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Model for the generation of leptonic mass

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A self-consistent model for the generation of leptonic mass is developed. In this model it is assumed that bare masses are zero, all of the (charged) leptonic masses being generated by the QED self-interaction. A perturbation expansion for the QED self-mass is formulated, and contact is made between this expansion and the work of Landau and his collaborators. In order to achieve a finite result using this expansion, it is assumed that there is a cutoff at the Landau singularity and that the functional form of the (self-mass) integrand is the same beyond that singularity as it is below. Physical interpretations of these assumptions are discussed. Self-consistency equations are obtained which show that the Landau singularity is in the neighborhood of the Planck mass. This result implies that, as originally suggested by Landau, gravitation may play a role in an ultraviolet cutoff for QED. These equations also yield estimates for the (effective) number of additional pointlike particles that electromagnetically couple to the photon. This latter quantity is consistent with present data from e^+e^- storage rings.

I. INTRODUCTION

Some time ago it was pointed out[1] that owing to the nonlinear nature of quantum field theory the symmetries which are manifest in the Lagrangian may not be present in solutions which satisfy the field equations obtained from that Lagrangian. Developing this idea along different lines, it has been proposed[2] that a dynamical symmetry breaking could lead to self-consistent solutions yielding a nonzero fermion mass. Following these ideas, Baker and Glashow[3] suggested that just such a mechanism might be responsible for the generation of the μ-e mass splitting. In second order they note that the self-mass integrals for the muon and the electron are independent of each other and, consequently, in this order one would not expect to obtain asymmetric solutions. They point out, however, that in fourth order the mass equations would be coupled through the fermion loop in the vacuum polarization graph, enabling the possibility of (self-consistent) asymmetric solutions, i.e., a mass splitting. Unfortunately, they did not actually detail the fourth- (or higher-) order equations nor determine if they actually admitted such asymmetric solutions.

The purpose of this paper is to develop a model for the generation of leptonic mass by the QED self-interaction. Only (self-consistent) symmetric solutions will be considered here; a study of possible asymmetric solutions, following the suggestion of Baker and Glashow, will be covered in a later paper.

It is assumed here that the bare, or mechanical, charged-lepton mass is zero, all of the mass being dynamically generated by the electromagnetic self-interaction. The assumption of a fermion with a null bare mass and dynamically generated physical mass follows Baker and Johnson,[4] who found in their model for the dynamical generation of fermion mass that consistency required a null bare mass. Now this scheme for the generation of fermion mass breaks the formal γ_5 invariance which obtains in a QED having fermions described by the massless Dirac equation. However, it has been shown[5] that the breaking of this γ_5 symmetry is immune to the Goldstone-boson dilemma.[6]

It is well known that QED, in general, and the self-mass, in particular, are divergent in the ultraviolet region. While Baker and Johnson obtain an eigenvalue equation, the postulated solution of which would solve this problem,[7] it is assumed here that there is a physical ultraviolet cutoff.

There is precedence for this assumption. Landau and his collaborators[8] have shown that the polarization of the vacuum will lead to a divergence of QED and serious difficulty for the theory beyond a certain point at very high energy, which we shall call the "Landau singularity." And consequently they suggested that cutting QED off at the Planck mass[9] might "save" QED from this "crisis." The Planck mass is so large (~10^{19} GeV) that it is well beyond any possible conflict with experiment.[10] Unfortunately, by the same token direct detection of effects at this energy likewise appear to be precluded from future experiments. That gravitation might furnish a cutoff for QED has also been investigated by others.[11,12]

More recently it has been shown by Lautrup[13] that this difficulty is present as well in QED phenomena not normally considered to be divergent (after renormalization). Specifically, he has shown that there are gauge-invariant sets of graphs for the anomalous magnetic moment of the electron that are not Borel summable, and that this difficulty is associated with the Landau singularity.

We see no reason to reject the evidence that there is such a singularity in renormalized QED. For example, to assume that the existence of a convergent renormalized QED theory (if the ultraviolet limit goes to infinity) depends upon fortuitous cancellations between gauge-invariant sets of graphs at and beyond the Landau singularity seems much too tenuous. (Of course, the grand unification schemes offer another possible resolution of this difficulty, but, as mentioned below, we do not pursue that avenue here.)

Thus, the existence of a physical cutoff actually would serve functions beyond those served in the renormalization procedure. First, it would save QED from the crisis pointed out by Landau[8] and further illuminated by Lautrup.[13] At the same time it would furnish a mass scale for QED,[12] which otherwise does not have one. In addition, if the QED ultraviolet cutoff were found to be near the Planck mass, this would imply a connection between the gravitational interaction and other forces of elementary particles, a notion offering some philosophical satisfaction and a possible avenue for an eventual unification of these interactions.

It is appropriate to remark here that it is conceivable that other interactions could be a source for leptonic mass. For example, the Baker-Glashow idea has been applied to gauge theories containing chiral $SU(n)\otimes SU(n)$ groups.[14] Along similar lines the recently developed unified gauge theories obtain lepton masses through a coupling of the lepton multiplets to various postulated Higgs. fields.[15] This latter approach is subject to the criticism that the coupling constants are arbitrary and as yet there is no experimental evidence indicating the existence of Higgs particles. Or, in a more mundane approach, one might consider weak-interaction self-mass diagrams involving the intermediate vector bosons. One should note, however, that if in order to treat leptons and quarks on a similar footing one assumes that the neutrinos are four-component Dirac spinors,[16] then one has a framework in which to argue that the weak-interaction contribution to the charged-lepton self-mass must be small; the neutrinos, which are assumed to participate in the weak interactions equally with the charged leptons, are observed to be massless, or nearly so.

These considerations and a general desire to avoid undue complications furnish motivation to keep this model (to the extent possible) within the confines of QED. Consequently, the possibility of non-QED interactions contributing to leptonic masses (aside from a hadronic component of vacuum polarization) is not considered in this paper.

As a basis for this study we assume (as did Baker and Glashow) that the Lagrangian and Hamiltonian are of the standard QED form and are symmetric in the bare muon and electron wave functions. (The only known physical difference between muon and electron is their rest mass.) Then a standard perturbation expansion for the self-mass is developed. As is well known, there are both infrared and ultraviolet divergence problems associated with this expansion. Of these, the former is less serious. In second order, the self-mass is not infrared divergent.[17] In fourth order there are infrared divergences in two graphs, but they have been shown to cancel,[18] leaving the fourth-order self-mass calculation without any infrared divergence. A similar result has been found in a careful study of the fourth-order calculations[19] for the Lamb shift. These cancellations are, in fact, special cases of the general statement by Gell-Mann and Low[20] that such cancellations must always appear in the self-energy problem. They reason that a perturbation expansion about a null bare mass using the bare charge cannot be infrared divergent; the physical momentum in the problem will furnish an appropriate infrared cutoff. And since the renormalization program does not change this aspect of the solution, the dressed fermion propagator is also not infrared divergent. Therefore, the self-mass cannot be infrared divergent. Thus, in the self-mass problem the renormalization program introduces the infrared divergences in the separate graphs and at the very same time the elaborate and complicated relationships between graphs which serve to just cancel out these divergences in the final result. This result has subsequently been demonstrated in general.[21]

Following this logic, in evaluating the contributions of the various graphs, infrared problems are ignored; for each graph, only the leading terms as a function of the ultraviolet cutoff are kept. As will be seen, dropping the nonleading terms leads to numerical errors of no more than a few percent.

The use of an ultraviolet cutoff ensures convergence of the perturbation expansion (for the sets of graphs summed here).[22] By symmetry, this expansion is the same function for both muon and electron. The possibility of other pointlike particles electromagnetically coupling to the photon is also taken into account. From this expansion are obtained equations of self-consistency for the relationship between the lepton mass scale and a "hard" ultraviolet cutoff, which is a sharp upper limit to the momentum integration, arbitrarily put in by hand.

Since the above formulation manifests a (Landau) singularity in the photon propagator at some large but finite energy, i.e., before the point of infinite energy is reached, the hard ultraviolet cutoff must always be maintained below this singularity. In order to eliminate this artifice and to extend the range of integration to infinity, it is naively assumed that by analytic continuation beyond the Landau singularity the integrand of the self-mass integral has the same functional form as it does below. At the same time, the hard ultraviolet cutoff, which ensured convergence of the perturbation expansion, is replaced by a Lorentz-invariant cutoff acting at the point of ultraviolet divergence, i.e., at the Landau singularity. One assumes this latter cutoff is due to physical causes. This leads to self-consistency equations relating the location of the singularity to the lepton masses with no free parameters (only mathematical uncertainties). These self-consistency equations also enable one to estimate the (effective) number of pointlike particles which electromagnetically couple to the photon. This quantity is experimentally accessible with e^+e^- storage rings.

II. PERTURBATION EXPANSION FOR THE SELF-MASS

Although the use of perturbation expansions is a standard technique in QED, for the sake of completeness and because the expansion for the (divergent) self-mass is less familiar, a brief development of that expansion is included here. Following the pioneering papers of Dyson[23] and Schwinger[24] the complete electron propagator $S_F'(p)$ is given by[25]

$$S_F'(p)=\frac{1}{\not p-m-\Sigma(\not p)}\,, \tag{1}$$

where the proper self-energy

$$\Sigma(\not p)=ie_0^{\,2}\int\frac{d^4k}{(2\pi)^4}D_F'(k)_{\mu\nu}\times\Gamma^\mu(p,p-k)S_F'(p-k)\gamma^\nu\,. \tag{2}$$

Throughout this paper the notation of Bjorken and Drell will be used. The quantities in Eqs. (1) and (2) are the usual renormalized quantities, e_0 being the bare coupling constant.

At this point one makes the standard substitutions, taking one to the renormalized quantities via the undetermined renormalization parameters, Z_i:

$$\begin{aligned}&e_0=Z_1Z_2^{-1}Z_3^{-1/2}e\,,\\&S_F'(p)=Z_2\tilde S_F'(p)\,,\\&D_F'(k)_{\mu\nu}=Z_3\tilde D_F'(k)_{\mu\nu}\,,\end{aligned} \tag{3}$$

and

$$\Gamma_\mu(p',p)=Z_1^{-1}\tilde\Gamma_\mu(p',p)\,,$$

where the tilde indicates the renormalized functions. The boundary condition[26]

$$\tilde\Gamma_\mu(p,p)\Big|_{\not p=m}=\gamma_\mu \tag{4}$$

is also understood. Substituting Eqs. (3) into Eq. (4) yields

$$\Sigma(\not p)=\tilde\Sigma(\not p)=ie^2\int\frac{d^4k}{(2\pi)^4}\tilde D_F'(k)_{\mu\nu}\times\tilde\Gamma^\mu(p,p-k)\tilde S_F'(p-k)\gamma^\nu\,, \tag{5}$$

where the Ward identity[27] $Z_1=Z_2$ has been used. Using Eqs. (1), (3), and (5), one may now write the renormalized electron propagator as

$$\tilde S_F'(p)=\frac{1/Z_2}{\not p-m-\tilde\Sigma(\not p)}\,. \tag{6}$$

To proceed, we note that $\tilde\Sigma(\not p)$ has a Taylor's series expansion about the point $\not p=m$:

$$\tilde\Sigma(\not p)=\tilde\Sigma_0(\not p)+(\not p-m)\tilde\Sigma_1(\not p)+(\not p-m)^2\tilde\Sigma_2(\not p)\,, \tag{7}$$

where $\tilde\Sigma_0$ and $\tilde\Sigma_1$ are coefficients which diverge as a function of an ultraviolet cutoff while $\tilde\Sigma_2$ is "finite" (i.e., not divergent) and also is to include all the higher-order terms. Employing the well-known Feynman rules,[28] the expression $\tilde\Sigma(p)$ may be expanded in a standard perturbation series about the observed or physical mass in powers of the renormalized charge e. This expansion for $\tilde\Sigma$ and Eq. (7) will then yield, order by order, expressions for $\tilde\Sigma_0$, $\tilde\Sigma_1$, and $\tilde\Sigma_2$.

The requirement that $\tilde S_F'$ be a suitable propagator puts certain conditions upon Z_2, $\tilde\Sigma_0$, and $\tilde\Sigma_1$. $\tilde S_F'$ must have a pole[29] at $\not p=m$ which requires

$$\tilde\Sigma_0(\not p)\Big|_{\not p=m}=0\,. \tag{8}$$

In the perturbation expansion, this is achieved by appropriately choosing the mass counterterm

$$\delta m=\delta m^{(2)}+\delta m^{(4)}+\delta m^{(6)}+\cdots \tag{9}$$

order by order; the superscripts indicate the order of e. The second condition, that the residue of the pole at $\not p=m$ must equal unity, requires

$$Z_2(1-\tilde\Sigma_1)=1 \tag{10}$$

or

$$Z_2^{-1} = 1 - \tilde{\Sigma}_1 , \tag{11}$$

a condition which can also be satisfied order by order.[30] It is welcome that we need concern ourselves here only with Eqs. (8) and (9), for this enables us to ignore the infrared problems associated with Z_2.

It is simplest to display graphically the perturbation expansion derived from Eq. (8). The second-order equation is depicted in Fig. 1. Using the Feynman rules to evaluate (in Feynman gauge) the graphs in Fig. 1 yields[31]

$$-i\tilde{\Sigma}^{(2)}(\not{p}) = i\delta m^{(2)} + (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \lambda^2 + i\epsilon} \times \gamma_\nu \frac{i}{\not{p} - \not{k} - m + i\epsilon} \gamma^\nu , \tag{12}$$

where λ is a small photon mass to allow for proper handling of the infrared divergences and ϵ is a small positive quantity defining the proper contour for the integral in the complex k_0 plane. In accordance with Eq. (8), $\not{p}$ will be set equal to m after the integration and $\delta m^{(2)}$ will then be defined by $\tilde{\Sigma}^{(2)}(m) = 0$. While at $\not{p} = m$, $\tilde{\Sigma}^{(2)}$ is not infrared divergent, it is ultraviolet divergent. Equation (12) has been shown[32] to yield

$$\delta m^{(2)} = \frac{3\alpha m}{4\pi}\left(\ln\frac{\Lambda^2}{m^2} + \tfrac{1}{2}\right), \tag{13}$$

where Λ is the ultraviolet cutoff mass and $\alpha = e^2/4\pi \approx \frac{1}{137}$ is the fine-structure constant.

We now observe that for $\Lambda = M_P$, the Planck mass (1.22×10^{19} GeV/c^2), the logarithm in Eq. (13) will dominate the factor $\frac{1}{2}$ by a factor of ~100. Similarly, in calculations of higher-order graphs ($2n$, say), terms having $\ln(\Lambda^2/m^2)$ to a power less than n will be dominated by the leading $[\alpha \ln(\Lambda^2/m^2)]^n$ term. Thus, to simplify the calculation, one need keep only the leading terms of the divergent integrals. That the resultant numerical uncertainties are small can be verified subsequent to the analysis. This step permits an enormous simplification in the evaluation of the various Feynman graphs. (See the appendices for details of these calculations.)

At this point it is assumed that all of the electron mass is due to QED and is dynamically generated. Thus, in this second-order calculation, one sets $\delta m^{(2)} = m$ and extracts from Eq. (13) the self-consistency condition relating m to Λ. Neglecting the $\frac{1}{2}$ relative to the logarithm, this relationship is

$$m = \Lambda \exp\left(-\frac{2\pi}{3\alpha}\right), \tag{14}$$

where Λ is seen to furnish a mass scale for the lepton mass. Owing to the exponential factor, however, m/Λ is a very small number. In fact, setting $\Lambda = M_P = (\hbar c/G)^{1/2}$, where G is the gravitational constant, yields $m = 3\times10^{-97}$ eV/c^2.

At first glance this extremely small mass value might lead one to conclude that this mechanism is far too feeble for the generation of a significant quantity of lepton mass (unless one contemplates an extremely high cutoff mass, well beyond the Planck mass). However, this is not necessarily the case. This calculation is extremely sensitive to the argument of the exponential. Recognizing the existence of higher-order terms, one can inquire what fraction of the total mass shift will $\delta m^{(2)}$ account for, given that other contributions will self-consistently make up the rest of the mass.

This question can be quantified by changing Eq. (14) to

$$m = \Lambda \exp\left(-\frac{2\pi}{3\alpha} f^{(2)}\right), \tag{14'}$$

where $f^{(2)}$ is the fraction of the mass (self-consistently) furnished by the second-order diagram in Fig. 1. Self-consistency then yields $f_e^{(2)} = 0.1795$ and $f_\mu^{(2)} = 0.1610$, for electron and muon, respectively. Thus, if the total of the contributions of the higher-order terms is only about five times that of the initial term, then a self-consistent formulation for the dynamic generation of the charged lepton mass becomes quite plausible. Furthermore, it is evident that the observed μ-e mass splitting would obtain when these higher-order contributions differ from each other by only a few percent.

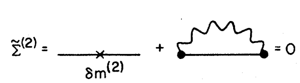

FIG. 1. The diagrammatic form of the equation which specifies the second-order quantity $\delta m^{(2)}$. In accordance with Eq. (8), $\not{p} = m$ is understood. The signs of the mathematical expressions represented by the graphs are determined by the Feynman rules.

III. SELF-CONSISTENCY EQUATIONS FOR THE SYMMETRIC SOLUTION

By letting $\delta m = m$ in several more detailed approximations, we now study self-consistency equations for the leptonic self-mass. This is a useful prelude to an investigation of asymmetric solutions, because associated with asymmetric solutions, one also expects symmetric solutions.[33] And one can determine appropriate mass scale relationships already from these symmetric solutions.

In this section the ultraviolet cutoff is set at the Planck mass. This step enables contact between the analysis of Landau and his collaborators and that of this work; the former uses direct functional analysis to find (approximate)solutions to the integral equations, while a standard perturbation approach is employed here. This contact is useful because it gives additional confidence in the application of the perturbation expansion to the self-mass problem. From both these approaches, one sees that vacuum polarization causes a divergence in the ultraviolet region *before* one reaches an infinite momentum. And the location of this divergent point, the Landau singularity, is found to be the same in both of these approaches.

In second order, the self-consistency equation for the mass is

$$\frac{\delta m^{(2)}}{m} = \frac{3\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} = \frac{9}{4}\xi = 1 , \qquad (15)$$

where the definition

$$\xi = \frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{m^2} \qquad (16)$$

has been introduced to replace m by the more convenient mass parameter ξ. When one uses the (dimensionless) ξ parameter, the explicit value of the cutoff need not be specified. Equation (15) has the solution $\xi = 0.4444$, the equivalent of Eq. (14).

In fourth order, the self-consistency condition for the electron mass is derived from the equation shown in Fig. 2 (plus the second-order equation shown in Fig. 1). Here, the first vacuum polarization graph is introduced. If one assumes that there is only the electron, then the vacuum polarization graph only appears once in the equation. If one assumes that there is only a pair of charged leptons, then it appears twice, once with an electron loop and once with a muon loop.

Now it is quite possible, even probable, considering the colliding beam results,[34] that there are other pointlike objects which electromagnetically couple to the photon. In this paper these objects will be taken into account by assuming that there is an effective number R of them. This contribution will be termed "hadronic," although it is recognized that it includes some heavy leptonic part.[35]

Using Eqs. (B5), (A9), (C8), and (D12), the equation in Fig. 2 (at $\not p = m$) yields

$$\delta m^{(4)} = -\frac{27m}{4}\xi^2 + \frac{9(R+2)m}{8}\xi^2 + \frac{81m}{32}\xi^2 + \frac{27m}{16}\xi^2 , \qquad (17)$$

where we have assumed that all mass parameters are equal. Using Eqs. (15) and (17), the fourth-order self-consistency condition on the self-mass becomes

$$\frac{\delta m^{(2)} + \delta m^{(4)}}{m} = \frac{9}{4}\xi - \frac{27}{4}\xi^2 + (R+2)\frac{9}{8}\xi^2 + \frac{81}{32}\xi^2 + \frac{27}{16}\xi^2 = 1 . \qquad (18)$$

Equation (18) leads to the quadratic equation

$$(36R - 9)\xi^2 + 72\xi - 32 = 0 . \qquad (19)$$

The appropriate positive root of Eq. (19) is plotted versus the (phenomenological) parameter R in Fig. 3.

One sees that in fourth order self-consistency requires an $(R+2)$ of ~115 to obtain the correct electron mass parameter ($\xi = \xi_e = 0.0798$ for $\Lambda = M_P$), in contrast to the value of ~12 implied by the analysis of Landau and his collaborators.[8] The reason for this disparity, of course, is that only a single loop of vacuum polarization has been taken into account. The self-consistency equation for the self-mass which one would associate with the Landau calculation would be one using a photon on which all concatenations of the proper photon

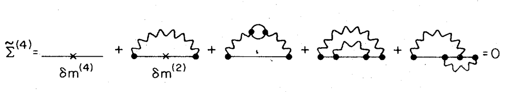

FIG. 2. The diagrammatic form of the equation which specifies the fourth-order quantity $\delta m^{(4)}$. As in Fig. 1, $\not p = m$ is understood.

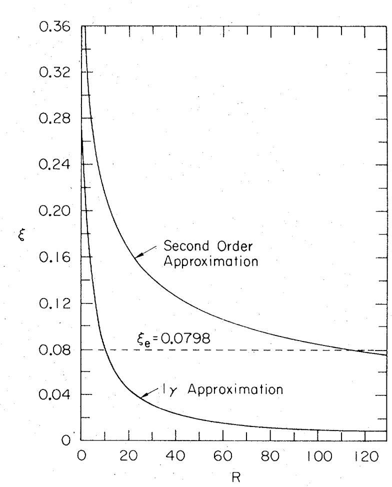

FIG. 3. Plot of the (appropriate positive) root of Eq. (19), the second-order approximation, and of the ξ of Eq. (21), the 1γ approximation, versus the parameter R. For orientation the value of ξ_e is shown when Λ is assumed to equal the Planck mass 1.22×10^{19} GeV/c^2.

self-mass blobs are summed. Since that calculation is a "zero approximation," the appropriate (proper) blob approximation to use here would be the single fermion loop. This calculation entails the sum of (gauge-invariant) graphs shown in Fig. 4(a). Since there is a contribution to δm for each Feynman graph, one employs the set of counterterm graphs shown in Fig. 4(b), which formally sum to (and define) $\delta m^{(1\gamma)}$, the one-photon estimate of the leptonic self-mass including vacuum polarization to all orders. [Each $\delta m^{(2n)}$ counterterm is tacitly partitioned into its $n\gamma, (n-1)\gamma, \ldots,$ and 1γ pieces according to the topologies of its associated graphs. $\delta m^{(1\gamma)}$, then, is only part of the full δm summation. This expansion will be described in more detail below.]

These expressions lead to the (one-photon) self-consistency equation shown in Fig. 4(c). An estimate using this set of graphs has been obtained in Appendix A. Using Eq. (A13) the self-consistency equation for this case becomes

$$\frac{\delta m^{(1\gamma)}}{m} = 1 = \frac{9}{4(R+2)} \ln\left(\frac{1}{1-(R+2)\xi}\right). \tag{20}$$

The solution to Eq. (20),

$$\xi = \frac{1}{R+2}\left[1-e^{-4(R+2)/9}\right], \tag{21}$$

is also plotted in Fig. 3. It can be seen that in this case the self-consistent relationship between the electron mass and an ultraviolet cutoff at the Planck mass is achieved when $R+2=12$, in accord with the results of Landau.

From these results one sees that in the self-mass problem the high-order vacuum polarization graphs are quite significant and, in fact, *cannot be ignored*. Consequently, in further analysis of the self-mass, the photons will include estimates of the vacuum polarization to all orders; the perturbation expansion, then, will be in the number of photons rather than in powers of the renormalized electric charge. In following the topologies of the usual perturbation expansion, not including the vacuum polarization, one can see that this new expansion will proceed by gauge-invariant sets of graphs, as does the usual perturbation theory, order by order, where order now denotes the number of photons. It is evident that this new expansion (recalling that the photon lines implicitly include their vacuum polarization) will include all of the graphs employed in the usual perturbation expansion. Therefore, the full summations of these two expansions are equivalent. As with usual perturbation calculations, we shall use low-order (gauge-invariant) summations to approximate the full expansion.

It is assumed here that the perturbation expansion, whether expanded using renormalized charge or the bare charge, is convergent. As pointed out by Bjorken and Drell,[36] it is conventional to make such an assumption for QED calculations even though convergence of either expansion has never been demonstrated. One notes that the expansion here (by the number of photons, each one including its own vacuum polarization) is akin to a perturbation expansion in the bare electric charge, for as is well known (since $Z_1 = Z_2$), only the renormalization constant for the photon propagator Z_3 (i.e., the vacuum polarization) enters into renormalization of the electric charge.

Having found the parameters of the self-consistent solution for the one-photon case, the next step is to investigate the two-photon case. However, at this point one finds an ambiguity on how to proceed. One approach is to evaluate

$$\delta m = \delta m^{(1\gamma)} + \delta m^{(2\gamma)} + \delta m^{(3\gamma)} + \cdots, \tag{22}$$

where Eq. (22) is constructed analogously to Eq. (9) and includes all contributions to δm. The equation from which $\delta m^{(2\gamma)}$ derives is depicted in Fig. 5, and the higher-order terms follow in the usual way. This approach leads to the expansion depicted in Fig. 6. For a two-photon calculation, one could then approximate δm by $\delta m^{(1\gamma)} + \delta m^{(2\gamma)}$ and derive the self-consistency equation from

$$\frac{\delta m^{(1\gamma)} + \delta m^{(2\gamma)}}{m} = 1. \tag{23}$$

This formulation, using Eqs. (B8), (C25), and

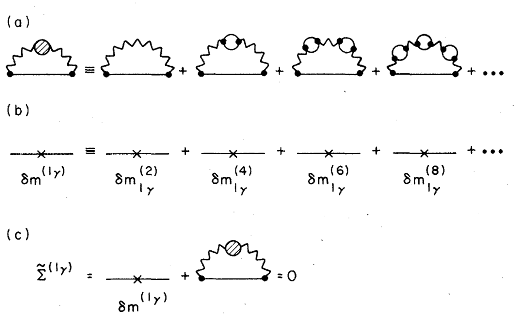

FIG. 4. (a) The diagrammatic form of the equation defining a leading approximation for the photon propagator including (the one-loop estimate for) vacuum polarization to all orders. The summation loop is hatched. (b) The appropriate fermion self-mass counterterm associated with (a). On the right-hand side the superscripts denote the order of e and the subscripts indicate that the counterterm is in the 1γ expansion. (c) The diagrammatic form of the equation which specifies $\delta m^{(1\gamma)}$, the one-photon approximation for the fermion self-mass. As with the equations depicted in Figs. 1 and 2, $\not{p}=m$ is understood.

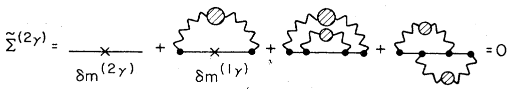

FIG. 5. The diagrammatic form of the equation which specifies $\delta m^{(2\gamma)}$, the two-photon contribution to the fermion self-mass. As before, $\not{p}=m$ is understood.

(D18), leads to

$$\frac{\delta m}{m}=A-\frac{4}{3}\frac{A\delta m^{(1\gamma)}}{m}+\frac{A^2}{2}+\frac{A^2}{3}$$

$$=A-\frac{A^2}{2}=1\,, \tag{24}$$

where the quantity A is defined by Eq. (A14). Equation (24) is dominated by the counterterm graph (second graph, Fig. 5). And since it has a large negative coefficient, one sees that there is no real value of A which will yield a solution.

Another approach might be called a topology summation. First, note that any graph containing more than one $\delta m^{(n\gamma)}$ cross (e.g., the last graph in Fig. 6) will have an m^j, where $j>1$, coefficient to the integral and hence will be negligible. This result may be obtained by a simple dimensional argument: One m factor sets the scale, but the additional factors of m are divided by factors proportional to Λ, rendering the graph negligible. As a consequence, all graphs containing more than one factor of $\delta m^{(n\gamma)}$ on the right-hand side of the equation depicted in Fig. 6 may be neglected. One can then sum over n, topology by topology, all of the graphs containing a $\delta m^{(n\gamma)}$ to yield single graphs of each topology, each with its (now summed) δm counterterm. See, for example, Fig. 7. Transferring these sum graphs to the left-hand side of the equation and then factoring out the δm leads to the equation depicted in Fig. 8.

Looking at Fig. 8, one can formulate a two-photon approximation to the self-consistency equation by

$$\frac{\delta m}{m}=\frac{A+\frac{5}{6}A^2}{1+\frac{4}{3}A}=1\,, \tag{25}$$

which has the solution $A=(1\pm\sqrt{31})/5$. The positive solution is $A=1.3136$, not too different from the one-photon solution $A=1$. One expects higher-order approximations to δm, in essence, to yield higher-order polynomial equations in A, specifying other values for A, which would still be on the order of unity.

Looking more closely at the difference between these two formulations, one sees that in this second case the approximation

$$\delta m=\delta m^{(1\gamma)}+\delta m^{(2\gamma)}/(1+\tfrac{4}{3}A) \tag{26}$$

has been made. Now the factor multiplying $\delta m^{(2\gamma)}$ is just what one would get for the geometric sum of a series whose recursion relation is given by Eq. (B8). Thus, the sum of the higher-order contributions of $\delta m^{(2\gamma)}$ implied by Eq. (B8) leads to an effective reduction of the initial (or direct) contribution. (All graphs involved in this summation are of the same topology; each order of $\delta m^{(n)}$ feeds into $\delta m^{(n+2)}$ via this counterterm graph, and then into $\delta m^{(n+4)}$ again by this graph, etc.)

One notes that the condition for convergence of the geometric series $|\frac{4}{3}A|<1$ is not met by the solution to Eq. (25). However, it is easy to see that since $-\frac{4}{3}A<0$, it is an alternating series which is Borel summable[37]; the sum is just the

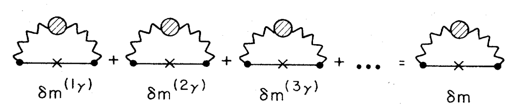

FIG. 7. Summation of the graphs of simplest topology which contain counterterm factors. This subset of graphs has been extracted from the general summation shown in Fig. 6. The sum graph on the right-hand side of this equation corresponds to the first graph in the denominator of the expression in Fig. 8.

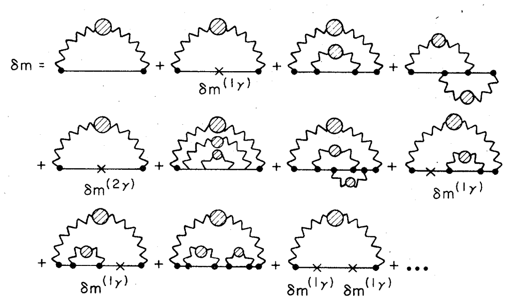

FIG. 6. First terms of a general expansion for δm, where the expansion is by the number of photons. As before, $\not{p}=m$ is understood. (Note that the sign of the δm standing alone will be properly determined by the Feynman rules.)

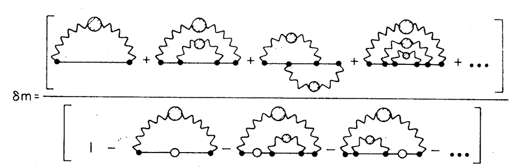

FIG. 8. Graphical expression for δm in which all non-negligible graphs containing δm counterterms have been transferred to the left-hand side of the equation, combined by factoring the δm out, and then divided out, becoming the denominator on the right-hand side. The small open circles in the fermion lines indicate the prior location of the δm factors.

same as that given by the formula for the geometric series ignoring the above restriction. This difficulty appears similar to other questions about the convergence of the perturbation series, and is thought to stem from the use of the perturbation expansion to solve the field equation problem. That is, the true solution is some function, and the series is only a representation of that function, entailing certain limitations. Such difficulties were recognized very early,[38] but they are not yet fully resolved. Here we shall employ Eq. (25) for the two-photon approximation and make the assumption that the use of the Borel summation technique is valid.

The solution to Eq. (25), given by

$$\xi = \frac{1}{R+2}\left(1 - e^{-4(R+2)A/9}\right), \tag{27}$$

called the 2γ solution, where A is set to 1.3136, is plotted in Fig. 9 near $R=10$. For comparison, the 1γ solution, Eq. (21), is also plotted. It is evident from Fig. 9 that variations in the value of A in the neighborhood of unity do not cause major excursions in the value of R required to derive the observed lepton mass scale from the ultraviolet cutoff mass.

Figure 9 is evidence that a reasonable estimate of the location of this divergence may be determined analytically by the summation of one-fermion loop graphs, the simplest graphs of vacuum polarization, on but a single photon; the two-photon calculation is not too different from the one-photon calculation. One expects that increasingly more accurate approximations will in effect specify different values of A, and hence different values of R (if one knows the cutoff). However, a glance at Eq. (27) shows that as $A \to \infty$, $\xi \to (R+2)^{-1}$, independent of A. Thus, the specific value of A does not influence the self-consistent value of ξ very much, unless higher-order approximations somehow conspire in such a way to require that A be small, which seems unlikely.

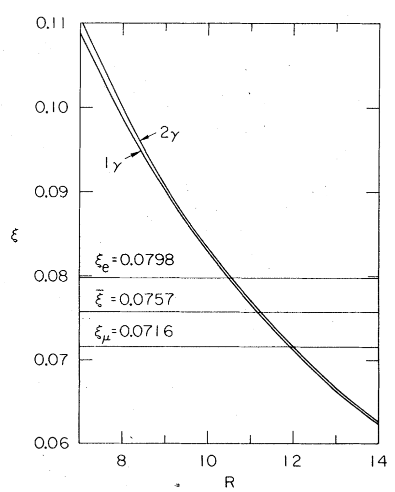

FIG. 9. Plots of the one-photon, Eq. (21), and two-photon, Eq. (27), approximations for mass parameter ξ as a function of the parameter R. The electron mass parameter ξ_e, muon mass parameter ξ_μ, and mean lepton mass parameter $\bar{\xi}$ (assuming that $\Lambda = M_P$) are indicated.

On this point it is relevant to recall that the 1γ perturbation summation used here (Fig. 4) is already in good agreement with the work of Landau and his collaborators.[8] In the latter analysis, which used functional representations for $D_F'(k)_{\mu\nu}$, Γ^μ, and $S_F'(p)$, the singularity was shown to be in the photon propagator; the renormalization effects in Γ^μ and $S_F'(p)$ canceled, as one would expect from the Ward identity. In the perturbation expansion developed here (as well as the one developed by Lautrup), this singularity is also manifest in the photon propagator. However, it does not originate in any one graph, all of which diverge at infinite momentum, but rather in a sum of (a subset of the) graphs comprising $\tilde{D}_{F\mu\nu}'$. An analogous sum of such graphs in the construction of $\tilde{S}_F'$ does not exhibit such a singularity [see Eq. (C12)]. Each term alternates in sign with successive iterations of the proper fermion self-energy contribution on the internal fermion line, and hence the series is Borel summable. (The Borel sum, of course, still has a divergence at infinite momentum.) This implies that an exact expression for $\tilde{S}_F'$ cannot eliminate the presence of the Landau singularity. The Ward identity then implies that a similar conclusion also applies to an exact expression for $\tilde{\Gamma}^\mu$.

If we look to a more complex internal structure of the proper fermion self-energy contribution for relief from the Landau singularity, we should bear in mind that the self-mass integral always remains of a logarithmically divergent form. Additional (internal) photons each add two vertices and two fermion propagators, which because of the Ward identity tend to cancel, leaving the vacuum polarization on the new photon as the major additional contribution. The effect of each additional photon diminishes the contribution of the associated parts of the graph by the additional flow of its momentum (owing to the additional integral) in the denominators of the neighboring propagators. Likewise, the contribution of this additional photon is attenuated by the flow of momentum from other closed loop integrations. As pointed out by Landau,[8] this statement is particularly relevant to nonplanar graphs. The initial divergence, that associated with the first photon [Fig. 4(c)], then appears to remain the controlling quantity in the self-mass expansion. The additional photons, in the formulation of this model, just serve to enable a better calculation of the quantity A from the self-consistency relationship $\delta m/m = 1$; their Landau singularity is already under control by the cutoff assumptions.

Similarly, to the extent that the proper self-energy insertion of the photon is known,[39] one sees no hint that higher-order effects here will make a qualitative modification of the result deriving from the simple one-fermion loop of vacuum polarization; the (known) higher-order components have the $\ln(Q^2/m^2)$ factor to a power lower than that of the associated α. One also notes that the coefficients diminish in magnitude with increasing powers of α, and that they are all of the same sign as the initial $\alpha\ln(Q^2/m^2)$ term, which would only serve to shift the location of the Landau singularity, not to eliminate it. These results on the higher-order structure of the proper photon self-energy graphs also serve to substantiate the above remarks on the higher-order structure of the proper fermion self-energy graphs.

In this section we have made contact between a perturbation expansion and the work of Landau, which sought solutions to the integral equations more directly. In both approaches, one sees that due to vacuum polarization, a divergence of the theory or Landau singularity will occur *before* the upper limit of momentum goes to infinity. And no obvious source of relief from this difficulty is in evidence.

In this regard, it is frequently stated that perturbation theory cannot be trusted because the quantity $(\alpha/\pi)\ln(Q^2/m^2)$ [in this analysis the appropriate quantity is $(R+2)(\alpha/\pi)\ln(Q^2/m^2]$ is getting too large. While it is true that the latter quantity does indeed become large, it is worthwhile to point out that it has been argued that in QED even a non-perturbative solution for the self-mass will diverge.[12] We observe that this non-perturbative divergence is of the same nature as that obtained here by a self-consistent solution using the perturbation expansion (linear in the cutoff momentum).

These results are strong evidence (although admittedly not a proof) that the problem of the Landau singularity is an enduring feature of QED which is not to be solved by going to higher order, employing Borel summations, or even by somehow obtaining an exact, nonperturbative solution. We see no reason to reject this evidence. Consequently, the simple straightforward application of a physical ultraviolet cutoff to solve this problem has been incorporated into this model. Some aspects of this cutoff are covered in the Sec. IV.

IV. DERIVATION OF A SELF-CONSISTENT VALUE FOR Λ

The results of Sec. III were obtained using a hard cutoff on the self-mass integrals. That is, the upper limit to the divergent integrals was just put in by hand at Λ (just below the Landau singularity). This approach implies that there are two significant mass values in the ultraviolet region, the cutoff point and the location of the Landau singularity. Furthermore, they must be close to each other in just the right relationship to achieve the self-consistent value for the self-mass integral.

We shall see below that this criticism, which stems from the use of a hard cutoff, can be averted. To do this, we shall merge the above two mass values into one by the use of a proper Lorentz-invariant cutoff at the Landau singularity. Motivated by the results of Sec. III, this cutoff is assumed to be physical; it is not later to be taken to infinity but remains at the Landau singularity. (The integration still ranges to infinity, however). We note that the form employed below for this cutoff is, of course, to be taken as a phenomonological representation of the physical processes which the model assumes to be operating in the neighborhood of the Landau singularity. The results of this section are (essentially) independent of the details of the form of this cutoff (which are a function of the specific processes involved), other than that it cuts off over a finite range of momentum, which would be a general feature of any physical process. (The problem with the hard cutoff stemmed from its "abrupt" character.)

This step leads to a self-consistency equation for the (effective) number R of hadronic objects which couple to the photon. From this equation one derives the ratio between the lepton mass and the requisite ultraviolet cutoff mass. Within the errors inherent in this analysis, the value of this cutoff is found to be consistent with the Planck mass.

From the analysis of Bjorken and Drell[40] one sees that the lower and upper cutoffs on the (logarithmically divergent) Lorentz-invariant self-mass integral can be cast into the form of the identity

$$\int_0^\infty \frac{d\gamma}{\gamma}\left(e^{ia\gamma}-e^{ib\gamma}\right)=\ln\frac{b}{a}\,, \qquad (28)$$

where γ is an inverse four-momentum squared parameter and a and b are the lower and upper cutoffs, respectively. The lower cutoff, while nominally at the photon mass, is effectively at the fermion mass (as discussed above, the self-mass is not infrared divergent). In Euclidean four-space, an appropriate form of the identity[41] is

$$\int_0^\infty \frac{d\gamma}{\gamma}(e^{-a\gamma}-e^{-b\gamma})=\ln\frac{b}{a}\,; \tag{29}$$

convergence is ensured by the Euclidean relation $K^2=1/\gamma>0$. [K^2 is defined by Eq. (A3).]

Including the vacuum polarization factor, i.e., the denominator,

$$\left[1-\frac{\alpha}{3\pi}(R+2)\ln(K^2/m^2)\right],$$

and setting the lower limit equal to m^2 (the use of the exponential form for the lower limit is not essential) in a logarithmic integral which is equivalent to that employed in the calculation of Eq. (A14) yields

$$A'=\frac{3\alpha}{4\pi}\int_{m^2}^\infty \frac{dK^2}{K^2}\,\frac{1-e^{-\Lambda^2/K^2}}{1-\dfrac{\alpha(R+2)}{3\pi}\ln(K^2/m^2)}\,, \tag{30}$$

where the prime denotes the physical cutoff modifications are employed in this calculation of A. Setting

$$\frac{\alpha(R+2)}{3\pi}\ln\frac{K^2}{m^2}=y \tag{31}$$

obtains

$$A'=\frac{9}{4(R+2)}\int_0^\infty \frac{dy}{1-y}\left[1-\exp\left(\frac{-\Lambda^2}{m^2e^{3\pi y/\alpha(R+2)}}\right)\right]. \tag{32}$$

Now to merge the divergence at $y=1$ (the Landau singularity) and the cutoff at $K^2=\Lambda^2$, one may set

$$\Lambda^2=m^2e^{3\pi/\alpha(R+2)}\,, \tag{33}$$

eliminating Λ^2 from Eq. (32). Thus,

$$A'=\frac{9}{4(R+2)}\int_0^\infty \frac{dy}{1-y}[1-\exp(-e^{3\pi(1-y)/\alpha(R+2)})]\,. \tag{34}$$

Now the value A' of the self-consistency integral is a function of R (and α) alone.

Equation (34) has been numerically integrated and the principal value of $A'(R)$ is plotted in Fig. 10. One observes that these calculations for A' are not particularly sensitive to the value of $\alpha(R+2)$ in the cutoff factor; the major functional dependence of A' upon R derives from the denominator of the fraction in front of the integral.

We note here that whereas the principal value prescription would solve the problem pointed out by Lautrup in connection with "convergent" quantities (it would furnish a definition of the otherwise improper integral), the self-mass integral, which is logarithmically divergent, needs more.

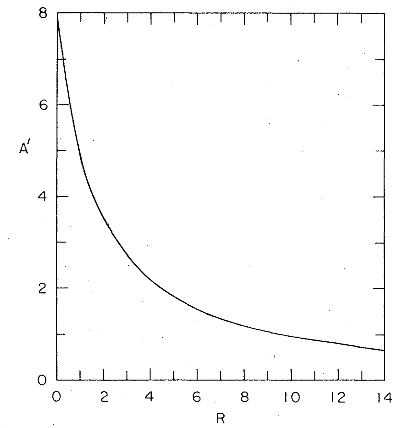

FIG. 10. Plot of A', the value of the one-gamma self-consistency integral, Eq. (34), cutoff by a (phenomenological) Lorentz-invariant cutoff located at the Landau singularity (at $y=1$) as a function of the parameter R.

Without a cutoff the negative integrand above $y=1$ would lead to a divergent, but negative self-mass. One possible approach, that of subtracting away the pole[42] at the Landau singularity, would still leave a logarithmically divergent self-mass. Thus, in this model the use of a physical cutoff at the Landau singularity is necessary to render the self-mass integral finite.

From the prior sections we suppose the appropriate self-consistent value of A' to be in the neighborhood of unity. From Fig. 10 it can be seen that $A'=1$, as given by the one-photon self-consistency relation, calls for $R=9.5$, while $A'=1.31$, the 2γ value, yields $R=7.1$. From Eq. (33), then (using $m=m_e$ and $A'=1$) one deduces the required QED cutoff mass to be at 1.24×10^{21} GeV/c^2, quite close, considering the sensitivity of exponential functions to their arguments, to the Planck mass of 1.22×10^{19} GeV/c^2. It should be noted that Eq. (33) is to some extent arbitrary. While variations in the details of Eq. (33) will not affect the deduced value of R very much because of its exponential form, the deduced value of Λ is affected by the form of Eq. (33). Thus, until one has a better understanding of the physics associated with the Landau singularity, the analysis of this model can only indicate that the Landau singularity is in the neighborhood of M_P (probably on the high side).

Aside from questions of the details of Eq. (33), a discrepancy between M_P and the self-consistent solution for Λ is not a philosophical problem. It should be kept in mind that while these two mass values may be associated (and hence forming a link between QED and gravitation), they need not actually be identical; the QED cutoff mass [which we shall denote as the "Landau mass" (M_L)] could entail factors in addition to the quantity defined as the Planck mass. In fact, in studying the relationship between the self-mass problem in QED and in classical mechanics, it has been suggested that gravitation will furnish a QED cutoff[12] at $\Lambda = 2M_P/\alpha^{1/2}$, which we note is in better agreement with the results of this section than is M_P.

It should be pointed out that in this derivation, the functional form $1/(1-y)$ of the integrand in Eq. (32), which was obtained from perturbation theory and which is valid below $y=1$, is naively assumed (by analytic continuation) to be the appropriate functional form to use throughout the range of integration, including the region beyond $y=1$. While not explicitly noted, this step is tacitly taken by other authors as well.[13,42]

This functional form for the self-mass integrand dictates that for $K^2 > M_L{}^2$, the contribution to the lepton mass by the (renormalized) photon depicted in Fig. 4(a) is negative, which in turn implies that there is a reversal in the sign of the effective coupling constant associated with photons having $K^2 > M_L{}^2$. It thus follows that the assumption concerning the functional form of self-mass integrand entails a novel physical consequence; at distances less than the "Landau length," like charges would attract and unlike charges would repel. Some comments on possible physical aspects of this notion will be made in Sec. V.

V. SUMMARY AND DISCUSSION

This paper has studied a model based upon the idea that all of the (charged) leptonic mass is due to QED self-interactions. A perturbation expansion for the electron QED self-mass is developed, and it is shown that this expansion gives results in agreement with the functional analysis of Landau and his collaborators. To obtain this agreement, the key feature which must be contained in the perturbation expansion is the one-loop vacuum polarization graph summed to all orders.

The fact that inclusion of higher-order proper vacuum polarization graphs, vertex insertions, or fermion propagator corrections does not appear to be essential or to significantly alter the simple one-photon results is taken as an argument in favor of the premise that—as originally suggested by Landau—there exists a divergence of the photon propagator (the Landau singularity) before the upper limit of photon momentum is taken to infinity. It is also noted that Lautrup has recently shown that this divergence at the Landau singularity is manifest in the anomalous magnetic moment as a non-Borel summable set of graphs. Lautrup's result is significant because it shows that the question of the Landau singularity is found also in "convergent" QED quantities as well as the divergent self-mass.

It is relevant to observe here that the perturbation sums obtained in the appendices offer a possible resolution to the conflict between the apparent Borel summability[43] of the general high-order QED term in the perturbation series (the general term evidently alternates in sign with increasing order) and the non-Borel-summability of the graphs of concatenations of vacuum polarization loops. It is simply that the Borel-summable components of the QED series [e.g., Eqs. (B8) and (C10)] have larger "recursion coefficients" than does the non-Borel-summable vacuum polarization component. Thus, with increasing order, the Borel-summable terms will exceed the non-Borel-summable terms by an ever increasing amount. This being the case, it is not surprising if the approximation methods now used to derive the general term in the QED series would miss this (smaller) non-Borel-summable component residing in the general higher-order perturbation term. Of course, a non-Borel-summable component, even if small, must still be reckoned with.

In this model the resultant singularity (in the photon propagator) is controlled by employing a physical cutoff at the Landau singularity and the assumption that, by analytic continuation, the mathematical form of the self-mass integrand above this singularity is the same as that below.[44] A principal-value prescription is then employed to evaluate the self-mass integral. These assumptions will not affect the perturbation theory calculation of "convergent" quantities. If the other divergences of the perturbation series are Borel summable, as may well be the case, then indeed, this model yields a convergent self-consistent formulation for the QED self-mass.

This model also leads to a self-consistency prediction with no free parameters that the Landau singularity is in the neighborhood of the Planck mass. From this, one may infer, as suggested by Landau, that gravitation plays some role in furnishing a cutoff for QED.

There are additional physical interpretations which one may associate with the assumptions used in this model. For example, one supposes that the assumption of a physical cutoff implies that the pointlike electronic charge would be actually characterized by the Landau length, an idea also explored by Landau,[8] while the analytic continuation of the self-mass integrand implies a reversal of electromagnetic coupling at the Landau singularity. This reversal of the electromagnetic coupling might be ascribed to the finite size of the pointlike electronic charge as follows. We first note that the vacuum polarization component of the renormalized electromagnetic coupling exceeds that of the Coulomb part when $y > \frac{1}{2}$, and completely dominates it as one approaches the Landau singularity. Keeping in mind a finite sized electronic charge, one imagines that the vacuum polarization cloud would form "around" a pair of (test) charges (i.e., the interaction points of the photon propagator) which are closer than the Landau length, rather than "between" them as one normally expects for larger separations. Since this change or "reversal" in the geometrical distribution of the vacuum polarization cloud relative to two interaction points will take place on a scale characterized by the Landau length, one is furnished with a physical rationale for the reversal in the electromagnetic coupling.

If the ideas explored in this model are relevant to the physical description of leptons, then it is appropriate to point out that some other approaches cannot be employed to investigate the question of leptonic self-mass—at least in their present form. In particular, the use of the renormalization group for this purpose is precluded because a basic assumption of that approach, that the leptonic masses may be neglected in the functional form of the propagators in the ultraviolet region,[45] is not fulfilled. In the perturbation theory result, the location of the Landau singularity is a direct function of the (mean) leptonic mass. Similarly, analyses of this question using the spectral representation are cast into doubt. If the leptonic charge has a structure characterized by the Landau length, then the notion of the light cone on this scale is undefined until that structure is better understood. As a consequence, a crucial step in the derivation of the spectral weight function[46] for this region is on uncertain ground.

The perturbation theory results obtained with this model, therefore, cannot be obtained (or refuted) by derivations based upon the renormalization group or the spectral function. A selection among them must be made on the basis of external criteria. While one may favor one or another of these approaches based upon one's view of "reasonable" physical assumptions, ultimately the decision must be made by experimental test.

Unfortunately, the prospects of direct tests of elementary particle theory at the Planck mass are very remote; we shall have to be content with other less direct predictions of the models or theories in question.

We have seen that vacuum polarization, including hadronic contributions, plays a crucial role in this model. The amount of vacuum polarization is described by a phenomenological parameter R which sets the scale of the mean lepton mass relative to cutoff mass. R is the (effective) number of pointlike objects which electromagnetically couple to the photon, and is essentially the same R as characterizes the e^+e^- hadronic cross section. R is determined by a self-consistency equation with no free parameters. A calculation using a one-photon approximation for the fermion self-mass indicates that R is on the order of ten while the two-photon approximation yields an R of about seven.

The error in these numbers due to variations in the form of the assumed physical cutoff should be no more than a few percent. This is because there are ~20 orders of magnitude between the leptonic mass scale and M_L, while on physical grounds one expects the cutoff to become effective in one order of magnitude of momentum range or less. However, since one is uncertain of the ultimate value of A' which a full perturbation sum will prescribe (higher-order estimates of the value of A' are well defined in this model, but tedious to calculate), one cannot yet estimate the error resulting from the truncation of the perturbation series. But the fact that the value of A' changes by only 30% in going from the 1γ to the 2γ solution is cause for optimism.

The experimental value of R is on the order of five at 10 GeV in the center of mass[34] and will soon be extended to higher energies by PEP and PETRA. Thus, at the present time, the self-consistency evaluation of R by this model does not conflict with experiment. In fact, it predicts the existence of (a limited number of) additional pointlike objects beyond those now indicated by the experimentally determined e^+e^- hadronic cross sections.

ACKNOWLEDGMENTS

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APPENDIX A: LEADING CONTRIBUTIONS TO THE VACUUM POLARIZATION GRAPHS

In this appendix estimates of the vacuum polarization contributions (fermion loops) to the one-photon fermion self-mass graph are made. These estimates are relatively simple once one exploits the simplifying assumptions: (1) Only the leading terms, i.e., those proportional to

$$\left(\alpha \ln \frac{\Lambda^2}{m^2}\right)^n$$

are required for order $2n$, and (2) the infrared problem may be ignored. (The rationale for the latter is discussed in Sec. I.) The approach is simply to effect a Wick rotation[47] on the energy variable of integration, converting the Minkowski space to a Euclidean space where, using symmetry, the integrals are simple to estimate.

To show the details of this approach we start with the second-order self-mass integral [obtained from Eq. (12)]

$$\begin{aligned}\delta m^{(2)} &= e^2 \int \frac{d^4k}{i(2\pi)^4} \frac{1}{k^2+i\epsilon} \gamma_\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2 + i\epsilon'} \gamma^\mu \\ &= e^2 \int \frac{d^4k}{i(2\pi)^4} \frac{1}{k^2+i\epsilon} \frac{-2\not{p} + 2\not{k} + 4m}{(p-k)^2 - m^2 + i\epsilon'}, \end{aligned} \tag{A1}$$

where the first two of the identities[48]

$$\begin{aligned} &\gamma_\mu m \gamma^\mu = 4m , \\ &\gamma_\mu \not{a} \gamma^\mu = -2\not{a} , \\ &\gamma_\mu \not{a}\not{b} \gamma^\mu = 4a\cdot b , \\ &\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a} , \end{aligned} \tag{A2}$$

have been employed to eliminate the γ matrices. In accordance with Eq. (8) the value of $\delta m^{(2)}$ will be determined when $\not{p}=m$.

In order to understand the effects of the Wick rotation, one must know the disposition of the poles of the integrand in the complex k_0 plane. For convenience we shall locate them in the rest frame in which $p_0=m$. One now sees that the poles of the integrand are distributed in the complex k_0 plane as shown in Fig. 11(a). That is, in the right half of the plane the poles are below the real axis, while in the left half they are above. The k_0 integrals over the range $-\infty \le k_0 \le \infty$ may be evaluated by means of a contour integration. For large k_0, the k_0 integrand goes like dk_0/k_0^n, where $n=3$ or 4, which means that the hemispherical contour (closing either above or below the real axis) can be added at infinity making no contribution to the integral. A contour closing below the real axis is also shown in Fig. 11(a). One now sees that the contour of this integral may be rotated counterclockwise 90° as shown in Fig. 11(b) without changing the value of the integral (no poles are crossed). Thus, one effects the Wick rotation and converts Eq. (A1) to a Euclidean integral over d^4K with the substitutions

$$k_0 = iK_4, \quad k_j = K_j, \text{ and } k^2 = -K^2, \tag{A3}$$

where $j=1,2,3$. It is now easy to evaluate these integrals in (the symmetric) Euclidean space; the contour integration and the $i\epsilon$ in the denominators are no longer required.

Now using $\not{p}=m$ and $p^2=m^2$, the integrals proportional to (the numerical factors) $-2\not{p}$ and $4m$ may be combined to yield

$$\frac{2\alpha m}{4\pi}\int_{m^2}^{\Lambda^2} \frac{dX}{X} = \frac{2\alpha m}{4\pi} \ln \frac{\Lambda^2}{m^2}, \tag{A4}$$

where $X=K^2$ and $d^4K = 2\pi^2 K^3 dK = \pi^2 K^2 dK^2$ have been employed.

Note that since we are ignoring the infrared problem, the lower limit has simply been set equal to m^2. Stopping the integrations at a specific value of Λ^2 is called in the text the use of a "hard" cutoff. (The upper limit is discussed in more detail in Sec. IV.) These approximations will lead to no errors to the accuracy to which we are working. Further, one notes that the term $2p\cdot k$ in the denominator of the fermion propagator has been dropped because by symmetry it averages out (to the accuracy of this estimation) to zero.

To evaluate the remaining integral, the one proportional to $2\not{k}$, one symmetrizes the denominator by the substitution $k=k'+p/2$. This yields a new denominator $(k'+p/2)^2(k'-p/2)^2$, which $\to k'^4$ within the accuracy to which we are working. The effect of this substitution is felt in the numerator in which $2\not{k} = 2(\not{k}'+\not{p}/2)$. One now drops the integral proportional to $\not{k}'$ as averaging to zero, and we have the effective substitution $2\not{k} \to \not{p} = m$ in the numerator. Thus, the integral deriving from the $2\not{k}$ term is

$$\frac{\alpha m}{4\pi}\int_{m^2}^{\Lambda^2} \frac{dX}{X} \tag{A5}$$

and the final result for the three integrals is

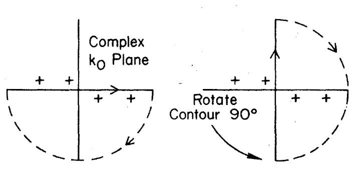

FIG. 11. (a) Disposition of the poles in the complex k_0 plane due to the zeros in the denominator of the integrand in Eq. (A1). The integral along real axis over $-\infty \le k_0 \le \infty$ may be calculated using a contour integral as shown. (b) The contour of (a) may be rotated counterclockwise by 90°, converting the Minkowski space for the integral in Eq. (A1) to a four-dimensional Euclidean space in which the integrals are easy to evaluate; e.g., Eqs. (A4), (A5), and (A9).

$$\delta m^{(2)} = \frac{3\alpha m}{4\pi} \ln \frac{\Lambda^2}{m^2}, \tag{A6}$$

in agreement with Eq. (13).

In order to determine the contribution of the single loop of vacuum polarization, one can simply make the appropriate changes in the photon propagator[49]

$$\frac{-i}{k^2} \to \frac{-i}{k^2}\left(1 + \frac{\alpha}{3\pi} \ln \frac{-k^2}{m^2}\right) \tag{A7}$$

in Minkowski space, or equivalently

$$\frac{1}{X} \to \frac{1}{X}\left(1 + \frac{\alpha}{3\pi} \ln \frac{X}{m^2}\right) \tag{A8}$$

in Euclidean space. Using Eq. (A8) to modify the integrals in Eqs. (A4) and (A5) yields

$$\frac{3\alpha m}{4\pi} \int \frac{dX}{X}\left(1 + \frac{\alpha}{3\pi} \ln \frac{X}{m^2}\right) = \frac{3\alpha m}{4\pi} \ln \frac{\Lambda^2}{m^2} + \frac{m}{8}\left(\frac{\alpha}{\pi} \ln \frac{\Lambda^2}{m^2}\right)^2. \tag{A9}$$

This expression evidently is appropriate for the sum of the first two graphs shown on the right-hand side of the equation in Fig. 4(a). The second term clearly is due to the single-fermion-loop graph and is in agreement with the leading term of a more detailed calculation.[18] Using Eq. (16), the one-fermion-loop contribution may be written as

$$9m\xi^2/8 \tag{A10}$$

for each fermion.

It is now a simple matter to extend this estimation, summed to all orders, using the substitution[50]

$$\frac{-i}{k^2} \to \frac{-i}{k^2}\left[\frac{1}{1 - \frac{\alpha}{3\pi} \ln \frac{-k^2}{m^2} - \frac{\alpha^2}{4\pi^2} \ln \frac{-k^2}{m^2} + \cdots}\right]. \tag{A11}$$

One now notes that the α^2 term in the denominator (associated with the vertex and fermion self-mass insertions in a single-fermion loop) has a logarithm to a power less than the power of α and hence will not contribute to the leading terms. Thus, the leading terms to all orders are evidently associated with concatenations of single loops rather than more complicated structure in the single loop itself. The graphical summation is shown in Fig. 4(a). Again, using the Euclidean space equivalent of Eq. (A11) the integration is easy, yielding[51]

$$\frac{9m}{4} \ln\left[\frac{1}{1 - \frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{m^2}}\right]. \tag{A12}$$

It is straightforward to include in this estimate the loops due to muons and other pointlike particles. We shall assume that there are (an effective number) R other pointlike particles, in addition to the two leptons. Taking all masses (the two leptons and others) to be equal, Eq. (A12) generalizes to

$$\frac{9m}{4(R+2)} \ln\left[\frac{1}{1 - (R+2)\frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{m^2}}\right]. \tag{A13}$$

For the purposes of subsequent analysis, it is useful to define the dimensionless quantity

$$A \equiv \frac{9}{4(R+2)} \ln\left[\frac{1}{1 - (R+2)\frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{m^2}}\right]. \tag{A14}$$

APPENDIX B: THE LEADING CONTRIBUTION TO THE COUNTERTERM GRAPHS

After the single-photon graph, the next simplest graph is the (fourth-order) counterterm graph (second graph, Fig. 2). The contribution of this graph is

$$-i\tilde{\Sigma}^{(4)}_{\mathrm{CT}}(\not p) = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} \times \frac{(-ie)^2 i^2 N}{[(p-k)^2 - m^2 + i\epsilon]^2} i\delta m^{(2)}, \tag{B1}$$

where

$$N = \gamma_\mu(\not p - \not k + m)(\not p - \not k + m)\gamma^\mu. \tag{B2}$$

In the single-photon graph, we have seen that a linear factor $\not k$ in the numerator (the denominator is even) upon integration gives a contribution proportional to m. Hence only the $\not k\not k = k^2$ term in Eq. (B2) need be kept; the k^2 term goes like Λ^2 upon integration and hence will dominate any term proportional to m. Thus,

$$N \cong \gamma_\mu k^2 \gamma^\mu = 4k^2, \tag{B3}$$

where $\cong$ is used to mean "equivalent to." Now, since the distribution of singularities for the integrand of Eq. (B1) is the same as it was for that in Eq. (A1), a Wick rotation in the complex k_0 plane may again be performed [see Eqs. (A3)]. As with the second-order integral, only the K^2 factors in the denominator need be kept, dropping the p and m factors. With these simplifications, the leading counterterm contribution becomes

$$\tilde{\Sigma}^{(4)}_{\mathrm{CT}} = -\frac{\alpha}{\pi} \int_{m^2}^{\Lambda^2} \frac{dX}{X} \delta m^{(2)} = -\frac{\alpha}{\pi} \ln \frac{\Lambda^2}{m^2} \delta m^{(2)}, \tag{B4}$$

where $X = K^2$. As with the second-order integral the upper limit was set equal to Λ^2 and the lower to m^2. (We have argued that the infrared contribution will cancel in the total of all of the graphs and hence may safely be ignored in the individual graphs.) Using Eq. (13) for the leading term of $\delta m^{(2)}$ yields

$$\tilde{\Sigma}_{\rm CT}^{(4)} = -\frac{3m}{4}\left(\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{m^2}\right)^2 = -\frac{27m}{4}\xi^2, \qquad \text{(B5)}$$

which is in agreement with Frank.[18] ξ is defined by Eq. (16).

It is now a simple matter to include the effect of vacuum polarization to all orders to obtain the appropriate contribution to $\delta m^{(2\gamma)}$. The first step is to note in Eq. (B4) that one makes the substitution

$$\delta m^{(2)} \to \delta m^{(1\gamma)}, \qquad \text{(B6)}$$

where the equations yielding $\delta m^{(1\gamma)}$, the one-photon self-mass contribution, are depicted in Fig. 4. Then by making use of the similarity in the form of the integrand in Eq. (B4) and that in Eqs. (A4) and (A5), one can make the substitution

$$\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{m^2} = \frac{4}{3m}\delta m^{(2)} \to \tfrac{4}{3}A, \qquad \text{(B7)}$$

where A is defined by Eq. (A14). Combining the substitutions in Eqs. (B6) and (B7), one has the result

$$\tilde{\Sigma}_{\rm CT}^{(2\gamma)} = -\tfrac{4}{3}A\,\delta m^{(1\gamma)}. \qquad \text{(B8)}$$

The result, given in Eq. (B8), is now in a form amenable to the factoring out of δm factor as is employed in Fig. 8.

APPENDIX C: THE LEADING CONTRIBUTIONS TO THE RAINBOW GRAPHS

The contribution of the fourth-order rainbow graph (fourth graph, Fig. 2) is

$$\tilde{\Sigma}_{\rm RG}^{(4)}(\not p) = e^2\int\frac{d^4k}{i(2\pi)^4}\frac{1}{k^2}\frac{\gamma_\mu(\not p-\not k+m)}{(p-k)^2-m^2}\left\{\,\right\}\gamma^\mu, \qquad \text{(C1)}$$

where

$$\left\{\,\right\} \equiv \tilde{\Sigma}_{\rm FP}^{(2)}(\not p-\not k)\frac{(\not p-\not k+m)}{(p-k)^2-m^2} \qquad \text{(C2)}$$

is a dimensionless factor,

$$\tilde{\Sigma}_{\rm FP}^{(2)} \equiv \tilde{\Sigma}^{(2)} + \delta m^{(2)}, \qquad \text{(C3)}$$

and the function $\tilde{\Sigma}^{(2)}$ is defined by Eq. (12). As indicated by Eq. (C3), $\tilde{\Sigma}_{\rm FP}^{(2)}$ is just the second-order fermion propagator graph without the $\delta m^{(2)}$ term, which is accounted for separately. [Recall that $\tilde{\Sigma}^{(2)}$ includes the graph for $\delta m^{(2)}$. Therefore, $\tilde{\Sigma}_{\rm FP}^{(2)}$ as defined by Eq. (C3) does not include the $\delta m^{(2)}$ graph.] The ϵ and λ factors have been omitted, as the resolution of the k_0 contour of integration and the photon mass is the same as in Appendices A and B. The integration of this graph[52] depends upon knowing the function $\tilde{\Sigma}_{\rm FP}^{(2)}$ off the mass shell, which is[53]

$$\tilde{\Sigma}_{\rm FP}^{(2)}(\not p-\not k) = \frac{\alpha}{4\pi}[4m-(\not p-\not k)]\ln\frac{\Lambda^2}{-k^2}, \qquad \text{(C4)}$$

where the region $-k^2 \gg m^2$ is assumed. Using Eq. (C4) in Eq. (C2) yields

$$\left\{\,\right\} \cong -\frac{\alpha}{4\pi}\left[1+\frac{3m\not k}{(p-k)^2-m^2}\right]\ln\frac{\Lambda^2}{-k^2}, \qquad \text{(C5)}$$

where terms proportional to p^2, pm, or m^2 in the numerator of the fraction in the square brackets have been dropped as insignificant relative to $m\not k$. (That $\not k$ must combine with the $\not k$ in the numerator of the first fermion propagator to be significant.) $\tilde{\Sigma}_{\rm RG}^{(4)}$ may now be evaluated in the same way as $\tilde{\Sigma}^{(2)}$ in Appendix A.

The first term in Eq. (C5) leads to the integral

$$-\left(\frac{\alpha}{4\pi}\right)^2 3m\int_{m^2}^{\Lambda^2}\frac{dX}{X}\ln\frac{\Lambda^2}{X} = -\frac{3m}{32}\left(\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{m^2}\right)^2. \qquad \text{(C6)}$$

The second term leads to

$$\left(\frac{\alpha}{4\pi}\right)^2 12m\int_{m^2}^{\Lambda^2}\frac{dX}{X}\ln\frac{\Lambda^2}{X} = \frac{12m}{32}\left(\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{m^2}\right)^2. \qquad \text{(C7)}$$

Combining Eqs. (C6) and (C7) gives the result

$$\tilde{\Sigma}_{\rm RG}^{(4)}(m) = \frac{9m}{32}\left(\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{m^2}\right)^2 = \frac{81m}{32}\xi^2, \qquad \text{(C8)}$$

where Eq. (16) has been used to introduce the parameter ξ.

It is of some interest to extend this result to include the more general diagram shown in Fig. 12 with n photons on the fermion propagator. To do this one simply makes the substitution $\{\,\}\to\{\,\}^n$ in Eq. (C1). This step leads to

$$\left\{\,\right\}^n \cong -\frac{\alpha}{4\pi}\left(1+\frac{3mn\not k}{(p-k)^2-m^2}\right)\ln^n\frac{\Lambda^2}{-k^2} \qquad \text{(C9)}$$

in place of Eq. (C5); terms with more than one factor of m have been dropped (see Sec. III). Equation (C9) in Eq. (C1) yields

$$\tilde{\Sigma}_{\rm RG}^{(2+2n)}(m) = (-1)^{n+1}\left(\frac{12n-3}{n+1}\right)\times m\left(\frac{\alpha}{4\pi}\ln\frac{\Lambda^2}{m^2}\right)^{n+1}, \qquad \text{(C10)}$$

which for $n=0$ can be seen to equal $\tilde{\Sigma}_{\rm FP}^{(2)}(m)$. The $(-1)^{n+1}$ factor will ensure convergence of the sum over n of these graphs provided

$$Y \equiv \frac{\alpha}{4\pi}\ln\frac{\Lambda^2}{m^2} = \tfrac{3}{4}\xi < 1. \qquad \text{(C11)}$$

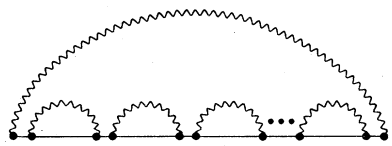

FIG. 12. Fermion self-mass graph containing a sequence of n internal one-photon subgraphs on the fermion propagator.

Equation (C11) shows that the "recursion coefficient" of the series associated with the diagrams of Fig. 12 is less than those in Fig. 4. Thus, in some sense a summation of these graphs is less divergent than the photon propagator summation; the criterion for summation of the former remains valid to momenta above the Landau singularity. And further, as remarked in the text, the Borel summation technique may be applied (the terms of the series alternate in sign) relaxing the requirement imposed by Eq. (C11). It is straightforward to sum over $0 \leq n \leq \infty$ the graphs represented by Eq. (C10). This sum is simply[54]

$$-\frac{12Y}{1+Y}+15\ln(1+Y)\,. \tag{C12}$$

Because of the alternating sign with each iteration, there is no analog to the Landau singularity in Eq. (C12).

To enable an estimation of $\delta m^{(2\gamma)}$ and hence obtain the two-γ self-consistency equation, it is also of interest to obtain the appropriate expression for $\tilde{\Sigma}_{\rm RG}^{(2\gamma)}$, which would be associated with the third graph in Fig. 5. As in Appendices A and B, this is effected by using the substitution

$$\frac{-i}{k^2}\to\frac{-i}{k^2}\left(1-\frac{\alpha}{3\pi}\sum_{j=1}^{R+2}\ln\frac{-k^2}{m_j{}^2}\right)^{-1}, \tag{C13}$$

which for equal-mass fermions is

$$\frac{-i}{k^2}\left[1-\frac{\alpha(R+2)}{3\pi}\ln\frac{-k^2}{m^2}\right]^{-1}$$

for the photon propagators. This step has the effect of modifying the logarithm factors in a way analogous to Appendices A and B.

Looking first at $\tilde{\Sigma}_{\rm FP}^{(2)}(\not p-\not k)$, we see that the integral from which Eq. (C4) is derived[55] may be approximated in the region $\Lambda^2>-k^2>-p^2=m^2$ by

$$\tilde{\Sigma}_{\rm FP}^{(2)}(\not p-\not k)=\frac{\alpha}{4\pi}[4m-(\not p-\not k)]\int_a^b\frac{dX}{X}\,, \tag{C14}$$

where $a=m^2/\Lambda^2$ and $b=m^2/-k^2$. For the equal-mass case, the substitution of Eq. (C13) into Eq. (C14) leads to

$$\tilde{\Sigma}_{\rm FP}^{(1\gamma)}(\not p-\not k)=\frac{\alpha}{4\pi}[4m-(\not p-\not k)]\times\int_a^b\frac{dX}{X}\,\frac{1}{1-(R+2)\frac{\alpha}{3\pi}\ln X}\,. \tag{C15}$$

Setting $y=\ln X$ and $dy=dX/X$ yields

$$\tilde{\Sigma}_{\rm FP}^{(1\gamma)}(\not p-\not k)=\frac{\alpha}{4\pi}[4m-(\not p-\not k)]\times\int_{a'}^{b'}\frac{dy}{1-(R+2)\frac{\alpha}{3\pi}y}\,, \tag{C16}$$

where $a'=\ln a$ and $b'=\ln b$. Equation (C16) may be integrated[56] to yield

$$\tilde{\Sigma}_{\rm FP}^{(1\gamma)}(\not p-\not k)=\frac{3}{4(R+2)}[4m-(\not p-\not k)]L\,, \tag{C17}$$

where

$$L\equiv\left[\ln\frac{1}{1-\frac{\alpha(R+2)}{3\pi}\ln\frac{\Lambda^2}{m^2}}-\ln\frac{1}{1-\frac{\alpha(R+2)}{3\pi}\ln\frac{-k^2}{m^2}}\right]. \tag{C18}$$

One verifies that as $\alpha\to 0$, one retrieves Eq. (C4) from Eqs. (C17) and (C18).

Using Eqs. (C17) and (C18) in place of Eq. (C4), i.e., making the substitution

$$\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{-k^2}\to\frac{3}{R+2}L\,, \tag{C19}$$

converts Eq. (C5) to

$$\Big\{\Big\}\cong-\frac{3}{4(R+2)}\left[1+\frac{3m\not k}{(p-k)^2-m^2}\right]L\,. \tag{C20}$$

In place of Eqs. (C6) and (C7) we must now evaluate

$$\tilde{\Sigma}_{\rm RB}^{(2\gamma)}(\not p-\not k)=\frac{27m\alpha}{16\pi(R+2)}\int_{m^2}^{\Lambda^2}\frac{dX}{X}\left[\frac{1}{1-\frac{\alpha(R+2)}{3\pi}\ln\frac{X}{m^2}}\right]\left[\ln\frac{1}{1-\frac{\alpha(R+2)}{3\pi}\ln\frac{\Lambda^2}{m^2}}-\ln\frac{1}{1-\frac{\alpha(R+2)}{3\pi}\ln\frac{X}{m^2}}\right]. \tag{C21}$$

The first term integrates as before yielding

$$\frac{81m}{16(R+2)^2}\left[\ln\frac{1}{1-\frac{\alpha(R+2)}{3\pi}\ln\frac{\Lambda^2}{m^2}}\right]^2. \tag{C22}$$

Using the substitution

$$Z = 1 - \frac{\alpha(R+2)}{3\pi} \ln \frac{X}{m^2}$$

enables an elementary integration of the second term:

$$-\frac{81m}{32(R+2)^2}\left[\ln \frac{1}{1 - \frac{\alpha(R+2)}{3\pi}\ln\frac{\Lambda^2}{m^2}}\right]^2 . \tag{C23}$$

The final result is

$$\tilde{\Sigma}_{\rm RB}^{(2\gamma)}(m) = \frac{81m}{32(R+2)^2}\left[\ln \frac{1}{1 - \frac{\alpha(R+2)}{3\pi}\ln\frac{\Lambda^2}{m^2}}\right]^2 , \tag{C24}$$

which as $\alpha \to 0$ retrieves Eq. (C8).

One may write this result as

$$\tilde{\Sigma}_{\rm RB}^{(2\gamma)}(m) = \tfrac{1}{2} m A^2 , \tag{C25}$$

where A is defined by Eq. (A14).

Although we do not do it here, this result may be extended in a straightforward way to describe the graph analogous to that in Fig. 12 but with all photons dressed. As with the "bare" photons, the terms in the summation of these graphs alternate in sign and the (Borel) summation converges. Again, there is no Landau singularity associated with this sum other than the original singularity associated with the photons themselves.

APPENDIX D: LEADING CONTRIBUTIONS TO THE VERTEX INSERTION GRAPHS

The contribution of the fourth-order vertex insertion graph (fifth graph, Fig. 2) is

$$-i\tilde{\Sigma}_{\rm VI}^{(4)}(\not p) = (-ie)^4 \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2}\int \frac{d^4q}{(2\pi)^4}\frac{-i}{q^2}\chi , \tag{D1}$$

where

$$\chi = \gamma_\mu \frac{i}{(\not p - \not k - m)}\gamma_\nu \frac{i}{(\not p - \not k - \not q - m)}\gamma^\mu \frac{i}{(\not p - \not q - m)}\gamma^\nu . \tag{D2}$$

As in Appendix C, the photon mass and the $i\epsilon$ factors in the propagators have been omitted. Equations (D1) and (D2) simplify at once to

$$\tilde{\Sigma}_{\rm VI}^{(4)} = e^4 \int \frac{d^4k}{i(2\pi)^4 k^2}\int \frac{d^4q}{i(2\pi)^4 q^2}\frac{N}{[(p-k)^2-m^2][(p-k-q)^2-m^2][(p-q)^2-m^2]} , \tag{D3}$$

where

$$N = \gamma_\mu(\not p - \not k + m)\gamma_\nu(\not p - \not k - q + m)\gamma^\mu(\not p - q + m)\gamma^\nu . \tag{D4}$$

While the point of view of this paper is that infrared divergences may safely be ignored, it is relatively easy to see by power counting that Eq. (D3) has no infrared divergences, even in the term going like m^3 in the numerator. First consider q large. Then for small k the integral over d^4k becomes (for p on the mass shell) proportional to $\int d^4k(k^2 p\cdot k)^{-1}$ which converges for small k. By symmetry the same result for large k but small q obtains. Now if we imagine that both k and q get small together, we will have an integral proportional to

$$\int d^4k\, d^4q [k^2 p\cdot k p\cdot(k+q) p\cdot q\, q^2]^{-1} .$$

This form still has one more power of the momentum variables in the numerator than in the denominator, and therefore also converges in the infrared region, as was found by Frank.[18]

We now return to the region of interest, the ultraviolet. The relation[18]

$$\gamma_\mu(\not A + a)\gamma_\nu(\not B + b)\gamma^\mu(\not C + c)\gamma^\nu = -8A\cdot C\not B + 4a\not C\not B + 4b\not C\not A + 4c\not B\not A + 4ab\not C + 4ac\not B + 4bc\not A - 8abc , \tag{D5}$$

is useful. We shall employ

$$a = b = c = m \quad\text{and}\quad A = p - k ,\quad B = p - k - q ,\quad C = p - q . \tag{D6}$$

The last four terms of Eq. (D5) (proportional to m^2 and m^3) may be neglected, yielding

$$N \cong -8A\cdot C\not B + 4m\not C\not B + 4m\not C\not A + 4m\not B\not A . \tag{D7}$$

Equations (D6) may be substituted into Eq. (D7) and multiplied out to give a sum of simple polynomials. Of these polynomials, those containing a p^2, mp, mqk, or pqk may also be neglected. This leaves

$$N \to -8[p\cdot q\not q + p\cdot k\not k - k\cdot q(\not k + \not q)] + 4m(q^2 + k^2) , \tag{D8}$$

which can be seen to be symmetric in k and q. Employing $\not p \to m$ and anticipating the use of the

Wick rotation into a Euclidean four-space, and noting that $\langle Q_4{}^2\rangle = \frac{1}{4}\langle Q^2\rangle$ by symmetry in Euclidean four-space (this result, while not intuitively obvious, also obtains in Minkowski space) reduces Eq. (D8) to

$$N \cong 8k\cdot q(\not{k}+\not{q}) + 2m(q^2+k^2)\,. \qquad \text{(D9)}$$

The integral associated with the first term may be evaluated by the change of variables $k=k' + p/2$, $q=q'+p/2$ which eliminates the linear terms such as $p\cdot q$ from the denominator. We then may drop all terms odd in q' and k' after this change of variable and obtain (dropping the primes)

$$N \cong +4p\cdot(k\not{k}+q\not{q}) + 2m(q^2+k^2)\,, \qquad \text{(D10)}$$

which in the rest frame is equivalent to $+3m(q^2+k^2)$. Thus the integral of Eq. (D3) may now be written simply as

$$\tilde{\Sigma}^{(4)}_{\text{VI}}(m) = e^4\int \frac{d^4k\,d^4q\,3m(q^2+k^2)}{i(2\pi)^4 i(2\pi)^4 k^4(k+q)^2q^4}\,. \qquad \text{(D11)}$$

As with the rainbow graph we know that we can perform a Wick rotation on both $d^4k = id^4K$ and $d^4q = id^4Q$ and perform a symmetric Euclidean integration. Since we have seen that the graph associated with $\tilde{\Sigma}^{(4)}_{\text{VI}}$ has no infrared divergence, it is legitimate to employ a lower cutoff at the fermion mass m for the Euclidean integration. To the accuracy of this analysis, the term $(k+q)^2$ in the denominator may be replaced simply by $-(K^2+Q^2)$, the effect of the product $2K\cdot Q$ averaging to zero.[57] The symmetric Euclidean integral then reduces to

$$\begin{aligned}\tilde{\Sigma}^{(4)}_{\text{VI}}(m) &= \frac{3m}{4}\left(\frac{\alpha}{\pi}\right)^2\int_m^\Lambda \frac{dK}{K}\int_m^\Lambda\frac{dQ}{Q}\\ &= \frac{3m}{4}\left(\frac{\alpha}{\pi}\ln\frac{\Lambda}{m}\right)^2 = \frac{3m}{16}\left(\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{m^2}\right)^2\\ &= \frac{27m}{16}\xi^2\,,\end{aligned} \qquad \text{(D12)}$$

where ξ is defined by Eq. (16).

While this leading term disagrees with that of Frank,[18] this analysis has the advantage of being simple and straightforward; his analysis, while intrinsically more accurate, was long and arduous and consequently more subject to mathematical error. One obtains further confidence in the above result by noting that if one discards the infrared divergent part of the vertex insertion calculated by Bjorken and Drell,[58] replacing γ^μ in the second-order graph $\tilde{\Sigma}^{(2)}$ by its one-photon approximation

$$\Lambda^\mu = \frac{\alpha}{4\pi}\gamma^\mu \ln\frac{\Lambda^2}{m^2}\,, \qquad \text{(D13)}$$

one obtains for the $\ln^2$ contribution

$$= \frac{3\alpha m}{4\pi}\ln\frac{\Lambda^2}{m^2}\left(\frac{\alpha}{4\pi}\ln\frac{\Lambda^2}{m^2}\right) \qquad \text{(D14)}$$

$$= \frac{3m}{16}\left(\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{m^2}\right)^2, \qquad \text{(D15)}$$

the same result as Eq. (D12).

At this point we note that the photon propagator modification, Eq. (A11), which applied prior to the integration converts

$$\tilde{\Sigma}^{(2)}(m) \to \tilde{\Sigma}^{(1\gamma)}(m)\,,$$

$$\tilde{\Sigma}^{(4)}_{\text{CT}}(m) \to \tilde{\Sigma}^{(2\gamma)}_{\text{CT}}(m)\,,$$

and

$$\tilde{\Sigma}^{(4)}_{\text{RB}}(m) \to \tilde{\Sigma}^{(2\gamma)}_{\text{RB}}(m)\,,$$

may also be effected by the substitution

$$\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{m^2} \to \frac{3}{(R+2)}\ln\frac{1}{1-\dfrac{\alpha(R+2)}{3\pi}\ln\dfrac{\Lambda^2}{m^2}} \qquad \text{(D16)}$$

after the integration. Contact between these two forms was seen to exist by letting $\alpha\to 0$, which eliminates the higher-order effects of the vacuum polarization. Using this same rule obtains

$$\tilde{\Sigma}^{(2\gamma)}_{\text{VI}}(m) = \frac{3m}{16}\left[\frac{3}{(R+2)}\ln\frac{1}{1-\dfrac{\alpha(R+2)}{3\pi}\ln\dfrac{\Lambda^2}{m^2}}\right]^2. \qquad \text{(D17)}$$

Equation (D17) may be written simply as

$$\tilde{\Sigma}^{(2\gamma)}_{\text{VI}}(m) \to \tfrac{1}{3}mA^2\,, \qquad \text{(D18)}$$

where Eq. (A14) defines A.

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[36]Ref. 25, Footnote 2, p. 312.

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[39]The α^2 contribution [Appendix A, Eq. (A11)] is given in Ref. 25, p. 357, while the α^3 part has been obtained from renormalization-group theory, Ref. 25, Eq. (19.161), p. 372.

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[45]Reference 15, p. 370. As pointed out by the authors, the argument used to support this assumption applies to the perturbation series on an order-by-order basis. They caution that such an argument would not apply to the sum of a perturbation series. And we note that the Landau singularity, in fact, arises from such a sum of terms. It is interesting to note that the derivation using the renormalization group of the α^3 term in the photon propagator (see Ref. 39) actually does not depend upon physics near the Landau singularity, but rather upon the relationship between expansion coefficients at lower energy before the scaling assumption becomes questionable. It is well known that the extrapolation of this functional form for the propagator to the Landau singularity contradicts the scaling assumption of the renormalization group.

[46]Reference 25, Eq. (16.31), p. 139.

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[48]Reference 30, p. 105.

[49]Reference 30, p. 159.

[50]Reference 25, p. 357.

[51]Reference 41, formula 90.1 using the substitution $X = 1-(\alpha/3\pi)\ln(K^2/m^2)$. It should be remarked that the Landau singularity does not appear in the individual perturbation terms, but only in the sum. Therefore, the Wick rotation to Euclidean four-space should be carried out term by term before summation.

[52]This method of evaluation $\tilde{\tilde{\Sigma}}^{(4)}_{RB}(m)$ differs from that of Frank (Ref. 18), who multiplied out the γ matrices in the numerator term $N \equiv \gamma_\mu(\not{p}-\not{k}+m)\gamma_\nu(\not{p}-\not{k}-\not{q}+m)\gamma^\nu \times(\not{p}-\not{k}+m)\gamma^\mu$, and then dealt with each (polynomial) term separately, as was done for $\tilde{\tilde{\Sigma}}^{(2)}$ in Appendix A. In examining this latter approach, after some algebra, one finds that N reduces to $(8m^3+8m^2\not{q}+8mk^2-4m\not{q}\not{k}-4m\not{k}\not{q}+4\not{k}\not{p}\not{k}-4k^2\not{k}-4\not{k}\not{q}\not{k})$, which agrees with Frank except for the last term, which he lists as $4\,\not{q}\not{k}\not{q}$, clearly in error since the original expressions for N have no quadratic term in $\not{q}$. This error evidently leads to the disagreement in the leading term of Frank's result and the one found in Appendix C.

[53]Cf., Ref. 30, Sec. 8.4.

[54]Reference 41, formulas 9.04 and 601.1.

[55]Reference 30, p. 162.

[56]Reference 41, formula 90.1.

[57]This assertion is easily verified by expanding the factor $(K+Q)^{-2}$ in terms of Gegenbauer polynomials, P. M. Morse and H. Feshbach, *Methods in Theoretical Physics, Part I* (McGraw-Hill, New York, 1953), p. 782. Such an expansion also enables an evaluation of the next leading term, going like $(\alpha/\pi)\xi$, which in this analysis is being neglected.

[58]Reference 30, Sec. 8.6.