PHYSICAL REVIEW D VOLUME 20, NUMBER 4 15 AUGUST 1979

Can coupling constants be related?

Satyanarayan Nandi and Ng Wing-Chiu
Universität Bonn, Physikalisches Institut, Bonn, West Germany
(Received 19 May 1978)

We analyze the conditions under which several coupling constants in field theory can be related to each other. When the relation is independent of the renormalization point, the relation between any g and g' must satisfy a differential equation $dg'/dg = \beta_{g'}(g,g')/\beta_g(g,g')$, as follows from the renormalization-group equations. Using this differential equation, we investigate the criteria for the feasibility of a power-series relation $g' = \sum_{n=1}^{\infty} K_n g^n$ for various theories, especially the Weinberg-Salam type (including Higgs bosons) with an arbitrary number of quark and lepton flavors. There is sometimes an arbitrariness in the higher $\{K_n\}$ arising from the integration constant of the differential equation, but K_1 is well determined in terms of only group theory for the theories we have investigated. We use the information on K_1 to compute approximately the value of the Weinberg angle θ_W, the magnitude of the Higgs self-coupling, and hence the mass of the Higgs boson m_H.

I. INTRODUCTION

Quantum field theory specifies the strength of particle interactions through one or several coupling constants. In general, when there are two or more coupling constants, each can be given an arbitrary value, and the theory is always presumed to be well defined.

Theories exist, of course, in which relationships exist among several *a priori* unrelated coupling constants. This is usually accomplished through the existence of *symmetries*. Gauge invariance, for example, demands that the gauge field be coupled to matter fields at the same strength as its self-coupling. Other unrelated fields are forced to couple with equal strength through the requirement of some discrete symmetry, for example. Supersymmetry, in general, enforces a slightly more intricate relationship of the couplings within the supermultiplet.

We also know of an example of coupling-constant relationships that arise dynamically. If the spontaneous symmetry breaking of a Higgs-gauge field system is attributed to radiative corrections, a relationship was found[1] between the Higgs field self-coupling λ and the gauge coupling g.

In this paper we propose to investigate a general *kinematical* requirement necessary for the existence of coupling-constant relations (CCR's). We restrict our attention to those CCR's that are *invariant* under a change in the renormalization point μ. This renormalization invariance at once tells us, via the renormalization-group equation,[2] the general condition under which a CCR may exist. The CCR discovered in Ref. 1 is explicitly dependent on the renormalization point, and as such lies outside the scope of our present investigation.

A constraint arises because when two coupling constants are related they must separately be able to absorb divergences in renormalization. A CCR must therefore be compatible with perturbation-theoretic renormalization, and this is guaranteed by the general condition arising from the renormalization-group equations.

The general condition takes the form of a differential equation, which for two coupling constants looks like

$$\frac{dg'}{dg} = B(g,g')\,. \tag{1.1}$$

Since a differential equation of this type, in general, specifies the solution up to an arbitrary integration constant, there is no useful relation resulting from the equation *per se*.

We have attempted to solve the differential equation (1.1) by (1) inserting the perturbation theory result for $B(g,g')$, and (2) *assuming* a *power-series*-type solution:

$$g' = \sum_n K_n g^n\,. \tag{1.2}$$

We concede that there need be no intrinsic special significance for a power-series solution, except for the following considerations. We found that certain well-defined conditions have to be satisfied by the theory for a power-series solution to exist, and these conditions single out those particular theories to form a special class. Furthermore, we can then essentially determine the integration constant of the power-series solution in some cases, thus giving us a definite power-series-type relationship between coupling constants. Lastly, we suspect that CCR's arising from symmetry or dynamics are probably all of the power-series type, and thus power-series relations allowed by the differential equation *may* correspond to a symmetry/dynamical mechanism as yet undiscovered. We thus feel that such power-series CCR's can be significant and may even be utilized in the construction of nature.

An attempt to discover supersymmetric relations by using the requirement of renormalization compatibility was in fact made by one of the authors.[3] The Wess-Zumino[4] Lagrangian for the fundamental supermultiplet containing a Majorana spinor field ψ, a scalar field ϕ, and a pseudoscalar field ϕ' was derived dynamically starting with the most general renormalizable Lagrangian for these fields, satisfying parity conservation. The assumption underlying the proof was that there exist relations among the bare couplings and that these relations remain preserved after renormalization. The result is that the only theory in which the assumed CCR is sustained to all orders in perturbation theory is precisely the Wess-Zumino supersymmetric Lagrangian. We shall rederive this result in Sec. III in example 5, within the framework of our present general approach.

At this point we have no reason to believe that power-series relations among couplings actually hold in nature. We can only appeal to the philosophical preference among physicists of taking the simplest case whenever a choice is to be made. A similar precedent may be the hope that the physical coupling constant is precisely that value determined[5] from the nontrivial fixed point of the β function. If we make the assumption that such relations are indeed true in nature, then we can make certain speculative predictions of measurable quantities. In the standard Weinberg-Salam model there exist the couplings g and g', and when Higgs bosons are introduced there also exist the scalar self-coupling constant λ. We shall be able to make speculative predictions of the values of g' and λ in terms of g for a certain range of group-theoretic choices in terms of group-theoretic parameters of the model. In other words, the Weinberg angle θ_W, the scalar coupling λ, and hence the mass of the yet to be discovered Higgs particle are determinable, if the power-series CCR's we discover are indeed valid.

In Sec. II we write down the differential equation relating coupling constants and discuss them in a general context for several field-theoretic examples in Sec. III. In Sec. IV we assume a power-series solution for the differential equation in these examples and obtain the respective criteria for their existence. In Sec. V we discuss some phenomenological implications of our speculative power-series CCR. For the Weinberg-Salam-type theories of flavor dynamics we calculate the value of the Weinberg angle, the magnitude of the λ scalar self-coupling, and hence the mass of the Higgs boson. Section VI concludes the paper.

II. DIFFERENTIAL EQUATION RELATING COUPLING CONSTANTS

We consider a field theory with n (*a priori* unrelated) coupling constants $g_1, \ldots, g_n$ determined by renormalization conditions at a common renormalization point μ. Then if one varies the renormalization point, the change in the g's is given by the renormalization-group equation

$$\frac{dg_i(t)}{dt} = \beta_i(g_1(t), \ldots, g_n(t)), \quad i = 1, \ldots, n \tag{2.1}$$

where

$$t = \ln\mu \,. \tag{2.2}$$

Since β_i functions are cutoff independent, they have no explicit dependence on t when we are dealing with a massless theory. When there are massive fields, then we must use the *mass-independent* approach[6] to renormalization-group equations, where β functions are always t (and mass) independent. We can eliminate the parametric dependence of the system of Eqs. (2.1) on t by forming quotients with the jth component (j can be chosen arbitrarily):

$$\frac{dg_i}{dg_j} = \frac{\beta_i(g_1, \ldots, g_n)}{\beta_j(g_1, \ldots, g_n)}, \quad \text{for } i = 1, \ldots, n, \quad i \neq j. \tag{2.3}$$

The solution of the ordinary differential Eqs. (2.3) gives a solution trajectory passing through any initial point $(g_1^{(0)}, \ldots, g_n^{(0)})$.[7] If there is to be any functional relation

$$g_i = F_i(\{g_j\}) \tag{2.4}$$

that has the same form under a change in t, that relation must be a solution trajectory of (2.3). Thus, by investigating (2.3) we shall find *all* possible functional relations among coupling constants if these relations are to be invariant under the renormalization group.

We emphasize right away that these differential equations in general do not imply that coupling constants can no longer be chosen with complete arbitrariness. Let us take the fictitious example

$$\frac{dg_2}{dg_1} = C \,, \tag{2.5}$$

where C is a constant. Then the solution is

$$g_2 = Cg_1 + K \,, \tag{2.6}$$

where K is the arbitrary integration constant. Thus, g_1 and g_2 can be chosen completely at will at any point t. Now suppose we wish to investigate whether a functional relation

$$g_2(t) = \gamma g_1(t) \tag{2.7}$$

is allowed by the renormalization invariance. Then we find that it is indeed allowed, since we can take $K=0$, but only if $\gamma=C$. Thus, the only allowable CCR is

$$g_2(t)=Cg_1(t). \tag{2.8}$$

III. SOME FIELD-THEORETIC EXAMPLES

We shall consider some examples of the differential equations for theories with two or more coupling constants. Clearly, the differential equations are simplest in the case with exactly two coupling constants, since then there is only one differential equation to solve. In theories with three or more coupling constants, there is in general a system of coupled differential equations which are rather more difficult to analyze.

Example 1. The theory consists of two scalar fields ϕ_1 and ϕ_2 with an interaction $\lambda_1\phi_1{}^4+\lambda_2\phi_2{}^4$. We have then the differential equation

$$\frac{d\lambda_2}{d\lambda_1}=\frac{\beta_2(\lambda_1,\lambda_2)}{\beta_1(\lambda_1,\lambda_2)}. \tag{3.1}$$

Since there is no interaction whereby ϕ_1 goes into ϕ_2, the β functions are in fact functions of only their respective coupling constants and the functional forms are identical:

$$\beta_2(\lambda_1,\lambda_2)=\beta(\lambda_2), \tag{3.2a}$$

$$\beta_1(\lambda_1,\lambda_2)=\beta(\lambda_1)=b\lambda_1{}^2+b_1\lambda_1{}^3+\cdots. \tag{3.2b}$$

Therefore, we have

$$\frac{d\lambda_2}{d\lambda_1}=\frac{\beta(\lambda_2)}{\beta(\lambda_1)}, \tag{3.3}$$

which equation is separable, and we get the solution

$$\int_{\lambda_2^{(0)}}^{\lambda_2}\frac{d\lambda_2}{\beta(\lambda_2)}=\int_{\lambda_1^{(0)}}^{\lambda_1}\frac{d\lambda_1}{\beta(\lambda_1)}, \tag{3.4}$$

where $(\lambda_1^{(0)},\lambda_2^{(0)})$ is the initial point. Clearly, the CCR

$$\lambda_1=\lambda_2 \tag{3.5}$$

will satisfy the equation, as is to be expected. Suppose we try to impose a CCR

$$\lambda_2=c\lambda_1, \tag{3.6}$$

with $c\neq1$. This means, by Eq. (3.3), that for a fixed c

$$\beta(c\lambda)=c\beta(\lambda). \tag{3.7}$$

Note that this behavior is a possible one; witness the model function

$$\beta(\lambda)=\sum_{n=-\infty}^{\infty}b_n\exp\left[\left(1+in\frac{2\pi}{\ln c}\right)\ln\lambda\right]. \tag{3.8}$$

But it can be *excluded* by appealing to perturbation theory for $\beta(\lambda)$ and comparing coefficients term by term, as we shall see in Sec. IV.

Example 2. This is a non-Abelian gauge theory with a direct-product gauge group $G\otimes G'$, and there are thus two coupling constants g and g'. The gauge fields can also be coupled to fermion matter fields, but we do not include scalar fields, since they always need their own $\lambda\phi^4$ self-coupling to be renormalizable and would necessitate a third coupling constant. The β functions are, in a perturbation expansion,

$$\beta_g(g,g')=bg^3+b_{10}g^5+b_{01}g^3g'^2+\cdots, \tag{3.9}$$

$$\beta_{g'}(g,g')=ag'^3+a_{10}g'^3g^2+a_{01}g'^5+\cdots, \tag{3.10}$$

where the a's and b's are constant coefficients dependent on the groups and representations chosen. It is noteworthy that there are no $g'g^2$ and gg'^2 terms present. If we take just the lowest order, the differential equation becomes

$$\frac{dg'}{dg}=\frac{ag'^3}{bg^3} \tag{3.11}$$

and can be integrated to give

$$g'^2=\frac{g^2}{a/b+Kg^2}, \tag{3.12}$$

where K is an arbitrary integration constant. It would seem that by setting $K=0$, it would be possible to have a CCR:

$$g'^2=\frac{b}{a}g^2. \tag{3.13}$$

This is, in general, *false* because of higher-order terms in perturbation theory, as we shall discuss fully in Sec. IV.

Example 3. Next we consider a *simple* gauge group G, but we allow scalar fields also. Thus the coupling constants are g (the gauge coupling) and λ (the scalar self-coupling). The differential equation is

$$\frac{d\lambda}{dg}=\frac{\beta_\lambda(\lambda,g)}{\beta_g(\lambda,g)}, \tag{3.14}$$

where the β functions have the expansion

$$\beta_\lambda(g,\lambda)=A\lambda^2+B\lambda g^2+Cg^4+\cdots, \tag{3.15}$$

$$\beta_g(g,\lambda)=bg^3+\cdots. \tag{3.16}$$

With the lowest-order expression the equation can be solved exactly by the substitution

$$x=g^2,\quad \lambda=ux \tag{3.17}$$

to give

$$u=\begin{cases}\dfrac{1}{2A}\left[-B'+(B'^2-4AC)^{1/2}\tanh\left(-\dfrac{(B'^2-4AC)^{1/2}}{4b}(\ln x+K)\right)\right], & \text{for } B'^2>4AC \quad (3.18a)\\ \dfrac{1}{2A}\left(-B'-\dfrac{4b}{\ln x+K}\right), & \text{for } B'^2=4AC \quad (3.18b)\\ \dfrac{1}{2A}\left[-B'+(4AC-B'^2)^{1/2}\tan\left(\dfrac{(4AC-B'^2)^{1/2}}{4b}(\ln x+K)\right)\right], & \text{for } B'^2<4AC \quad (3.18c)\end{cases}$$

where $B'\equiv B-2b$ and K is an integration constant.

The guess for an implementable CCR is only slightly less obvious. Choose $K=+\infty$, $-\infty$, and when (3.18a) holds, a possible CCR is

$$\lambda=\begin{cases} g^2\dfrac{1}{2A}[-B'-(B'^2-4AC)^{1/2}], & \text{for } K=+\infty \quad (3.19a)\\ g^2\dfrac{1}{2A}\;-B'+(B'^2-4AC)^{1/2}], & \text{for } K=-\infty\end{cases} \tag{3.19b}$$

When (3.18b) holds, a possible relation is

$$\lambda=\frac{-B'}{2A}g^2. \tag{3.20}$$

When (3.18c) holds, no useful result emerges. Again, we defer to Sec. IV the modification required by going to higher orders in perturbation theory.

Example 4. Lastly, we consider a more realistic gauge model with $G\otimes G'$ as gauge group, and scalar Higgs fields are also included. The β functions in perturbation theory take the form

$$\beta_\lambda(g,g',\lambda)=A\lambda^2+B\lambda g^2 + B'\lambda g'^2+Cg^4+C'g'^4+\cdots, \tag{3.21}$$

$$\beta_g(g,g',\lambda)=bg^3+b_{100}g^5 + b_{010}g^3g'^2+b_{001}g^3\lambda+\cdots, \tag{3.22}$$

$$\beta_{g'}(g,g',\lambda)=ag'^3+a_{100}g'^3g^2 + a_{010}g'^5+a_{001}g'^3\lambda+\cdots. \tag{3.23}$$

If we choose g as the independent variable, the differential equations are

$$\frac{d\lambda}{dg}=\frac{\beta_\lambda}{\beta_g},\qquad \frac{dg'}{dg}=\frac{\beta_{g'}}{\beta_g}, \tag{3.24}$$

with the β given by Eq. (3.21)–(3.23). We have not been able to find any exact solution to this system of equations, even when only the lowest-order expressions for the β are used.

Example 5. In this example we rederive the supersymmetric result of Ref. 3 using our general approach. The case we consider is the fundamental supermultiplet of Wess and Zumino containing a Majorana spinor field ψ, a scalar field ϕ, and a pseudoscalar field ϕ'. The most general Lorentz-invariant and renormalizable Lagrangian with these fields satisfying parity conservation is given by

$$\mathcal{L}_I(\phi,\phi',\psi)=g_1\bar\psi\psi\phi+g_2\bar\psi i\gamma^5\psi\phi'+m(g_3\phi^3+g_4\phi\phi'^2) +g_5\phi^4+g_6\phi^2\phi'^2+g_1\phi'^4, \tag{3.25}$$

where m is the common mass of these fields. The Wess-Zumino supersymmetric Lagrangian corresponds to

$$g_1=g_2=g_3=g_4,\qquad g_5=\tfrac12 g_6=g_7=\tfrac12 g_1^2. \tag{3.26}$$

For the seven coupling constants we have six differential equations corresponding to Eq. (2.3). In the lowest nontrivial order (one-loop approximation) the differential equation for g_1 and g_2 is

$$\frac{dg_2}{dg_1}=\frac{g_2(8g_2^2-2g_1^2)}{g_1(8g_1^2-2g_1^2)}. \tag{3.27}$$

Using the same technique as in our previous examples, one finds that a power-series CCR can be obtained for Eq. (3.27) with the result

$$g_2^2=\sum_1^\infty K_n(g_1^2)^n, \tag{3.28}$$

where $K_1=1$ and $K_n=0$ for $n\neq1$.

Similarly, analyzing the differential equations for the other coupling constants, one obtains the result (3.26). When we go to the higher loops, it turns out that the result (3.26) is not modified. This is, of course, due to the fact that the Lagrangian (3.25) possesses a global supersymmetry when the relations (3.26) are satisfied.

Finally, we mention some additional features when we are dealing with massive fields. There we must use a mass-independent[6] renormalization procedure, and besides Eq. (2.1) there also exists an equation for the mass parameter (s):

$$\frac{d}{dt}m(t)=-[1+\gamma_\theta(g(t))]m(t), \tag{3.29}$$

where the notation of Ref. 6 has been used. To simplify the discussion let us assume there is only one coupling constant g, and then we get the differential equation

$$\frac{dm}{dg} = -\frac{1+\gamma_\theta(g)}{\beta(g)}\, m. \tag{3.30}$$

This separable equation can be immediately integrated to give

$$\ln\frac{m}{m_0} = -\int_{g_0}^{g} dg\, \frac{1+\gamma_\theta(g)}{\beta(g)}. \tag{3.31}$$

For an asymptotically free theory with

$$\beta(g) = -bg^3 + \cdots \tag{3.32}$$

we find

$$\frac{m}{m_0} = \exp\left(-\frac{1}{2g^2}\right)\exp\left(\frac{1}{2g_0^{\,2}}\right), \tag{3.33}$$

and hence as $g\to 0$,

$$\frac{m}{m_0} \sim e^{-g^{-2}/2} \xrightarrow[g\to 0]{} 0, \tag{3.34}$$

which agrees with the conclusion of Ref. 6.

In any theory in which the electron and muon fields, say, appear symmetrically, their γ_θ are identical functions of g. Thus, the masses must satisfy the equation

$$\frac{dm_\mu}{dm_e} = \frac{m_\mu}{m_e}\,\frac{1+\gamma_\theta^\mu(g)}{1+\gamma_\theta^e(g)} = \frac{m_\mu}{m_e}, \tag{3.35}$$

giving (K = integration constant)

$$m_\mu(t) = K m_e(t). \tag{3.36}$$

Thus, (3.36) is an allowable relation between masses.

IV. POWER-SERIES SOLUTIONS

We investigate these differential equations restricting possible CCR's in greater detail in this section. We shall assume a perturbative expansion for the β functions, and we shall investigate whether a CCR in the form of an infinite power series is compatible with perturbation theory order by order.

Power series are certainly not the only kind of CCR's allowed by the differential equations. To demand a power-series solution to the differential equations is in effect to pick out a certain subclass of solutions. But it is the only class in which definitive statements can be made in conjunction with perturbation theory. Also, power-series CCR's would encompass those CCR's arising from a symmetry, and they are generally linear or quadratic in form.

We expect these power-series CCR's to be at best asymptotic series valid for small values of the coupling constants involved, as perturbation theory itself is in a similar situation. Power-series relations are at least formally invertible in terms of power series. Let y be expandable as

$$y = a_1x + a_2x^2 + a_3x^3 + \cdots. \tag{4.1}$$

We write the inverse

$$x = b_1y + b_2y^2 + b_3y^3 + \cdots, \tag{4.2}$$

and we can determine

$$b_1 = \frac{1}{a_1},\quad b_2 = -\frac{a_2}{a_1^{\,3}}, \ldots. \tag{4.3}$$

Thus, it is irrelevant which particular variable is chosen as the independent one in the power-series CCR's we investigate.

We now discuss the examples of Sec. III in turn.

Example 1. The differential equation is

$$\frac{d\lambda_2}{d\lambda_1} = \frac{b\lambda_2^{\,2} + b_1\lambda_2^{\,3} + \cdots}{b\lambda_1^{\,2} + b_1\lambda_1^{\,3} + \cdots} = \left(\frac{\lambda_2}{\lambda_1}\right)^2 \frac{b + b_1\lambda_2 + \cdots}{b + b_1\lambda_1 + \cdots}. \tag{4.4}$$

This equation, just like the differential equations in our other examples, is characterized by the fact that the point $\lambda_1\to 0$, $\lambda_2\to 0$ is a *singular* one, in that the right-hand side of the equation takes on the indeterminate form 0/0. A long-known procedure of finding power-series solutions to such equations exists,[8] and we shall follow it in our investigation of the examples.

It is clear that in order for the equation not to become singular, λ_2 must approach zero as $\lambda_1\to 0$. So we make the substitution

$$\lambda_2 = u(\lambda_1)\lambda_1 \tag{4.5}$$

and we get

$$u + \lambda_1\frac{du}{d\lambda_1} = u^2\,\frac{b + b_1u\lambda_1 + \cdots}{b + b_1\lambda_1 + \cdots}. \tag{4.6}$$

Writing

$$u = u_0 + O(\lambda_1), \tag{4.7}$$

we have

$$u_0 = u_0^{\,2}, \tag{4.8}$$

and so

$$u_0 = 1 \tag{4.9a}$$

or

$$u_0 = 0. \tag{4.9b}$$

Let

$$u = u_0 + v(\lambda_1), \tag{4.10}$$

and consider first $u_0 = 1$. If we expand the right-hand side up to order λ_1^2 we get

$$1 + v + \lambda_1 \frac{dv}{d\lambda_1} = (1 + 2v + v^2) \frac{1 + \frac{b_1}{b}(1+v)\lambda_1 + \frac{b_2}{b}(1+v)^2 \lambda_1^2 + \cdots}{1 + \frac{b_1}{b}\lambda_1 + \frac{b_2}{b}\lambda_1^2 + \cdots}$$

$$= 1 + 2v + v^2 + \frac{b_1}{b} v \lambda_1 + \cdots , \qquad (4.11)$$

and so

$$-v + \lambda_1 \frac{dv}{d\lambda_1} = v^2 + \frac{b_1}{b} v \lambda_1 + \cdots . \qquad (4.12)$$

Now we assume a power-series solution

$$v = K_1 \lambda_1 + K_2 \lambda_1^2 + K_3 \lambda_1^3 + \cdots , \qquad (4.13)$$

and the left-hand side is

$$-v + \lambda_1 \frac{dv}{d\lambda_1} = \sum_{n=1}^{\infty} (n-1) K_n \lambda_1^n . \qquad (4.14)$$

It is crucial to note that since the coefficient of v is -1 in this case K_1 is multiplied by zero and *drops out* of the left-hand side. So comparing coefficients order by order, we have

$$0 = (1-1)K_1 = 0 , \qquad (4.15a)$$

$$(2-1)K_2 = K_1^2 + \frac{b_1}{b} K_1 , \qquad (4.15b)$$

$$\cdots .$$

The right-hand side of (4.15a) *happens to be* zero, and so K_1 can be any *arbitrary* number and the equations will still be consistent. This corresponds to an arbitrary integration constant of the differential equation. K_2 is, however, no longer arbitrary once the choice for K_1 is made, and since $(n-1)$ vanishes only for $n = 1$, all subsequent equations in (4.15) are well defined and should yields $\{K_n\}$ in terms of one arbitrary constant K_1. There is the other possibility (4.96), but that is the trivial case leading to $\lambda_2 = 0$.

Thus, we obtain the most general power-series CCR possible to be

$$\lambda_2 = \lambda_1 [1 + K_1 \lambda + K_2(K_1)\lambda^2 + K_3(K_1)\lambda^3 + \cdots] , \qquad (4.16)$$

where $K_n(K_1)$ indicates that all higher K_n are determined in terms of an arbitrary K_1. The initial term is *not* arbitrary and the only consistent possibility is the coefficient 1.

In particular, the obvious discrete symmetry $\lambda_1 = \lambda_2$ is a special case of Eq. (4.16) if we set the arbitrary K_1 equal to zero. In the expansion (4.11) all higher terms must contain at least one v factor, from the requirement that the right-hand side must be unity when $v = 0$. Thus, the equation for K_n in the system (4.15) must have a right-hand side in terms of $K_{n-1}, K_{n-2}, \ldots$ only, without any constant term independent of K's. When K_1 is set equal to zero, all higher K_n must then also vanish.

Example 2. We rewrite the differential equation in terms of $y = g'^2$, $x = g^2$, and so

$$\frac{dy}{dx} = \frac{ay^2 + a_{10}xy^2 + a_{01}y^3 + \cdots}{bx^2 + b_{10}x^3 + b_{01}x^2 y + \cdots} . \qquad (4.17)$$

We use the same technique as in example 1 to analyze this equation: Let

$$y = ux \qquad (4.18)$$

and

$$u + x \frac{du}{dx} = u^2 \frac{a + a_{10}x + a_{01}ux + \cdots}{b + b_{10}x + b_{01}ux + \cdots} . \qquad (4.19)$$

Taking

$$u = u_0 + O(x) , \qquad (4.20)$$

we determine u_0 by

$$u_0 = u_0^2 \frac{a}{b} , \qquad (4.21)$$

and so

$$u_0 = \frac{b}{a} , \qquad (4.22a)$$

or

$$u_0 = 0 . \qquad (4.22b)$$

Only (4.22a) is interesting [(4.22b) is the trivial case] and we get the equation for v, where

$$u = u_0 + v(x) , \qquad (4.23)$$

to be

$$u_0 + v + x \frac{dv}{dx} = (u_0^2 + 2u_0 v + v^2) \frac{a}{b} \frac{1 + (a_{10}/a)x + (a_{01}/a)(u_0 + v)x + \cdots}{1 + b_{10}/bx + b_{01}/b(u_0 + v)x + \cdots}$$

$$= \left(u_0 + 2v + \frac{a}{b} v^2\right) \frac{1 + (a_{10}/a)x + (a_{01}/a)(u_0 + v)x + \cdots}{1 + (b_{10}/b)x + (b_{01}/b)(u_0 + v)x + \cdots} , \qquad (4.24)$$

where in the last equality we have used Eq. (4.22a). We again try

$$v=K_1x+K_2x^2+\cdots \tag{4.25}$$

and compute coefficients up to $O(x^2)$. We obtain

$$\begin{aligned}-v+x\frac{dv}{dx}&=\left(\frac{b}{a^2}a_{10}-\frac{b_{10}}{a}+\frac{b^2}{a^3}a_{01}-\frac{b}{a^2}b_{01}\right)x\\&\quad+\left(\frac{b}{a^2}a_{01}-\frac{b_{01}}{a}+D+E\frac{b}{a}\right)vx+\frac{a}{b}v^2\\&\quad+(F+Gu_0+Hu_0^2)x^2+\cdots,\end{aligned}\tag{4.26}$$

where

$$\begin{aligned}D&=2\left(\frac{a_{10}}{a}-\frac{b_{10}}{b}\right),\\E&=2\left(\frac{a_{01}}{a}-\frac{b_{01}}{b}\right),\\F&=\frac{b}{a}\left(\frac{a_{20}}{a}-\frac{a_{10}b_{10}}{ab}-\frac{b_{20}}{b}+\frac{b_{10}^2}{b^2}\right),\\G&=\frac{b}{a}\left(\frac{a_{11}}{a}-\frac{a_{01}b_{10}}{ab}-\frac{a_{10}b_{01}}{ab}-\frac{b_{11}}{b}+2\frac{b_{10}b_{01}}{b^2}\right),\\H&=\frac{b}{a}\left(\frac{a_{02}}{a}-\frac{a_{01}b_{01}}{ab}-\frac{b_{02}}{b}+\frac{b_{01}^2}{b^2}\right).\end{aligned}\tag{4.27}$$

We then deduce by comparing coefficients that

$$0=(1-1)K_1=\frac{b}{a^2}a_{10}-\frac{b_{10}}{a}+\frac{b^2}{a^3}a_{01}-\frac{b}{a^2}b_{01},\tag{4.28a}$$

$$\begin{aligned}(2-1)K_2&=K_1\left(\frac{b}{a^2}a_{01}-\frac{b_{01}}{a}+D+E\frac{b}{a}\right)\\&\quad+\frac{a}{b}K_1^2+\left(F+G\frac{b}{a}+H\frac{b^2}{a^2}\right),\end{aligned}\tag{4.28b}$$

$$\cdots.$$

In this case the right-hand side of (4.28a) is no longer automatically zero, but is determined by group-theoretic parameters. Consistency of the power-series solution demands, of course, that it vanish. Thus, it is a well-defined group-theoretic exercise to search for pairs of groups G and G' in which

$$l\equiv\frac{b}{a^2}a_{10}-\frac{b_{10}}{a}+\frac{b^2}{a^3}a_{01}-\frac{b}{a^2}b_{01}=0.\tag{4.29}$$

This has in fact been done by Levin,[9] who was interested in seeing whether it is consistent with perturbation theory up to two loops to impose the CCR

$$g'^2=\frac{b}{a}g^2.\tag{4.30}$$

Our results (4.28) show that indeed only for the groups satisfying (4.29) (several G, G' pairs were found by Levin) is it possible to have a power-series CCR between g'^2 and g^2. But it is clearly not enough to look at two loops. Suppose $l=0$, then as before we conclude from (4.28a) that K_1 is an arbitrary (integration) constant. However, even if $K_1=0$, K_2 from Eq. (4.28b) is not automatically zero, and so on, for the higher $\{K_n\}$. Thus, it is not sufficient to demand $l=0$ if one wants to have an exact relation $g'^2=(b/a)g^2$. The groups G and G' must conspire in such a way that $K_2, K_3, \ldots$ all vanish if K_1 is set equal to zero.

Our conclusion is that if (4.29) is satisfied, the most general power-series CCR for example 2 is of the form

$$g'^2=g^2\left(\frac{b}{a}+K_1g^2+K_2(K_1)g^4+K_3(K_1)g^6+\cdots\right),\tag{4.31}$$

where K_1 is arbitrary (can be zero), while all other K_n are determined in terms of K_1. Even if K_1 is chosen to vanish, the higher K_n in general do not vanish, and so there are always higher-order corrections to the conjectured relation (3.13).

For group pairs G, G' in which (4.29) is *not* satisfied, there is no possible power-series CCR consistent with perturbation theory. It is interesting to see the nature of the solution of the differential Eq. (4.26) in that case. On the right-hand side of (4.26) there is only one term of $O(x)$, and we may in some mathematical sense regard the higher-order terms as a "small" perturbation. Thus, we write

$$-v+x\frac{dv}{dx}=lx+\sum_{\substack{i,j\\i+j\geq 2}}^{\infty}c_{ij}x^iv^j,\tag{4.32}$$

and when we substitute

$$v=\bar{v}x\tag{4.33}$$

we have

$$\begin{aligned}x\frac{d\bar{v}}{dx}&=l+\sum_{\substack{i,j\\i+j\geq 2}}^{\infty}c_{ij}x^{i+j-1}\bar{v}^j\\&\equiv l+\Phi(x,\bar{v}).\end{aligned}\tag{4.34}$$

This is equivalent to the integral equation

$$\bar{v}(x)-\bar{v}(x_0)=\int_{x_0}^{x}\frac{dx}{x}[l+\Phi(x,\bar{v})].\tag{4.35}$$

We can then iterate by

$$\bar{v}^{(0)}(x)=\bar{v}^{(0)}(x_0)+l\ln\frac{x}{x_0}\,,$$

$$\bar{v}^{(1)}(x)-\bar{v}^{(1)}(x_0)=\int_{x_0}^{x}\frac{dx}{x}[l+\Phi(x,\bar{v}^{(0)}(x))]\,,$$

$$\bar{v}^{(2)}(x)-\bar{v}^{(2)}(x_0)=\int_{x_0}^{x}\frac{dx}{x}[l+\Phi(x,\bar{v}^{(1)}(x))]\,, \tag{4.36}$$

$$\cdots .$$

Since $\bar{v}^{(0)}(x)$ is only a logarithm, clearly the iteration using Φ as a double power series will only yield powers of x and $\ln x$. Thus, the solution is a power series in x modified by powers of $\ln x$. If $l=0$ [(4.29) holds] then by our previous result there are only powers of x.

Even in the case $l\neq 0$ we have

$$g'^2=g^2\left(\frac{b}{a}+g^2\bar{v}(g^2)\right)\,, \tag{4.37}$$

where $\bar{v}(g^2)$ begins with $O(\ln g^2)$, and so the coefficient of g^2 is always b/a and *never arbitrary*.

Example 3. The equation takes the form

$$\frac{d\lambda}{dx}=\frac{A\lambda^2+B\lambda x+Cx^2+\sum_{i+j\geq 3}^{\infty}A_{ij}x^i\lambda^j}{2bx^2+x^2\sum_{i+j\geq 1}^{\infty}b_{ij}x^i\lambda^j}\,, \tag{4.38}$$

which the usual substitution

$$\lambda=ux \tag{4.39}$$

converts to

$$u+x\frac{du}{dx}=\frac{Au^2+Bu+C+\sum_{i+j\geq 3}^{\infty}A_{ij}x^{i+j-2}u^j}{2b+\sum_{i+j\geq 1}^{\infty}b_{ij}x^{i+j}u^j}\,. \tag{4.40}$$

Taking

$$u=u_0+O(x) \tag{4.41}$$

we determine u_0 by

$$0=-u_0+\frac{Au_0^2+Bu_0+C}{2b} \tag{4.42}$$

or

$$u_0^{(\pm)}=\frac{-B'\pm(B'^2-4AC)^{1/2}}{2A}\,, \tag{4.43}$$

$$B'=B-2b\,. \tag{4.44}$$

Note that this exactly corresponds to the solution (3.19).

As before we have

$$u=u_0+v\,, \tag{4.45}$$

and up to $O(x^2)$ the differential equation is

$$Pv+x\frac{dv}{dx}=Qx+\frac{A}{2b}v^2+Rvx+Sx^2\,, \tag{4.46}$$

where

$$P=\frac{-1}{2b}(2Au_0+B')\,, \tag{4.47}$$

$$Q=\frac{1}{2b}(A_{30}+A_{21}u_0+A_{12}u_0^2+A_{03}u_0^3-b_{10}u_0-b_{01}u_0^2)\,, \tag{4.48}$$

and R and S are given in the appendix and are uninteresting. [S involves coefficients such as A_{40}, which corresponds to an $O(g^8)$ in $\beta\lambda$.]

Thus, the solution is given by

$$(P+1)K_1=Q\,,$$

$$(P+2)K_2=\frac{A}{2b}K_1^2+RK_1+S\,, \tag{4.49}$$

$$\cdots$$

and so

$$K_1=\frac{Q}{P+1}\,,$$

$$K_2=\frac{1}{P+2}\left[\frac{A}{2b}\left(\frac{Q}{P+1}\right)^2+R\left(\frac{Q}{P+1}\right)+S\right]\,. \tag{4.50}$$

The allowable power-series CCR is thus given by

$$\lambda^{(\pm)}=g^2(u_0^{(\pm)}+K_1g^2+K_2g^4+\cdots)\,, \tag{4.51}$$

where $u_0^{(\pm)}$ is given by the two solutions (4.43) of

This is by far the most esthetically appealing case investigated so far. In (4.51) all K_n are determined solely by group theory. There is *no* arbitrary integration constant involved. This arises because of an $O(v)$ contribution with coefficient P. P is precisely the negative of the derivative of the quadratic form in (4.42) at the point u_0, where the quadratic vanishes. If (4.42) has two real roots, the two derivatives have opposite signs and one of them must be positive. Thus $P+n$ in (4.49) can be made nonzero for all n, and there is then always a solution for K_n $\forall n$, and no arbitrariness. There is also no group-theoretic constraint such as (4.29) to be satisfied for the existence of a power-series solution. The higher-order corrections to the low-order relation (3.19) exist and are completely *computable*.

In practice one would like to have u_0 real and preferably positive. The reality condition means

$$B'^2>4AC\,, \tag{4.52}$$

precisely the condition (3.18a) to give an interesting linear CCR in lowest order. The relation (4.51), of course, simply includes the higher-order corrections to the low-order CCR (3.19).

The reason that there is no arbitrary (integration) constant in our solution (4.51) can be seen from (3.18a). That low-order solution already is never analytic in g^2 unless the choice $K=\pm\infty$ for the integration constant is made. Thus, the requirement that the solution be a power series essentially forces us to choose a particular integration constant already.

Example 4. We call $g'^2 = y$ and use $g^2 = x$ as the independent variable, and we have

$$\frac{d\lambda}{dx} = \frac{A\lambda^2 + Bx\lambda + B'\lambda y + Cx^2 + C'y^2 + \cdots}{bx^2 + b_{100}x^3 + b_{010}x^2 y + b_{001}x^2\lambda + \cdots}, \quad (4.53)$$

$$\frac{dy}{dx} = \frac{ay^2 + a_{100}xy^2 + a_{010}y^3 + a_{001}y^2\lambda + \cdots}{bx^2 + b_{100}x^3 + b_{010}x^2 y + b_{001}x^2\lambda + \cdots}, \quad (4.54)$$

where

$$\frac{dg^2}{dt} = bg^4 + \cdots, \quad \frac{dg'^2}{dt} = ag'^4 + \cdots. \quad (4.55)$$

As usual, we let

$$\lambda(x) = u(x)x, \quad u(x) = u_0 + v(x), \quad v(x) = \sum_{n=1}^{\infty} H_n x^n \quad (4.56)$$

and

$$y(x) = w(x)x, \quad w(x) = w_0 + z(x), \quad z(x) = \sum_{n=1}^{\infty} K_n x^n. \quad (4.57)$$

Then we determine that

$$u_0 = \frac{1}{b}[Au_0^2 + (B + B'w_0)u_0 + (C + C'w_0^2)] \equiv \frac{1}{b}(Au_0^2 + \bar{B}u_0 + \bar{C}), \quad (4.58)$$

where u_0 once again has two nontrivial solutions and

$$w_0 = \frac{aw_0^2}{b}. \quad (4.59)$$

As usual we have then

$$u_0 + v + x\frac{dv}{dx} = [A(u_0^2 + 2u_0 v + v^2) + B(u_0 + v) + B'(u_0 + v)(w_0 + z) + C + C'(w_0 + z)^2 + \cdots] \times [b + b_{100}x + b_{010}(w_0 + z)x + b_{001}(u_0 + v)x + \cdots]^{-1} \quad (4.60)$$

and

$$w_0 + z + x\frac{dz}{dx} = [a(w_0 + z)^2 + a_{100}(w_0 + z)^2 x + a_{010}(w_0 + z)^3 x + a_{001}(w_0 + z)^2(u_0 + v)x + \cdots] \times [b + b_{100}x + b_{010}(w_0 + z)x + b_{001}(u_0 + v)x + \cdots]^{-1}. \quad (4.61)$$

Thus, (4.60) bears some resemblance to the equation of example 2, and (4.61) to that of example 3. Clearly, by construction the constant w_0 and u_0 terms cancel on both sides. From previous experience we need only investigate the $O(x)$ terms in order to determine the feasibility of a power-series solution.

Thus, we get the equations

$$v + \frac{dv}{dx} = Tv + Ux, \quad (4.62)$$

$$z + \frac{dz}{dx} = 2z + Vx, \quad (4.63)$$

and so the coefficients are determined by

$$[(1 - T) + 1]H_1 = U, \quad [(1 - T) + 2]H_2 = \cdots, \quad \cdots \quad (4.64)$$

and

$$0 = (-1 + 1)K_1 = V, \quad (-1 + 2)K_2 = \cdots, \quad \cdots. \quad (4.65)$$

The constants T, U, and V are

$$T = 2Au_0 + \bar{B}, \quad (4.66)$$

$$U = -\frac{1}{b^2}(Au_0 + \bar{B}u_0 + \bar{C})(b_{100} + b_{010}w_0 + b_{001}u_0), \quad (4.67)$$

$$V = \frac{1}{b}\left(a_{100}w_0^2 + a_{010}w_0^3 + a_{001}w_0^2 u_0 - aw_0^2\frac{b_{100}}{b} - aw_0^3\frac{b_{010}}{b} - aw_0^2 u_0\frac{b_{001}}{b}\right). \quad (4.68)$$

To the lowest order in the x expansion those two sets of Eqs. (4.64) and (4.65) are uncoupled. Each displays (by now) familiar features: In (4.64), H_1 is determinable and H_n is iteratively determined; in (4.65), the solution can be a power series *only* if

$$V = 0. \quad (4.69)$$

In that case, K_1 is arbitrary, while K_n can be iteratively determined in terms of K_1. To higher orders in x, the two sets of equations become coupled, and so the arbitrariness in K_1 will also be "transmitted" to the higher H_n as well as K_n.

Thus, the allowable CCR, when $V=0$, is

$$\lambda = g^2[u_0 + H_1 g^2 + H_2(K_1) g^4 + \cdots], \tag{4.70}$$

$$g'^2 = g^2[w_0 + K_1 g^2 + K_2(K_1) g^4 + \cdots], \tag{4.71}$$

where the dependence of H_n, K_n on K_1 is shown. If $V \neq 0$, then the nonanalyticity in g^2 will be "transmitted" also to λ, and we get

$$\lambda = g^2[u_0 + g^2 \bar{v}(g^2)] \tag{4.72}$$

$$g'^2 = g^2[w_0 + g^2 \bar{z}(g^2)], \tag{4.73}$$

where $\bar{v}$ and $\bar{z}$ will be double powers series in g^2 as well as $\ln g^2$.

The validity of (4.69) is thus seen to be of crucial importance to the type of allowable CCR in example 4. Most of the required coefficients are not known now and so we cannot yet settle the question of whether $V=0$ for a given model.

V. PHENOMENOLOGICAL APPLICATIONS

In this section we shall discuss some phenomenological applications of our ideas. We consider only the unified theories of weak and electromagnetic interactions based on a simple gauge group G or on a semisimple group $G_1 \times G_2$. Except for the confusing status of parity violation in atomic physics, the simple SU(2)×U(1) Salam-Weinberg model has demonstrated a phenomenal success. However, a few fundamental questions remain unanswered: (A) How many leptons and quark flavors exist? (B) What is the mass of the Higgs boson which one needs to introduce in all such theories to break the symmetry spontaneously?

According to our power-series CCR approach, the gauge coupling constants g, g' and the Higgs ϕ^4-self-coupling λ are related by Eqs. (4.70)–(4.73). Since b is [defined by $dg/dt = \frac{1}{2} b g^3$] a function of the number of leptons and quark flavors only, the experimental measurement of the Weinberg angle, within our approach, determines this number. Also, the mass of the Higgs boson is given by

$$m_H = 2\sqrt{\lambda}\ \frac{m_{W^\pm}}{g}. \tag{5.1}$$

So, using Eqs. (4.70)–(4.73) *to lowest order* in g, we can calculate the Higgs boson mass from the experimental values of $m_{W^\pm}$ and $\sin^2\theta_W$.

A. Number of leptons and quark flavors

For simplicity, we consider the SU(2)×U(1) Salam-Weinberg model, extended only to include more quarks and leptons. Let n_1 (n_2) be the number of lepton (quark flavor) doublets, n_3 be the number of lepton singlets, and n_4 (n_5) be the number of quark flavor singlets of charge $\frac{2}{3}$ ($-\frac{1}{3}$), respectively.

We also restrict ourselves to only left-handed doublets. Then $n_1 = n_3$ and $n_2 = n_4 = n_5$. Also, in order that the model be triangle-anomaly-free, one must have $n_1 = n_2 = n$ (say).

In this case, one has

$$\frac{b}{2} = -\frac{1}{16\pi^2}\left(\frac{22}{3} - \frac{4n}{3} - \frac{m}{6}\right), \tag{5.2}$$

$$\frac{a}{2} = \frac{1}{16\pi^2}\ \frac{2}{3}\left(\frac{10}{3} n + \frac{m}{4}\right), \tag{5.3}$$

where m is the number of complex Higgs doublets.

In order for the relation between g and g' to make sense, it must be satisfied for all t. So both couplings must be either asymptotically free or nonfree. Since g' is always asymptotically nonfree, g must also be so. From Eq. (5.2) this then gives the restriction

$$\tfrac{4}{3} n + \tfrac{1}{6} m \geq \tfrac{22}{3}$$

or (5.4)

$$n + \tfrac{1}{8} m \geq \tfrac{11}{2}.$$

If there is only the Higgs doublet ($m=1$), then from (5.4) one obtains

$$n \geq \tfrac{43}{8} \ \text{ or } \ n \geq 6, \tag{5.5}$$

since n is an integer.

We emphasize that the lower bound on n ($n=6$ quark flavor doublets means 12 quark flavors) is independent of any experimental number. This follows purely from our philosophy of power-series CCR's and from the SU(2)×U(1) Salam-Weinberg model.

The relation between g and g' in lowest order is given by

$$g'^2 = \frac{b}{a}\ g^2. \tag{5.6}$$

For g and g' both ultraviolet- or infrared-free, the solutions of the renormalization-group equations yield

$$g^2(t) \sim -\frac{1}{bt} \ \text{ and } \ g'^2(t) \sim -\frac{1}{at} \tag{5.7}$$

in the ultraviolet and infrared limit, respectively. Thus, we can deduce directly the validity of (5.6) in such limits.[10] The Weinberg angle θ_W is given by

$$\sin^2\theta_W = \frac{g'^2}{g^2 + g'^2}. \tag{5.8}$$

Then, using (5.6) and the expression for b and a as given by (5.2) and (5.3), we get[11]

$$\sin^2\theta_W = \frac{3}{8}\ \frac{n - \frac{11}{2} + \frac{1}{8} m}{n - \frac{33}{16} + \frac{3}{32} m}. \tag{5.9}$$

Note that $d\sin^2\theta_W/dn$ is positive, and for $m=1$ one obtains the bounds

$$0.06 \leq \sin^2\theta_W \leq 0.38\,. \tag{5.10}$$

The lower bound is obtained for $n=6$ and the upper bound for $n=\infty$.

If we use the experimental value of the Weinberg angle $\sin^2\theta_W \simeq 0.24$, then for $m=1$ we obtain from Eq. (5.9)

$$n=11\,. \tag{5.11}$$

This would mean that there are about 22 leptons and 22 quark flavors, so that counting colors, the total number of elementary fermions is about 88. It is amusing to note that this is about equal to the number of elements in the periodic table. The asymptotically free color SU(3) gauge theory of strong interactions restricts n to be less than or equal to 8. We emphasize that the present experimental uncertainty in the measurement of the Weinberg angle and also the uncertainty in our lowest-order approximation do not exclude the value $n=8$. Also, if there are more than one Higgs doublet, the value of n will be reduced.

If the SU(2)×U(1) turns out to be a subgroup of a grand simple group G unifying other interactions, then it is possible to obtain a group-theoretic value C relating g and g':

$$g'=g/C\,. \tag{5.12}$$

It is generally believed that G is strongly broken down to SU(2)×U(1), and so there exist superheavy particles which do not participate in giving rise to phenomena observed today. For present-day energies it suffices to use only the "observed" part of the group, presumably SU(2)×U(1), as input to calculate the β functions. The result we obtain for the Weinberg angle would therefore correspond to the value *observed* at present-day energies. The value of C imposed by the unifying group G can be "discovered" by our approach, but only by including the full set of superheavy particles in the input from β, and we have no idea what sort of such extra particles is to be included.

Finally, we mention that, as shown in Sec. IV, since we have not computed the value of V in (4.68), strictly speaking, power-series CCR's do not exist for the SU(2)×U(1) g, g' case. The approximate solution at the two-loop level is given by [see Eqs. (4.72) and (4.73)]

$$g'^2=g^2\left(\frac{b}{a}+K_1'g^2\ln g^2+K_1g^2\right), \tag{5.13}$$

where K_1 is an arbitrary integration constant. Though the second and the third terms are higher-order terms in perturbation series, the arbitrary integration constant K_1 can, in principle, be very large, so that the term K_1g^2 could be equal to or even larger than b/a for the experimental value of g. We emphasize, however, that in that case, the ratio g'^2/g^2 would be substantially t dependent. Since the measurement of the Weinberg angle does not show any detectable Q^2 dependence over the measured range of $Q^2 \simeq 5$ to 50 GeV^2, we conclude that $g'^2=g^2b/a$ is a good approximation to Eq. (5.13).

B. Higgs boson mass

We consider again the simplest possibility, i.e., SU(2)×U(1) Salam-Weinberg model with only one complex Higgs doublet. In this case there is only one Higgs self-coupling λ, and after spontaneous symmetry breaking there remains only one scalar field (the Higgs boson) whose mass[12] is given by

$$m_H=\sqrt{2\lambda}\,\langle\phi_0\rangle=2\sqrt{\lambda}\;\frac{m_{W^\pm}}{g}\,, \tag{5.12}$$

where $\langle\phi_0\rangle$ is the vacuum expectation value of the neutral member of the Higgs doublet. There have been some attempts[13,14] to estimate the Higgs mass or to give bounds on it. We emphasize that our approach is completely different from these.

As we have seen in Sec. IV, in this case it is possible to implement a power-series CCR between λ and the gauge coupling. *In the lowest order* the relation is

$$\lambda=u_0^{(\pm)}g^2\,, \tag{5.13}$$

and thus the Higgs boson mass

$$m_H=2\sqrt{u_0^{(\pm)}}\,m_W\,, \tag{5.14}$$

where $u_0^{(\pm)}$ are the two solutions of (4.58):

$$u_0^{(\pm)}=\frac{1}{2A}\,\{-(\overline{B}-b)\pm[(\overline{B}-b)^2-4A\overline{C}]^{1/2}\}\,. \tag{5.15}$$

We emphasize that our approach predicts that the Higgs boson mass is of the same order as the intermediate vector bosons' masses.

The constants A, $\overline{B}$, $\overline{C}$, and b are completely fixed by the group [in this case SU(2)×U(1)] and by the representations (in this case only left-handed doublets and right-handed singlets) and the number (n) of leptons or quark doublets. For the case under consideration they are

$$\begin{aligned} A&=\frac{3}{4\pi^2}\,,\\ \overline{B}&=\frac{-1}{8\pi^2}\left(\tfrac{9}{2}+6\tan^2\theta_W\right),\\ \overline{C}&=\frac{1}{8\pi^2}\left(\tfrac{9}{8}+6\tan^4\theta_W\right),\\ \tfrac{1}{2}b&=\frac{-1}{16\pi^2}\left(\tfrac{22}{3}-\tfrac{4}{3}n-\tfrac{1}{6}\right)\,. \end{aligned} \tag{5.16}$$

Using the experimental value of $\sin^2\theta_W \simeq 0.24$ and hence the value of $n=11$ [as determined from Eq. (5.8)], we predict

$$m_H \simeq 56 \text{ or } 224 \text{ GeV}. \tag{5.17}$$

As already noted in Ref. 13, it will be extremely difficult to detect the Higgs boson of such a high mass with the present generation experimental facility.

VI. CONCLUSION

Because of our heavy dependence on perturbation theory, the possible CCR we established can only be expected to have validity when all coupling constants concerned are sufficiently small. Thus, if both g and g' are infrared and ultraviolet free, respectively, then both are expected to be small in the infrared and ultraviolet regions, respectively, and one might expect the existence of power-series CCR's we envisage. If g is ultraviolet-free and g' is infrared-free, then in the infrared region when g' is small, g may be very large, and so we cannot expect any CCR as we propose. The same consideration applies to the (λ, g) case. If (3.18a) holds (tanh function), then λ and g can be simultaneously small in the asymptotic domain, whereas if (3.18c) holds (tan function), then λ is violently oscillatory if g is only slightly changed, and indeed, in this case we find no possible power-series CCR.

Coleman and Weinberg,[1] using completely different dynamical assumptions, found that λ is of order g^4 in the one-loop approximation. We emphasized in the Introduction that their result is outside our scope because the functional form they obtained is explicitly dependent on the renormalization point. They investigated also a possible improvement of their result by the renormalization group, and obtained for their theory with λ and g a relation of the tangent type, which, being oscillatory, enables them to choose their renormalization point judiciously to make λ be of order g^4. With our approach, the tangent case forbids the emergence of any power-series CCR. We thus consider that our results are complementary, rather than contradictory to theirs.

Our phenomenological application in Sec. V is necessarily speculative, because, among other things, we have little control over the higher-order (in g) coefficients, being dependent on an arbitrary integration constant. If one can measure experimentally the variation of the (effective) coupling constants over a range of momentum where they are all small, then we may able to disentangle from Eq. (4.37) relating g and g' the coefficient b/a of g^2. This would be a direct measurement of the ratio b/a. Equation (4.37) is independent of any power-series assumptions and is valid for a Weinberg-Salam-type theory.

Our method can be applied to groups other than $SU(2)\otimes U(1)$ and more complicated representations for the Higgs particle. There might be more coupling constants encountered, but the lowest-order coefficients in the CCR obtained should still be independent of arbitrariness. We did not illustrate these more complicated cases in view of a lack of consensus on which group-theoretic structure should be employed.

The feasibility of power-series CCR's should be connected to some special feature of the theory, perhaps a symmetry of some kind. Supersymmetry normally imposes a CCR of the monomial type. When infinite power series are involved, we might attribute it to the existence of perhaps a "hypersymmetry" in the theory. This may be connected to a mathematical criterion of the existence of power-series solutions of our differential equations. A discovery of this type would shed more light on power-series CCR's.

ACKNOWLEDGMENTS

We are grateful to H. Rollnik for many helpful discussions throughout this work and to K. Symanzik for an illuminating communication.

APPENDIX

R and S of Eq. (4.46) are given by

$$R=\frac{1}{2b}\left(A_{21}+2A_{12}u_0+3A_{03}u_0^{\,2}-\frac{b_{10}}{2b}\,2Au_0-\frac{b_{10}}{2b}\,B-\frac{b_{01}}{2b}\,2Au_0^{\,2}-\frac{b_{01}}{2b}\,u_0B\right), \tag{A1}$$

$$\begin{aligned}S=\frac{1}{2b}\Bigg\{&A_{40}+A_{31}u_0+A_{22}u_0^{\,2}+A_{12}u_0^{\,3}+A_{04}u_0^{\,4}-\frac{b_{10}}{2b}\,A_{30}-\frac{b_{10}}{2b}\,A_{21}u_0-\frac{b_{10}}{2b}\,A_{12}u_0^{\,2}\\&-\frac{b_{10}}{2b}\,A_{03}u_0^{\,3}-\frac{b_{01}}{2b}\,u_0A_{30}-\frac{b_{01}}{2b}\,u_0^{\,2}A_{21}-\frac{b_{01}}{2b}\,u_0^{\,3}A_{12}-\frac{b_{01}}{2b}\,A_{03}u_0^{\,4}\\&+2bu_0\left[\frac{b_{20}}{2b}+\frac{b_{11}}{2b}\,u_0+\frac{b_{20}}{2b}\,u_0^{\,2}+\frac{1}{4b^2}(b_{10}+b_{01}u_0)^2\right]\Bigg\}.\end{aligned} \tag{A2}$$

[1]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).
[2]M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954); C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).
[3]S. Nandi, Nucl. Phys. B97, 178 (1975).
[4]J. Well and B. Zumino, Phys. Lett. 49B, 52 (1972).
[5]K. Wilson, Phys. Rev. D 3, 1818 (1970); S. L. Adler, *ibid.* 5, 3021 (1972).
[6]S. Weinberg, Phys. Rev. D 8, 3497 (1973). The generalization to the spin-zero case has been given by C. G. Callan (unpublished).
[7]There exists a unique continuous function $y(x)$ which satisfies the equation $dy/dx = F(x,y)$ and passes through the initial point (x_0,y_0), provided that $F(x,y)$ is continuous in a neighborhood of (x_0,y_0) and satisfies the Lipschitz condition. (See any book on ordinary differential equations.) This statement is not valid at the origin in the coupling-constant plane, which is a *singular point* of the differential equation.
[8]E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956), p. 293 ff.
[9]D. Levin, Phys. Rev. D 17, 2695 (1978).
[10]This was noticed by E. Ma, Prog. Theor. Phys. 58, 1896 (1977).
[11]Curiously, a similar result for the Weinberg angle was obtained by A. Zee, Phys. Rev. D 18, 2600 (1978), using entirely different considerations.
[12]If we use the mass-independent renormalization-group-equation approach of Ref. 6, then this expression for the Higgs mass is tru only to the lowest order in the coupling constant. The m's and g's used in Ref. 6 differ from the conventional ones by higher-order terms.
[13]J. Ellis, M. K. Gaillard, and D. V. Nanopoulos, Nucl. Phys. B106, 292 (1976).
[14]A. D. Linde, Pis'ma Zh. Eksp. Teor. Fiz. 23, 73 (1976) [JETP Lett. 23, 64 (1976)]; S. Weinberg, Phys. Rev. Lett. 36, 294 (1976); P. H. Frampton, *ibid.* 37, 1378 (1976); B. W. Lee, C. Quigg, and H. B. Thacker, *ibid.* 38, 883 (1977); M. Veltman, Acta Phys. Pol. B8, 475 (1977); Phys. Lett. 70B, 253 (1977).