one particles with a conventional potential would be in an s wave and thus if the angular momentum interactions are not too large, one indeed should have the vector lie above the scalar and tensor.

A naive guess could be that having a light link-meson mass μ^2 —which is tantamount to β near zero (strong longitudinal coupling)—would pull together the bare-hadron masses and help them mix. However, as was shown in Sec. III by reducing β to zero one cannot generate even one zero-mass hadron (unlike the situation in the bare quark-antiquark meson sector). The ground states of the various representations become increasingly heavier because of the “welding energy” at each transverse site. This *a priori* spacing of the low-lying bare particles means that a nonnegligible coupling should exist between two- and four-body states, in particular, g_{NLB} would have to be large. However, for small β the lowest-lying even two-body state has a small mass relative to its separation from the four-body states. Thus, a large g_{NLB} falls outside the allowed region. We are thus pushed to values of β around 0.2 and $g_{\text{NLB}} \approx -3.5$ (the values used in Fig. 12) before a reasonable four-body mixing occurs. Even in this range some improvement may be desired. The energy difference between the bare two-body even state and the first four-body $B2$ state is rather large and it mixes much more into the second excited two-body even state only for $\beta \approx 0.45$. g_{NLB} can be made large enough to overcome the energy difference and reach a 20% mixture. This reflects itself in the dispersion relations for nonzero transverse momentum.

Before studying the $k_T \neq 0$ case in detail we note that the local-box term has a negative eigenvalue and thus cannot serve to increase significantly the allowed region. By studying the structure of theory at $k_T \neq 0$ one gets additional information on the “continuumlike” behavior of the excitations. The Hamiltonian described in Sec. IV is diagonalized for nonzero transverse momentum. In Fig. 14 we plot the dispersion relation: The energy of the hadrons versus their transverse momentum. The x - and y -transverse-momentum components are equal.

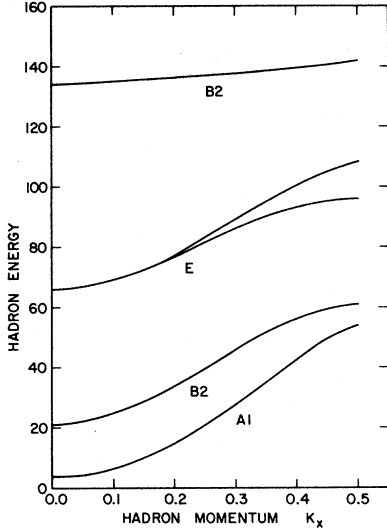


FIG. 14. Dispersion relation; the energy vs transverse momentum for $\beta = 0.45$; $g_{\text{NLB}}^2 = -46$; $g_{\text{LB}} = 0$. The transverse momentum is in the k_x - k_y direction.

We first note that each of the lowest-lying states involving two- and four-link meson mixing satisfies a physical acoustic dispersion. This allows an appropriate transverse motion for the hadrons. The next information available relates to the extent to which the hadrons in this approximation have continuumlike properties. We can study several aspects of this question; in the infinite momentum frame the dispersion relation is

$$E = m^2 + ck_T^2,$$

where c stands for the velocity of light. In the continuum limit all states should have the same coefficient c and in addition the rotational invariance of the theory should be restored. In our case one compares c in the x or y direction to c in the 45° direction ($k_x = k_y$). In Table I we list c for the lowest-lying A1, B2, and one of the E states (for $k_T \neq 0$, the E states are no longer exactly degenerate). The large β and g_{NLB} region is again more continuumlike. For the 45° direction, c is the same for all representations within 20%. For A1 and B2 states the 45° rotation changes c by less than 10%. The E states are not yet behaving in a satisfying manner. One state (shown in the table) has a very large value of c and the other (not shown) is essentially flat [they change roles when going from $(k_x, 0)$ to $(0, k_y)$].

For smaller values of β the situation is somewhat different for B2 states, which hardly mix with the high-energy four-body state. As a final observation we note that for large β both the A1 and B2 masses are much smaller than the value

TABLE I. The velocity of light of various hadrons. The units of C are determined by $(g^2 C_N)/(\pi a^2) = 1$.

$\beta = 0.2$; $g_{\text{NLB}}^2 = -3.65$; $g_{\text{LB}} = 0$		
	$C_{k_x=k_y}$	$C_{k_x, k_y=0}$
A1	23	22
B2	5	6
E	19	36
$\beta = 0.45$; $g_{\text{NLB}}^2 = -46$; $g_{\text{LB}} = 0$		
	$C_{k_x=k_y}$	$C_{k_x, k_y=0}$
A1	145	135
B2	167	177
E	135	270

of the energy for large k_T . This is in the right direction as the edge of the Brillouin zone moves to infinity for small values of the lattice spacing.

VI. DISCUSSION

The transverse-lattice, infinite momentum frame version of QCD attempts to deal directly with the physical degrees of freedom of QCD, maintaining and imposing the full internal symmetry structure of the theory at the cost of the full Lorentz symmetry. The main thrust of this paper is in actually implementing this program. This was done by treating the longitudinal dynamics nonperturbatively and perturbing in the transverse motion dynamics.

The hadrons emerging from the analysis are composed of a superposition of bare hadrons. Each bare hadron is a weakly bound system of link mesons (β is rather large). Link-meson-number violation is large enough to allow a reasonable transverse motion but is small enough to validate the expansion in terms of link-meson number. Strongly bound bare hadrons cannot be supported by our approximation to the effective potential. An improvement of our understanding of the linear version of QCD is needed before it can be established as a faithful effective theory at some hadronic distance scale.

We wish to conclude with some remarks on the general characteristics of the gluon-bound-state spectrum. Gluon-bound states have the rather unique property of being formed from gauge particles which are color nonsinglets but have triality zero. One would like to know in what way will the gluon-bound-state spectrum reflect these special facets of its constituents.

In our analysis the Lorentz pattern of the low-lying excitations seems to be explained by a valence-gluon picture. It is however not clear that this description is faithful. The first source of

doubt is the gauge dependence of the separation between the dependent potential and the independent degrees of freedom. However, if there exists any gauge in which a valence-gluon picture serves to classify the gauge invariant physical states, it is a useful concept. The more serious problem is the strong-couplinglike bias inherent in our analysis. In both the straightforward strong-coupling calculations in the $A^0=0$ gauge⁷ and in our more complicated approach the valence structure is a strong-coupling feature. The expansion in link-meson number dictates (as was shown in Sec. V) the order of the Lorentz excitations.

To leading order of the $1/N_c$ expansion an infinite number of stable gluon-bound states was obtained. This is consistent with expectations¹⁰ from the continuum limit. In this sense there exists a limit in which gluons (link mesons) are confined and attract with constant forces. In our scheme there was no algebraic characterization of the states save that they are color singlets. One may wonder if these states form some degeneracy patterns associated with a surviving global symmetry. Such a symmetry should be explicit in a string theory of hadrons.

ACKNOWLEDGMENT

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APPENDIX A: SOME DETAILS OF THE SOLUTION OF THE FOUR-BODY BOUND-STATE EQUATION

The four-body bound-state equation is simplified when decomposed in terms of states with a definite D_4 symmetry. D_4 has four different one-dimensional representations $A1, A2, B1, B2$ and one two-dimensional representation E . The character table of the group D_4 can be found in standard books on finite dimensional groups.¹¹ It is

	Identity	C_2	$2C_4$	$2C_2'$	$2C_2''$
A1	1	1	1	1	1
A2	1	1	1	-1	-1
B1	1	1	-1	1	-1
B2	1	1	-1	-1	1
E	2	-2	0	0	0

where the various group members of D_4 generate the transformations on square whose sides are denoted (in order) by x, y, z, w ,

$$C_2: x \leftrightarrow z, y \leftrightarrow w,$$

$$C_4(90^\circ): xyzw \rightarrow yzwx,$$

$$C_4(270^\circ): xyzw \rightarrow wxyz,$$

$$C_{2'I}: xyzw \rightarrow wzyx,$$

$$C_{2'II}: xyzw \rightarrow yxwz,$$

$$C_{2''I}: xyzw \rightarrow zyxw,$$

$$C_{2''II}: xyzw \rightarrow xwzy.$$

In the two-dimensional E representation one has

$$C_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C_4^{90^\circ} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C_4^{270^\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$C_{2'I} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad C_{2'II} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C_{2''I} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_{2''II} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The following trial functions were constructed as irreducible representations of the various representations (the constraint $x+y+z+w=1$ was imposed). The matrix elements of the kinetic energy and the potential energy were calculated using Eq. (3.14) and the algebra was kept under control with the help of MACSYMA.

APPENDIX B: THE HAMILTONIAN FOR GENERAL TRANSVERSE MOMENTUM

The Hamiltonian is given in the basis of two- and four-body states as defined in Fig. 2. A dictionary for the symbols is given below and an example was done in Eq. (4.5):

$$\phi_{\alpha n}(x_1, x_2, x_3, x_4) = (x_1 x_2 x_3 x_4)^{\beta} P_n^{\alpha}(x_1, x_2, x_3, x_4). \quad (3.21)$$

For $\alpha=A1$:

$$P_0 = 1,$$

$$P_1 = x^2 + y^2 + z^2 + w^2 + xy + yz + zw + wx, \quad (B1)$$

$$P_2 = x^3 + y^3 + z^3 + w^3,$$

$$P_3 = x^4 + y^4 + z^4 + w^4 + x^2 z^2 + y^2 w^2 + x^3 z + z^3 x + y^3 w + w^3 y + xyzw.$$

For $\alpha=A2$:

$$P_0 = xy(y-x) + yz(z-y) + zw(w-z) + wx(x-w). \quad (B2)$$

For $\alpha=B1$:

$$P_0 = (x-z)(y-w). \quad (B3)$$

TABLE II. The appropriate matrix elements for the Hamiltonian for general transverse momentum. The following notation is used: M =kinetic energy, C =direct Coulomb, LB =local box, B =nonlocal box, CP =Coulomb production, CA =Coulomb annihilation, $e_x=e^{ik_x a}$, and $e_y=e^{ik_y a}$.

A_1		A_2	
A_1	$M_1+M_2+2C_{12}+LB_{12}$		0
A_2	0	$M_1+M_2+2C_{12}+LB_{12}$	
B_1	$B(234, 2')\delta_{11'}+e_y B(412, 1')\delta_{32'}$	$B(123, 1')\delta_{42'}+e_x B(341, 2')\delta_{21'}$	
B_2	$B(123, 1')\delta_{42'}+e_y B(341, 2')\delta_{21'}$	$B(234, 2')\delta_{11'}+e_x B(412, 1')\delta_{32'}$	
C_1	$CP(1234, 1'2')$	$e_y CP(3412, 2'1')$	
C_2	$CP(3412, 2'1')$	$CP(1234, 2'1')$	
C_3	$e_x CP(3412, 1'2')$	$e_y CP(1234, 1'2')$	
C_4	$e_x CP(1234, 2'1')$	$CP(3412, 1'2')$	
D_1	$CP(1234, 1'2')+e_y CP(3412, 2'1')$		0
D_2	0	$CP(1234, 1'2')+e_x CP(3412, 2'1')$	
E_1	0	$\sqrt{2}CP(1234, 1'2')+\sqrt{2}CP(4123, 2'1')$	
		$+\sqrt{2}LB(123, 1')\delta_{42'}+\sqrt{2}LB(234, 2')\delta_{11'}$	
E_2	$\sqrt{2}CP(1234, 1'2')+\sqrt{2}CP(4123, 2'1')$		0
	$+\sqrt{2}LB(123, 1')\delta_{42'}+\sqrt{2}LB(234, 2')\delta_{11'}$		
B_1		B_2	
A_1	$B(2, 2'3'4')\delta_{11'}+e_x^* B(1, 4'1'2')\delta_{23'}$	$B(1, 1'2'3')\delta_{24'}+e_x^* B(2, 3'4'1')\delta_{12'}$	
A_2	$B(1, 1'2'3')\delta_{24'}+e_y^* B(2, 3'4'1')\delta_{12'}$	$B(2, 2'3'4')\delta_{11'}+e_y^* B(1, 4'1'2')\delta_{23'}$	
B_1	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$		0
B_2	0	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$	
C_1	$B(34, 3'4')\delta_{11'}\delta_{22'}$	$B(12, 1'2')\delta_{33'}\delta_{44'}$	
C_2	$B(34, 2'3')\delta_{21'}\delta_{14'}$	$B(12, 2'3')\delta_{34'}\delta_{41'}$	
C_3	$B(34, 4'1')\delta_{12'}\delta_{23'}$	$B(12, 4'1')\delta_{32'}\delta_{43'}$	
C_4	$B(34, 1'2')\delta_{24'}\delta_{13'}$	$B(12, 3'4')\delta_{31'}\delta_{42'}$	
D_1	0		0
D_2	0		0
E_1	0		0
E_2	0		0
C_1		C_2	
A_1	$CP(12, 1'2'3'4')$	$CP(21, 3'4'1'2')$	
A_2	$e_y^* CP(21, 3'4'1'2')$	$CP(21, 1'2'3'4')$	
B_1	$B(34, 3'4')\delta_{11'}\delta_{22'}$	$B(23, 3'4')\delta_{12'}\delta_{41'}$	
B_2	$B(12, 1'2')\delta_{33'}\delta_{44'}$	$B(23, 1'2')\delta_{14'}\delta_{43'}$	
C_1	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$	$e_y CA(41, 2'3')\delta_{31'}\delta_{24'}$	
	$+LB_{23}+LB_{41}+CA_{23}+CA_{41}$		
C_2	$e_y^* CA(23, 4'1')\delta_{13'}\delta_{42'}$	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$	
		$+LB_{23}+LB_{41}+CA_{23}+CA_{41}$	
C_3	$e_x CA(41, 2'3')\delta_{31'}\delta_{24'}$		0
C_4	0	$e_x CA(23, 4'1')\delta_{13'}\delta_{42'}$	
D_1	$CA(23, 2'3')\delta_{11'}\delta_{44'}$	$e_y CA(41, 4'1')\delta_{33'}\delta_{22'}$	
D_2	$e_x e_y^* CA(41, 4'1')\delta_{22'}\delta_{33'}$	$e_x CA(41, 2'3')\delta_{24'}\delta_{31'}$	
E_1	$e_y^* \sqrt{2}CA(41, 4'1')\delta_{22'}\delta_{33'}$	$\sqrt{2}CA(23, 2'3')\delta_{11'}\delta_{44'}$	
E_2	$\sqrt{2}CA(23, 2'3')\delta_{11'}\delta_{44'}$	$\sqrt{2}CA(34, 4'1')\delta_{12'}\delta_{23'}$	
C_3		C_4	
A_1	$e_x^* CP(12, 3'4'1'2')$	$e_x^* CP(21, 1'2'3'4')$	
A_2	$e_y^* CP(12, 1'2'3'4')$	$CP(12, 3'4'1'2')$	
B_1	$B(41, 3'4')\delta_{21'}\delta_{32'}$	$B(12, 3'4')\delta_{31'}\delta_{42'}$	
B_2	$B(41, 1'2')\delta_{23'}\delta_{34'}$	$B(34, 1'2')\delta_{13'}\delta_{24'}$	
C_1	$e_x^* CA(23, 4'1')\delta_{13'}\delta_{42'}$		0
C_2	0	$e_x^* CA(41, 2'3')\delta_{24'}\delta_{31'}$	
C_3	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$	$e_y CA(23, 4'1')\delta_{13'}\delta_{42'}$	
	$+LB_{23}+LB_{41}+CA_{23}+CA_{41}$		

TABLE II. (Continued.)

C_4	$e_y^* CA(41, 2'3')\delta_{31'}\delta_{24'}$	$M_1 + M_2 + M_3 + M_4 + C_{12} + C_{23} + C_{34} + C_{41}$ $+ LB_{23} + LB_{41} + CA_{41} + CA_{23}$
D_1	$e_x^* CA(23, 4'1')\delta_{13'}\delta_{42'}$	$e_x^* e_y CA(41, 2'3')\delta_{31'}\delta_{24'}$
D_2	$e_y^* CA(23, 2'3')\delta_{11'}\delta_{44'}$	$CA(23, 4'1')\delta_{13'}\delta_{42'}$
E_1	$e_x^* \sqrt{2} CA(12, 2'3')\delta_{34'}\delta_{41'}$	$\sqrt{2} CA(12, 4'1')\delta_{32'}\delta_{43'}$
E_2	$e_x^* \sqrt{2} CA(41, 4'1')\delta_{22'}\delta_{33'}$	$e_x^* \sqrt{2} CA(34, 2'3')\delta_{14'}\delta_{21'}$
	D_1	D_2
A_1	$CP(12, 1'2'3'4') + e_y^* CP(21, 3'4'1'2')$	0
A_2	0	$CP(12, 1'2'3'4') + e_x^* CP(21, 3'4'1'2')$
B_1	0	0
B_2	0	0
C_1	$CA(23, 2'3')\delta_{11'}\delta_{44'}$	$e_x^* e_y CA(41, 4'1')\delta_{22'}\delta_{33'}$
C_2	$e_y^* CA(41, 4'1')\delta_{33'}\delta_{22'}$	$e_x^* CA(23, 4'1')\delta_{42'}\delta_{13'}$
C_3	$e_x CA(41, 2'3')\delta_{31'}\delta_{24'}$	$e_y CA(23, 2'3')\delta_{44'}\delta_{11'}$
C_4	$e_x e_y^* CA(23, 4'1')\delta_{42'}\delta_{13'}$	$CA(41, 2'3')\delta_{31'}\delta_{24'}$
D_1	$M_1 + M_2 + M_3 + M_4 + C_{12} + C_{23} + C_{34} + C_{41}$ $+ LB_{23} + LB_{41} + CA_{23} + CA_{41}$	0
D_2	0	$M_1 + M_2 + M_3 + M_4 + C_{12} + C_{23} + C_{34} + C_{41}$ $+ LB_{23} + LB_{41} + CA_{23} + CA_{41}$
E_1	0	$\sqrt{2} CA(23, 3'4')\delta_{41'}\delta_{12'} + e_x \sqrt{2} CA(41, 2'3')\delta_{31'}\delta_{24'}$
E_2	$\sqrt{2} CA(23, 2'3')\delta_{11'}\delta_{44'}$ $+ e_x^* \sqrt{2} CA(34, 4'1')\delta_{12'}\delta_{23'}$	0
	E_1	E_2
A_1	0	$\sqrt{2} CP(12, 1'2'3'4') + \sqrt{2} CP(21, 4'1'2'3')$ $+ \sqrt{2} LB(1, 1'2'3')\delta_{24'} + \sqrt{2} LB(2, 2'3'4')\delta_{11'}$
A_2	$\sqrt{2} CP(12, 2'3'4'1') + \sqrt{2} CP(21, 1'2'3'4')$ $+ \sqrt{2} LB(1, 4'1'2')\delta_{23'} + \sqrt{2} LB(2, 3'4'1')\delta_{12'}$	0
B_1	0	0
B_2	0	0
C_1	$e_y \sqrt{2} CA(41, 4'1')\delta_{22'}\delta_{33'}$	$\sqrt{2} CA(23, 2'3')\delta_{12'}\delta_{44'}$
C_2	$\sqrt{2} CA(23, 2'3')\delta_{11'}\delta_{44'}$	$\sqrt{2} CA(41, 3'4')\delta_{21'}\delta_{32'}$
C_3	$e_y \sqrt{2} CA(23, 1'2')\delta_{43'}\delta_{14'}$	$e_x \sqrt{2} CA(41, 4'1')\delta_{22'}\delta_{33'}$
C_4	$\sqrt{2} CA(41, 1'2')\delta_{23'}\delta_{34'}$	$e_x \sqrt{2} CA(23, 3'4')\delta_{41'}\delta_{12'}$
D_1	0	$\sqrt{2} CA(23, 2'3')\delta_{11'}\delta_{44'} + e_y \sqrt{2} CA(41, 3'4')\delta_{21'}\delta_{32'}$
D_2	$\sqrt{2} CA(23, 3'4')\delta_{41'}\delta_{12'}$ $+ e_x \sqrt{2} CA(41, 2'3')\delta_{31'}\delta_{24'}$	0
E_1	$M_1 + M_2 + M_3 + M_4 + C_{12} + C_{23} + C_{34}$ $+ C_{41} + LB_{12} + LB_{23} + LB_{34} + LB_{41}$ $+ CA(12) + CA(23) + CA(34) + CA(41)$	0
E_2	0	$M_1 + M_2 + M_3 + M_4 + C_{12} + C_{23} + C_{34}$ $+ C_{41} + LB_{12} + LB_{23} + LB_{34} + LB_{41}$ $+ CA(12) + CA(23) + CA(34) + CA(41)$

For $\alpha = B2$:

$$\begin{aligned}
 P_0 &= x + z - (y + w), \\
 P_1 &= x^2 + z^2 - (y^2 + w^2), \\
 P_2 &= x^3 + z^3 - (y^3 + w^3), \\
 P_3 &= (x + z)yw - (y + w)xz.
 \end{aligned} \tag{B4}$$

For $\alpha = E$:

$$\begin{aligned}
 P_0 &= (x - z, y - w), \\
 P_1 &= (x^2 - z^2, y^2 - w^2), \\
 P_2 &= (x^3 - z^3, y^3 - w^3), \\
 {}_1P_3 &= [yw(x - z), xz(y - w)], \\
 {}_3P_2 &= [xz(x - z), yw(y - w)].
 \end{aligned} \tag{B5}$$

The appropriate matrix elements for the Hamiltonian are tabulated in Table II.

APPENDIX C: DECOMPOSITION OF HADRONS, AT ZERO TRANSVERSE MOMENTUM, IN TERMS OF BARE-HADRON SYMMETRIES

One can classify the hadrons with $k_T = 0$ according to the symmetry D_4 . However, only certain configurations of bare hadrons are allowed in a hadron of a given symmetry. The following is a list of such a decomposition. The letters $a, b, c,$

d , e correspond to the states $A_1, A_2, B_2, B_2, C_1, C_2, C_3, C_4, D_1, D_2, E_1, E_2$ appearing in Fig. 2:

$$A1: \text{even}\{a(++)\},$$

$$A1\{b(++), c(++++) , d(++), e(++)\},$$

$$A2\{b(++), c(++++) , d(++), e(++)\},$$

$$B1\{c(++++) , d(++)\}, \quad B2\{c(++++) , d(++)\}.$$

$$A2: A1\{b(+ -)\}, \quad A2\{b(+ -)\},$$

$$E(1)\{c(++++)\}, \quad E(2)\{c(++++)\}.$$

$$B1: A1\{c(+ - -+)\}, \quad A2\{c(+ - -+)\},$$

$$B1\{b(++), c(+ - -+)\},$$

$$B2\{b(++), c(+ - -+)\}.$$

$$B2: \text{even}\{a(+ -)\},$$

$$A1\{d(+ -), e(+ -)\}, \quad A2\{d(+ -), e(+ -)\},$$

$$B1\{b(+ -), d(+ -)\}, \quad B2\{b(+ -), d(+ -)\},$$

$$E(1)\{c(+ - -+)\}, \quad E(2)\{c(+ - -+)\}.$$

$$E: \text{odd} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad A1 \left\{ \begin{pmatrix} c(-+-+) \\ c(++--) \end{pmatrix} \right\},$$

$$A2 \left\{ \begin{pmatrix} c(-+-+) \\ c(++--) \end{pmatrix} \right\},$$

$$B1 \left\{ \begin{pmatrix} e_2 \\ -e_1 \end{pmatrix}, \begin{pmatrix} c(-+-+) \\ c(++--) \end{pmatrix} \right\},$$

$$B2 \left\{ \begin{pmatrix} e_2 \\ -e_1 \end{pmatrix}, \begin{pmatrix} c(-+-+) \\ c(++--) \end{pmatrix} \right\},$$

$$\begin{pmatrix} E(1)b_1 + E(2)b_2 \\ E(2)b_1 + E(1)b_2 \end{pmatrix}, \quad \begin{pmatrix} E(1)b_2 - E(2)b_1 \\ E(1)b_1 - E(2)b_2 \end{pmatrix},$$

$$E1 \left\{ \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} c(++--) \\ c(+ - -+) \end{pmatrix} \right\},$$

$$E2 \left\{ \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} c(++--) \\ c(+ - -+) \end{pmatrix} \right\}.$$

This assumes the following sign conventions for the E states: $E(1)(1234) = E(2)(2341) = -E(2)(4123)$ and $E(2)(1234) = -E(1)(2341) = E(1)(4123)$.

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