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Parameter counting for multimonopole solutions

Erick J. Weinberg
Department of Physics, Columbia University, New York, New York 10027
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SU(2) magnetic monopole solutions are considered in the Prasad-Sommerfield limit. It is shown that any solution with n units of magnetic charge belongs to a $(4n-1)$-parameter family of solutions. It is conjectured that these parameters correspond to the positions and relative U(1) orientations of n noninteracting unit monopoles.

I. INTRODUCTION

It has been known for some time that certain spontaneously broken gauge theories possess classical solutions which may be interpreted as magnetic monopoles.[1,2] However, only a rather limited number of such solutions have been found; in the SU(2) theory which we consider in this paper, only the spherically symmetric 't Hooft-Polyakov monopole and its dyon generalization[3] are known. It is not known whether the theory has any other classical solutions. In this paper we attempt to gain some insight into this question by determining some of the properties which such solutions would possess.

Throughout we restrict ourselves to the Prasad-Sommerfield[4] limit of vanishing scalar field potential. Bogomol'nyi[5] and Coleman, Parke, Neveu, and Sommerfield[6] have shown that for this case the energy of a static configuration with electric charge Q and magnetic charge g is bounded from below by

$$E \geqslant v(Q^2+g^2)^{1/2}\,, \tag{1.1}$$

where v is the vacuum expectation value of the Higgs field. This bound is achieved by fields which satisfy a set of first-order differential equations; such fields, being minima of the energy, will also be solutions of the Euler-Lagrange equations of the theory. Equation (1.1) implies that a solution with $Q=0$ and n units of magnetic charge will have exactly n times the energy of a unit monopole. This suggests that such solutions correspond to n noninteracting unit monopoles. Since there is a long-range magnetic force between monopoles, this would at first sight seem unlikely. However, in the Prasad-Sommerfield limit the scalar field becomes massless and can mediate a long-range force; Manton[7] has shown that at least to order $1/r^2$ this cancels the magnetic repulsion. It is tempting to conjecture that this cancellation is exact.

One way to test this conjecture is to count the parameters entering an arbitrary solution with n units of magnetic charge. If the conjecture is true, these should include the $3n$ needed to specify the positions of n independent monopoles. In this paper we perform this parameter count by studying infinitesimal perturbations about an arbitrary solution and find that there are $4n-1$ physical parameters. We conjecture that $3n$ of these are position parameters while the remaining $n-1$ specify relative orientations in the space corresponding to the unbroken U(1) subgroup.

The remainder of this paper is organized as follows: In Sec. II we review some properties of the theory and of the Prasad-Sommerfield limit. Section III contains our calculation of the number of parameters. Our results are an extension of an index theorem due to Callias,[8] whose methods we follow. Section IV contains a discussion of our results. There are three appendices. In the first we discuss some questions raised by the comparison between our methods and those used in some previous index calculations. The second appendix contains some results concerning the continuum spectra of certain operators which are needed in Sec. III. The third appendix contains a discussion of the normalizability of rotational zero modes.

II. REVIEW OF THE THEORY

We consider an SU(2) gauge theory containing an isotriplet of Higgs scalars, with Lagrangian density[9]

$$\mathcal{L} = -\tfrac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} + \tfrac{1}{2}D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} - \lambda(\vec{\phi}^2-v^2)^2\,, \tag{2.1}$$

where

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + \vec{A}_\mu\times\vec{A}_\nu \tag{2.2}$$

and

$$D_\mu\vec{\phi} = \partial_\mu\vec{\phi} + \vec{A}_\mu\times\vec{\phi}\,. \tag{2.3}$$

Vector notation refers to SU(2) indices, and the scale of the gauge field has been chosen so that the gauge coupling constant is unity. Two isospin components of the gauge field acquire a mass via the Higgs mechanism, while the third, corresponding to the unbroken U(1) subgroup, remains massless. Electric and magnetic charges coupled to this massless field may be defined by

$$Q = \int dS_i \hat{\phi} \cdot \vec{E}_i \,, \qquad g = \int dS_i \hat{\phi} \cdot \vec{B}_i \,, \tag{2.4}$$

where

$$\vec{E}_i = \vec{F}_{0i} \,, \qquad \vec{B}_i = \tfrac{1}{2}\epsilon_{ijk}\vec{F}_{jk} \,, \qquad \hat{\phi} = \frac{\vec{\phi}}{|\vec{\phi}|} \,, \tag{2.5}$$

and the integration is over a surface at spatial infinity. While Q may take on any value, g must be of the form $4\pi n$ for some integer n. In fact, the magnetic charge is a topological invariant—it is unchanged by continuous variations of the fields which preserve the boundary conditions imposed by the requirement of finite energy.

The Prasad-Sommerfield limit is obtained by setting $\lambda = 0$ in the Lagrangian (2.1), but retaining the boundary condition that $|\phi|$ approach v at spatial infinity. In this limit the energy of a static configuration of fields may be written as[5,6]

$$\begin{aligned} E &= \tfrac{1}{2}\int d^3x\,[\vec{E}_i^{\,2} + \vec{B}_i^{\,2} + (D_i\vec{\phi})^2 + (D_0\vec{\phi})^2] \\ &= \tfrac{1}{2}\int d^3x\,[(\vec{E}_i - \sin\alpha\, D_i\vec{\phi})^2 \\ &\qquad + (\vec{B}_i - \cos\alpha\, D_i\vec{\phi})^2 + (D_0\vec{\phi})^2] \\ &\quad + vg\cos\alpha + vQ\sin\alpha \geq v(g^2+Q^2)^{1/2} \,, \end{aligned} \tag{2.6}$$

where α is an arbitrary angle. In obtaining the second equality we have used the Bianchi identity

$$0 = D_i \vec{B}_i \,, \tag{2.7}$$

and have assumed that the fields are constrained by Gauss's law

$$0 = D_i\vec{E} - D_0\vec{\phi}\times\vec{\phi} \,. \tag{2.8}$$

For fixed Q and g the lower bound on the energy will be achieved if

$$\vec{B}_i = \cos\alpha\, D_i\vec{\phi} \,, \qquad \vec{E}_i = \sin\alpha\, D_i\vec{\phi} \,, \qquad D_0\vec{\phi} = 0 \tag{2.9}$$

with

$$\alpha = \tan^{-1}(Q/g) \,.$$

Fields satisfying these conditions will be solutions of the equations of motion. There may of course be other solutions of the second-order field equations, which do not satisfy these first-order equations, but these will be of higher energy. Such solutions will not be considered in this paper.

Finally, we point out a very simple relationship between the dyon solutions, with both electric and magnetic charge, and those with only magnetic charge. Given any solution with $Q = 0$, the substitution

$$\tilde{\phi}(\tilde{x}) = \phi(x) \,, \qquad \tilde{A}_i(\tilde{x}) = \cos\alpha\, A_i(x) \,, \qquad \tilde{A}_0(\tilde{x}) = \sin\alpha\,\phi(x) \,, \qquad \tilde{x} = \frac{1}{\cos\alpha}x \tag{2.10}$$

gives a solution with the same magnetic charge but with electric charge

$$Q = g\tan\alpha \,. \tag{2.11}$$

Furthermore, it is clear that all solutions with nonzero Q can be obtained in this manner. Thus, to determine the number of parameters entering an arbitrary solution, it is sufficient to consider solutions with $Q = 0$, satisfying

$$\vec{B}_i = D_i\vec{\phi} \,. \tag{2.12}$$

III. COUNTING ZERO MODES

We now wish to determine the number of physical zero modes about an arbitrary Prasad-Sommerfield solution with magnetic charge $4\pi n$, i.e., the number of infinitesimal perturbations which leave the energy unchanged. In view of the remarks at the end of Sec. II, we need only consider solutions with $Q = 0$. Expanding Eq. (2.12) about such a solution and keeping terms linear in the perturbation, we obtain

$$0 = D_i\delta\phi - \phi\delta A_i - \epsilon_{ijk}D_j\delta A_k \,, \tag{3.1}$$

where

$$D_i = \partial_i + A_i \,. \tag{3.2}$$

Here we have adopted a notation in which $\delta\phi$ and δA_i are vectors, while ϕ and A_i are 3×3 matrices

$$\phi = \vec{\phi}\cdot\vec{T} \,, \qquad A_i = \vec{A}_i\cdot\vec{T} \,, \tag{3.3}$$

where the T^a are anti-Hermitian and satisfy

$$[T^a, T^b] = \epsilon^{abc}T^c \,, \qquad \mathrm{tr}\,T^aT^b = -2\delta^{ab} \,, \qquad (T^a)_{ij} = \epsilon_{iaj} \,. \tag{3.4}$$

We will later make use of the fact that for any unit vector $\hat{n}$,

$$(\hat{n}\cdot\vec{T})^3 = -(\hat{n}\cdot\vec{T}) . \qquad (3.5)$$

Among the solutions to Eq. (3.1) will be many which simply correspond to gauge transformations and are of no physical interest. These are most conveniently excluded by imposing the background gauge condition

$$0 = D_i\delta A_i + \phi\delta\phi . \qquad (3.6)$$

Our problem then is to count the number of linearly independent solutions of Eqs. (3.1) and (3.6).

The algebraic manipulations are considerably simplified if these equations are replaced by an equivalent Dirac equation.[10,11] If

$$\psi = I\delta\phi + i\sigma_j\delta A_j , \qquad (3.7)$$

then Eqs. (3.1) and (3.6) are equivalent to

$$\begin{aligned} 0 &= \mathfrak{D}\psi \\ &= (-i\sigma_j D_j + \phi)\psi . \end{aligned} \qquad (3.8)$$

Note, however, that the δA_i and ϕ corresponding to a given ψ are linearly independent of those corresponding to $i\psi$; the desired number of zero modes is thus twice the number of normalizable zero eigenmodes of $\mathfrak{D}$.

Callias[8] has derived an index theorem for Dirac operators of this type. However, his results are derived only for the case where ϕ has no zero eigenvalues at spatial infinity (or, equivalently, for the case where there is a nonzero mass term). In the present case, this condition does not hold, but as we will show, it is still possible to derive an expression for the number of normalizable zero modes in terms of the magnetic charge. Except for the complications introduced by the absence of a mass term, we follow the methods of Callias.

We begin by defining[12]

$$\mathcal{I} = \lim_{M^2\to 0} \mathcal{I}(M^2) , \qquad (3.9)$$

where

$$\mathcal{I}(M^2) = \mathrm{Tr}\left(\frac{M^2}{\mathfrak{D}^*\mathfrak{D}+M^2}\right) - \mathrm{Tr}\left(\frac{M^2}{\mathfrak{D}\mathfrak{D}^*+M^2}\right) . \qquad (3.10)$$

Here

$$\mathfrak{D}^* = -i\sigma_j D_j - \phi \qquad (3.11)$$

is the adjoint of $\mathfrak{D}$. Making use of Eq. (2.12), we find that

$$\begin{aligned} \mathfrak{D}^*\mathfrak{D} &= -(D_j)^2 - \phi^2 - 2i\sigma_j(D_j\phi) , \\ \mathfrak{D}\mathfrak{D}^* &= -(D_j)^2 - \phi^2 . \end{aligned} \qquad (3.12)$$

Clearly $\mathcal{I}$ counts the zero eigenvalues of these two operators. The normalizable zero eigenfunctions of $\mathfrak{D}^*\mathfrak{D}$ (which are the same as those of $\mathfrak{D}$) each contribute 1. Similarly, there would be a contribution of -1 from each normalizable zero mode of $\mathfrak{D}\mathfrak{D}^*$, but it is evident from Eq. (3.12) that $\mathfrak{D}\mathfrak{D}^*$ is positive and has no such modes. (Note that ϕ is anti-Hermitian, so the eigenvalues of ϕ^2 are negative semidefinite.) Finally, since the continuum portions of the spectra extend down to zero, we must consider the possibility of a contribution from this source of the form

$$\mathcal{I}^{\text{cont}} = \lim_{M^2\to 0} M^2 \int_{|k|<\epsilon} \frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2+M^2}\left[\rho_{\mathfrak{D}^*\mathfrak{D}}(k^2) - \rho_{\mathfrak{D}\mathfrak{D}^*}(k^2)\right] . \qquad (3.13)$$

Here $\rho_O(k^2)$ is the density of continuum eigenvalues of O and ϵ is an arbitrary positive number. Clearly $\mathcal{I}^{\text{cont}}$ will vanish unless the $\rho_O(k^2)$ are quite singular at $k^2=0$. In Appendix B we argue that in fact these behave like the density of states for a nonrelativistic particle in an exponentially decreasing potential, and are thus nonsingular, so $\mathcal{I}^{\text{cont}}=0$. Thus, taking into account the factor of 2 introduced by converting to a spinor problem, we conclude that there are $2\mathcal{I}$ normalizable zero modes about the given monopole solution.

We now wish to express $\mathcal{I}$ in terms of the magnetic charge. We begin by defining a set of Euclidean Dirac matrices

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad (3.14)$$

obeying

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} . \qquad (3.15)$$

It is also convenient to define a fictitious gauge field

$$W_i = A_i , \quad W_4 = \phi \qquad (3.16)$$

with a corresponding field strength

$$G_{ij} = F_{ij} , \quad G_{i4} = D_i\phi . \qquad (3.17)$$

Note that Eq. (2.12) is equivalent to requiring $G_{\mu\nu}$ to be self-dual. Although W_μ has four components, it must be remembered that we are considering a three-dimensional problem, so the fourth component of the covariant derivative is simply

$$D_4 = W_4 = \phi . \qquad (3.18)$$

In this notation,

$$\gamma\cdot D\equiv\gamma_\mu D_\mu=\begin{pmatrix}0 & \mathfrak{D}\\ -\mathfrak{D}^* & 0\end{pmatrix} \tag{3.19}$$

and

$$\begin{aligned}\mathfrak{g}(M^2)&=-\operatorname{Tr}\gamma_5\frac{M^2}{-(\gamma\cdot D)^2+M^2}\\ &=-\int d^3x\,\operatorname{tr}\left\langle x\left|\gamma_5\frac{M^2}{-(\gamma\cdot D)^2+M^2}\right|x\right\rangle ,\end{aligned} \tag{3.20}$$

where tr indicates a trace over only SU(2) and spinor indices.

We now define a nonlocal current

$$J_i(x,y)=\operatorname{tr}\left\langle x\left|\gamma_5\gamma_i\frac{1}{\gamma\cdot D+M}\right|y\right\rangle . \tag{3.21}$$

A straightforward calculation using the cyclic property of the trace and the identities

$$\begin{aligned}\delta(x-y)&=\left[\gamma_i\frac{\partial}{\partial x_i}+\gamma_\mu W_\mu(x)+M\right]\left\langle x\left|\frac{1}{\gamma\cdot D+M}\right|y\right\rangle\\ &=\left\langle x\left|\frac{1}{\gamma\cdot D+M}\right|y\right\rangle\left[-\frac{\overleftarrow{\partial}}{\partial y_i}\gamma_i+\gamma_\mu W_\mu(y)+M\right]\end{aligned} \tag{3.22}$$

yields

$$\begin{aligned}\left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_i}\right)J_i(x,y)&=-2M\operatorname{tr}\left\langle x\left|\gamma_5\frac{1}{\gamma\cdot D+M}\right|y\right\rangle\\ &\quad-\operatorname{tr}\gamma_5\gamma_\mu[W_\mu(x)-W_\mu(y)]\\ &\quad\times\left\langle x\left|\frac{1}{\gamma\cdot D+M}\right|y\right\rangle .\end{aligned} \tag{3.23}$$

We now let y approach x; in three dimensions the γ_5's are sufficient to render all the Green's functions finite in this limit. In particular, the second term on the right-hand side vanishes, rather than giving an anomaly as it would in four dimensions. Thus

$$\begin{aligned}\partial_i J_i(x,x)&=-2M\operatorname{tr}\left\langle x\left|\gamma_5\frac{1}{\gamma\cdot D+M}\right|x\right\rangle\\ &=-2\operatorname{tr}\left\langle x\left|\gamma_5\frac{M^2}{-(\gamma\cdot D)^2+M^2}\right|x\right\rangle .\end{aligned} \tag{3.24}$$

It then follows from Eq. (3.20) that

$$\begin{aligned}\mathfrak{g}(M^2)&=\tfrac{1}{2}\int d^3x\,\partial_i J_i(x,x)\\ &=\tfrac{1}{2}\lim_{R\to\infty}\int_R dS_i\,J_i(x,x),\end{aligned} \tag{3.25}$$

where the surface of integration is a sphere of radius R.

To evaluate this expression we note that $J_i(x,x)$ may be rewritten as

$$J_i(x,x)=-\operatorname{tr}\left\langle x\left|\gamma_5\gamma_i(\gamma\cdot D)\frac{1}{-(\gamma\cdot D)^2+M^2}\right|x\right\rangle , \tag{3.26}$$

and that

$$-(\gamma\cdot D)^2+M^2=-D_j{}^2-\phi^2+M^2-\tfrac{1}{4}[\gamma_\mu,\gamma_\nu]G_{\mu\nu}. \tag{3.27}$$

We now insert into Eq. (3.26) the expansion, valid for any $M^2\neq 0$,

$$\frac{1}{-(\gamma\cdot D)^2+M^2}=\frac{1}{-D_j{}^2-\phi^2+M^2}+\frac{1}{-D_j{}^2-\phi^2+M^2}\left(\tfrac{1}{4}[\gamma_\mu,\gamma_\nu]G_{\mu\nu}\right)\frac{1}{-D_j{}^2-\phi^2+M^2}+\cdots . \tag{3.28}$$

Since $G_{\mu\nu}$ falls like $1/x^2$, only the first two terms in this expansion can give nonzero contributions when $R\to\infty$ in Eq. (3.25). Taking the trace over the spinor indices, we find that the first term in J_i vanishes, while the second leads to

$$\hat{x}_i J_i(x,x)=-2\hat{x}_i\epsilon_{i\lambda\mu\nu}\operatorname{tr}\left\langle x\left|D_\lambda\frac{1}{-D_j{}^2-\phi^2+M^2}G_{\mu\nu}\frac{1}{-D_j{}^2-\phi^2+M^2}\right|x\right\rangle+O\left(\frac{1}{|x|^3}\right), \tag{3.29}$$

where tr now indicates a trace only over SU(2) indices. The leading asymptotic behavior of $G_{\mu\nu}$ is determined by the magnetic charge [see Eq. (B7)] to be

$$\begin{aligned}G_{ij}(x)&=\epsilon_{ijk}\frac{n\hat{x}_k}{x^2}\hat{\phi}\cdot\vec{T}+O\left(\frac{1}{|x|^3}\right),\\ G_{i4}(x)&=\frac{n\hat{x}_i}{x^2}\hat{\phi}\cdot\vec{T}+O\left(\frac{1}{|x|^3}\right).\end{aligned} \tag{3.30}$$

If we substitute these expressions into Eq. (3.29), we see that the leading terms are those with $\lambda=4$; asymptotically, $D_4=\phi=v\hat{\phi}\cdot\vec{T}$. Next, we substitute for the propagators the asymptotic form

$$\frac{1}{-D_j{}^2-\phi^2+M^2}=(i\hat{\phi}\cdot\vec{T})\frac{1}{-\nabla^2+v^2+M^2}(i\hat{\phi}\cdot\vec{T})+[I-(i\hat{\phi}\cdot\vec{T})^2]\frac{1}{-\nabla^2+M^2}[I-(i\hat{\phi}\cdot\vec{T})^2]+O\left(\frac{1}{|x|}\right). \tag{3.31}$$

Because of the identity (3.5) and the fact that asymptotically $\vec{G}_{\mu\nu}$ is parallel to $\hat{\phi}$, only the first term in the propagator contributes to Eq. (3.29). We obtain

$$\hat{x}_i J_i(x,) = -\frac{4nv}{x^2}\operatorname{tr}(\hat{\phi}\cdot\vec{T})^6\left\langle x\left|\left(\frac{1}{-\nabla^2+v^2+M^2}\right)^2\right|x\right\rangle + O\left(\frac{1}{|x|^3}\right)$$
$$= \frac{8vn}{x^2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+v^2+m^2)^2}+O\left(\frac{1}{|x^3|}\right)$$
$$= \frac{n}{x^2\pi}\frac{v}{(v^2+M^2)^{1/2}}+O\left(\frac{1}{|x^3|}\right). \quad (3.32)$$

The surface integral in Eq. (3.25) is now trivial, leading to

$$\mathscr{I}(M^2) = \frac{2nv}{(v^2+M^2)^{1/2}}. \quad (3.33)$$

We now take the limit $M^2=0$ and find the number of normalizable zero modes to be

$$2\mathscr{I} = 2\lim_{M^2\to 0}\mathscr{I}(M^2) = 4n. \quad (3.34)$$

For the case $n=1$, this result is verified by the work of Mottola[11] and Adler,[13] who have obtained explicit expressions for the zero modes about the Prasad-Sommerfield monopole. They find four normalizable modes, of which three correspond to translations of the original solution. The fourth is a gauge mode which is not eliminated by the background gauge condition (3.6); it corresponds to an infinitesimal transformation generated by the electric charge. Such a mode will occur about any solution, and will be given by

$$\vec{\Lambda} = \vec{\phi},$$
$$\delta\vec{A}_i = D_i\vec{\Lambda} = D_i\vec{\phi}, \quad (3.35)$$
$$\delta\vec{\phi} = \vec{\phi}\times\vec{\Lambda} = 0.$$

Since an overall charge rotation is of no physical significance, we subtract this mode from our total, and conclude that there are $4n-1$ physical normalizable zero modes.

IV. DISCUSSION

We have considered infinitesimal perturbations about an arbitrary Prasad-Sommerfield solution with magnetic charge $4\pi n$ and have shown that there are $4n-1$ nongauge zero modes. Although the existence of a zero mode only requires that the energy be stationary to first order in the perturbation, the positivity of $\mathfrak{D}\mathfrak{D}^*$ [see Eq. (3.12)] allows us to extend this result to all orders. If

$$A_i = A_i^{(0)} + \sum_{n=1}^{\infty}\lambda^n\delta A_i^{(n)},$$
$$\phi = \phi^{(0)} + \sum_{n=1}^{\infty}\lambda^n\delta\phi^{(n)}, \quad (4.1)$$

and

$$\psi^{(n)} = I\delta\phi^{(n)} + i\sigma_j\delta A_j^{(n)}, \quad (4.2)$$

where $A_i^{(0)}$ and $\phi^{(0)}$ satisfy Eq. (2.12), then the conditions that A_i and ϕ satisfy this equation to all orders in λ may be written in the form

$$D\psi^{(1)} = 0,$$
$$D\psi^{(n)} = f_n(\psi^{(1)},\psi^{(2)},\ldots,\psi^{(n-1)}) \quad (n>1). \quad (4.3)$$

Thus once a zero mode $\psi^{(1)}$ is given, all higher $\psi^{(n)}$ can be constructed successively according to

$$\psi^{(n)} = \mathfrak{D}^*(\mathfrak{D}\mathfrak{D}^*)^{-1}f_n(\psi^{(1)},\psi^{(2)},\ldots,\psi^{(n-1)}). \quad (4.4)$$

The positivity of $\mathfrak{D}\mathfrak{D}^*$ ensures the existence of its inverse. We therefore conclude that every solution with magnetic charge $4\pi n$ belongs to a $(4n-1)$-parameter family of solutions.

This number has a simple interpretation for solutions corresponding to n noninteracting monopoles. Each monopole, viewed in isolation, would be specified by four parameters—three to specify its position and one to specify its orientation with respect to the unbroken U(1) gauge group. Because the background gauge condition allows an overall charge rotation, the orientation parameters have no absolute significance; only the $n-1$ relative orientations are physically significant. (The situation is analogous to that of the group orientation parameters which appear in multi-instanton solutions.)

Although our calculations have been done in terms of $Q=0$ solutions, the results apply equally well to electrically charged solutions, with a similar interpretation of the parameters for a multidyon solution. It should be noted that the electric charges of the individual dyons in such a solution would not be independent parameters; Eq. (2.10) requires that each dyon have the same ratio of electric to magnetic charge. Indeed, only if this is the case can the repulsive electric and magnetic forces between each pair of dyons be canceled by the attractive scalar force.

The number of zero modes about a multidyon solution may also be understood by considering the configurations obtained by letting the corresponding collective coordinates vary with time. These configurations may be characterized by the momenta and electric charges of the n dyons. Imposing the constraint that the total electric charge be Q leaves $4n-1$ independent variables.

We should note that our results have a further implication for the case $n=1$. Adler[14] has suggested that there may be $n=1$ solutions other than the Prasad-Sommerfield solution. These new solutions would not be spherically symmetric, so there would be rotational zero modes in addition to the three translational modes. Since we have shown that there are only three normalizable nongauge zero modes about an $n=1$ solution, the conjectured solutions can only exist if they are sufficiently singular such that the rotational modes are non-normalizable when put into the background gauge. We show in Appendix C that continuity of the field strengths is sufficient to guarantee that these modes are normalizable in the background gauge.

Added note. Adler has pointed out that, since the proof of the positivity of $\mathfrak{D}\mathfrak{D}^*$ requires an integration by parts, nonspherically symmetric $n=1$ solutions might exist if the rotational modes do not belong to the class of functions for which the surface term vanishes. This can occur if the derivatives of the field strength are discontinuous on a surface or if they diverge sufficiently rapidly at a line or point singularity.

Finally, we should stress that while we have shown that any solution with magnetic charge $4\pi n$ must belong to a $(4n-1)$-parameter family, we have not demonstrated that any such solution exists for $n>1$. Whether there are in fact any multimonopole solutions remain an unsolved problem.[15]

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APPENDIX A

Those familiar with the use of methods similar to those of this paper to count parameters in multi-instanton[10,16] and multivortex solutions will recall that in those applications it is shown that $\mathscr{I}(M^2)$ is independent of M^2. In this appendix we show why these arguments fail in the case at hand.

We begin by recalling the argument of Brown, Carlitz, and Lee.[10] If

$$\gamma_5=\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix} \tag{A1}$$

and

$$K=\begin{pmatrix} 0 & \mathfrak{D}^* \\ \mathfrak{D} & 0\end{pmatrix}, \tag{A2}$$

then

$$\mathscr{I}(M^2)=\mathrm{Tr}\gamma_5\frac{M^2}{K^2+M^2} \tag{A3}$$

and

$$\begin{aligned}\frac{d\mathscr{I}(M^2)}{dM^2}&=\mathrm{Tr}\gamma_5\frac{1}{K^2+M^2}K^2\frac{1}{K^2+M^2}\\ &=\mathrm{Tr}\gamma_5 K\frac{1}{K^2+M^2}K\frac{1}{K^2+M^2}\\ &=\mathrm{Tr}K\frac{1}{K^2+M^2}\gamma_5 K\frac{1}{K^2+M^2}\\ &=-\mathrm{Tr}\gamma_5 K\frac{1}{K^2+M^2}K\frac{1}{K^2+M^2}\ .\end{aligned} \tag{A4}$$

In the third equality we have assumed the cyclic property of the trace; we must examine matters more closely to see whether this assumption is justified. In terms of $\mathfrak{D}$ and $\mathfrak{D}^*$ we have

$$\begin{aligned}\frac{d\mathscr{I}(M^2)}{dM^2}&=\mathrm{Tr}\gamma_5 K\frac{1}{K^2+M^2}K\frac{1}{K^2+M^2}\\ &=\int d^3x\,d^3y\,\mathrm{tr}\left[-\left\langle x\left|\mathfrak{D}\frac{1}{\mathfrak{D}^*\mathfrak{D}+M^2}\right|y\right\rangle\left\langle y\left|\mathfrak{D}^*\frac{1}{\mathfrak{D}\mathfrak{D}^*+M^2}\right|x\right\rangle+\left\langle x\left|\mathfrak{D}^*\frac{1}{\mathfrak{D}\mathfrak{D}^*+M^2}\right|y\right\rangle\left\langle y\left|\mathfrak{D}\frac{1}{\mathfrak{D}^*\mathfrak{D}+M^2}\right|x\right\rangle\right],\end{aligned} \tag{A5}$$

where tr indicates a trace over internal indices only. If the integral is sufficiently convergent, we may interchange the order of integration in the second term and conclude that $d\mathscr{I}(M^2)/dM^2$ vanishes. Because of the nonzero mass in the Green's functions, the integrand falls exponentially as $|x-y|$ increases; we must investigate the behavior as x and y approach infinity with $x-y$ fixed.

From Eqs. (3.8), (3.11), and (3.12) we have

$$\begin{aligned}&\mathfrak{D}=-i\sigma_j D_j+\phi\ ,\\ &\mathfrak{D}^*=-i\sigma_j D_j-\phi\ ,\\ &\mathfrak{D}\mathfrak{D}^*+M^2\equiv\Delta=-D_j{}^2-\phi^2\ ,\\ &\mathfrak{D}^*\mathfrak{D}=\Delta-2i\sigma_j D_j\phi\ .\end{aligned} \tag{A6}$$

Since $D_j\phi$ falls like $1/r^2$, terms of second or higher order in an expansion of $(\mathfrak{D}^*\mathfrak{D}+M^2)^{-1}$ about Δ^{-1} will be convergent. Furthermore, the zeroth-order term in such an expansion vanishes, as the two terms in the integrand are identical once the trace over internal indices is taken. There remains only the term linear in $D_j\phi$; after taking the trace over spinor indices, we obtain

$$\frac{d\mathcal{I}(M^2)}{dM^2} = -4\int d^3x\, d^3y\, \mathrm{tr}[\epsilon_{ijk}\langle x|D_i\Delta^{-1}D_j\phi\Delta^{-1}|y\rangle\langle y|\phi\Delta^{-1}|x\rangle + \langle x|D_i\Delta^{-1}D_i\phi\Delta^{-1}|y\rangle\langle y|\phi\Delta^{-1}|x\rangle - \langle x|\phi\Delta^{-1}D_i\phi\Delta^{-1}|y\rangle\langle y|D_i\Delta^{-1}|x\rangle - (x \leftrightarrow y)]\,. \tag{A7}$$

Since each term in the integrand falls only like $1/x^2$ for $x,y\to\infty$, $(x-y)$ fixed, we cannot interchange the order of integration in the second term and cannot conclude that $\mathcal{I}$ is independent of M^2.

Let us now compare the analogous argument for the instanton case. The equations are similar to those given above, except that space is now four dimensional and one must make the replacements

$$\phi \to D_4\,,\quad D_j\phi \to F_{4j}\,.$$

Equation (A7) is then replaced by

$$\frac{d\mathcal{I}(M^2)}{dM^2} = -4\int d^4x\, d^4y\, \mathrm{tr}[\bar\eta^j_{\mu\nu}\langle x|D_\mu\Delta^{-1}F_{4j}\Delta^{-1}|y\rangle\langle y|D_\nu\Delta^{-1}|x\rangle - (x \leftrightarrow y)]\,. \tag{A8}$$

Here $\bar\eta^j_{\mu\nu}$ is the usual anti-self-dual symbol. $F_{\mu\nu}$ falls like $1/x^4$, so the two halves of the integral might appear to be logarithmically divergent. However, the antisymmetry of $\bar\eta^j_{\mu\nu}$ causes the $1/x^4$ term to vanish, so the integrand falls at least as fast as $1/x^5$ and is convergent.

An alternative argument[16,17] that $\mathcal{I}(M^2)$ is independent of M^2 begins by noting that if ψ is an eigenfunction of $\mathfrak{D}^*\mathfrak{D}$ with nonzero eigenvalue, then $\mathfrak{D}\psi$ is an eigenfunction of $\mathfrak{D}\mathfrak{D}^*$ with the same eigenvalue. Because of this correspondence between the nonzero eigenvalues of $\mathfrak{D}^*\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}^*$, the only contribution to $\mathcal{I}(M^2)$ is from the zero eigenvalues, and is manifestly independent of M^2. On a compact space the eigenvalues are discrete and there is no difficulty with this argument; in an open space the spectrum becomes continuous and one might worry about making a point-by-point correspondence between the spectra. The most natural way of proceeding would be to put the system in a box with periodic boundary conditions and then to take the size of the box to infinity. Since the functions appearing in $\mathfrak{D}^*\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}^*$ are not periodic, this can be done only if these operators are modified slightly near the edge of the box. A simple perturbation theory argument shows that this procedure will be valid only if the physically meaningful fields fall sufficiently rapidly at large distances; in particular, it fails for the monopole case with its $1/r^2$ magnetic field. In contrast, the long-range fields in the instanton and vortex problems are pure gauge transformations, and this method may be applied.

APPENDIX B

In Sec. III it was necessary to show that the continuum portions of the spectra of $\mathfrak{D}^*\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}^*$ did not contribute to $\mathcal{I}$. This result followed from the assertion that the density of continuum eigenvalues was nonsingular at zero. In this appendix we justify this assertion.

We must first establish some results concerning the asymptotic behavior of solutions of Eq. (2.12). We begin by considering

$$D_i\hat\phi = \partial_i\hat\phi + \vec{A}_i\times\hat\phi\,. \tag{B1}$$

This is most easily studied in a gauge in which $\hat\phi$ is constant.[18] In such a gauge, the components of $\vec{A}_i$ orthogonal to $\hat\phi$ correspond to a particle of mass v and therefore vanish asymptotically as e^{-vr}. Since the first term in Eq. (B1) is identically zero in this gauge, we obtain the gauge-invariant result

$$D_i\hat\phi = O(e^{-vr})\,. \tag{B2}$$

It follows immediately that

$$D_i\vec\phi = \hat\phi\partial_i|\phi| + (D_i\hat\phi)|\phi| = \hat\phi\partial_i|\phi| + O(e^{-vr}) \tag{B3}$$

and

$$D_i{}^2\vec\phi = \hat\phi\partial_i{}^2|\phi| + O(e^{-vr})\,. \tag{B4}$$

Combining Eqs. (2.7) and (2.12) we obtain

$$0 = D_i{}^2\vec\phi\,. \tag{B5}$$

The last two equations and the boundary condition on $|\phi|$ imply that

$$|\phi| = v - c/r + O(1/r^2) \tag{B6}$$

for some constant c. Consequently

$$D_i\vec\phi = \vec{B}_i = \hat\phi\left[\frac{c\hat r_i}{r^2} + O(1/r^3)\right] + O(e^{-vr})\,. \tag{B7}$$

If the magnetic charge is $4\pi n$, then clearly $c = n$.

We now consider the spectra of $\mathfrak{D}^*\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}^*$. The eigenfunctions of these operators satisfy

$$[-D_j{}^2 - \phi^2 - 2is\sigma_j(D_j\phi)]\psi = k^2\psi\,, \tag{B8}$$

where s is one for $\mathfrak{D}^*\mathfrak{D}$ and zero for $\mathfrak{D}\mathfrak{D}^*$. Here ϕ and $D_j\phi$ are 3×3 matrices defined as in Eq.

(3.3), and D_j is given by Eq. (3.2). The continuum spectra correspond to eigenfunctions with oscillatory behavior at infinity.

It is convenient to decompose the isovector ψ according to

$$\psi^a + \hat{\phi}^a\eta + \chi^a \tag{B9}$$

with

$$0 = \hat{\phi}^a\chi^a\,. \tag{B10}$$

It follows from Eq. (B2) that

$$(D_j\psi)^a = \hat{\phi}^a\partial_j\eta + (D_j\chi)^a + O(e^{-vr})\,, \tag{B11}$$

and that

$$(D_j{}^2\psi)^a = \hat{\phi}^a\partial_j{}^2\eta + (D_j{}^2\chi)^a + O(e^{-vr})\,. \tag{B12}$$

From Eq. (B3) we see that

$$\begin{aligned} D_j\dot{\phi} &= (D_j\phi)^a T^a \\ &= \phi\left(\frac{\partial_j|\phi|}{|\phi|}\right) + O(e^{-vr})\,. \end{aligned} \tag{B13}$$

Finally

$$(\phi\hat{\phi})^a = \phi^b (T^b)_{ac}\hat{\phi}^c = 0\,, \tag{B14}$$

where $(T^b)_{ac}$ is given by Eq. (3.4). Thus Eq. (B8) may be rewritten as

$$-\hat{\phi}^a\partial_j{}^2\eta + \left[\left(-D_j{}^2 - \phi^2 - 2is\phi\sigma_j\frac{\partial_j|\phi|}{|\phi|}\right)\chi\right]^a + (e^{-vr}) = k^2(\hat{\phi}^a\eta + \chi^a)\,. \tag{B15}$$

We now note that

$$\begin{aligned} \hat{\phi}^a(D_j\chi)^a &= \partial_j(\hat{\phi}^a\chi^a) - (D_j\hat{\phi})^a\chi^a \\ &= O(e^{-vr})\,, \end{aligned} \tag{B16}$$

and consequently that

$$\begin{aligned} \hat{\phi}^a(D_j{}^2\chi)^a &= \partial_j[\hat{\phi}^a(D_j\chi)^a] - (D_j\hat{\phi})^a(D_j\chi)^a \\ &= O(e^{-vr})\,. \end{aligned} \tag{B17}$$

Equation (B15) can now be separated into components parallel to and orthogonal to $\hat{\phi}$, yielding

$$-\nabla^2\eta + O(e^{-vr}) = k^2\eta \tag{B18}$$

and

$$[-D_j{}^2 - \phi^2 - 2is\sigma_j(D_j\phi)]\chi + O(e^{-vr}) = k^2\chi\,. \tag{B19}$$

The latter equation may be written as

$$[-\nabla^2 + v^2 + O(1/r)]\chi = k^2\chi\,, \tag{B20}$$

so χ must vanish exponentially at infinity if $k^2 < v^2$. In contrast, η will have oscillatory behavior at infinity for all $k^2 > 0$. We now note that Eq. (B18) is similar to the Schrödinger equation for a nonrelativistic particle in an exponentially vanishing potential. Since such a potential leads to a nonsingular phase shift and thus a nonsingular density of scattering states, we conclude that the densities of continuum eigenvalues of $\mathfrak{D}^*\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}^*$ are nonsingular at zero.

APPENDIX C

In Sec. IV it was pointed out that nonspherically symmetric $n=1$ solutions could exist only if the zero modes corresponding to rotation were nonnormalizable when put into the background gauge. In this appendix we show that continuity of the field strengths is sufficient to guarantee that these modes are square-integrable. (In Ref. 13 it is shown that normalizability of the translation modes follows from the weaker condition that the energy be finite.)

The desired mode corresponding to rotation about the ith axis is given by

$$\delta A_\mu = \epsilon_{ijk}x_j F_{k\mu} + D_\mu[x_i(\phi - vf\hat{\phi})] + D_\mu\Lambda \tag{C1}$$

where $A_4 = \phi$, f is a smooth function satisfying

$$f(\vec{r}) = \begin{cases} 0\,, & r < R \\ 1\,, & r > 2R \end{cases} \tag{C2}$$

with R chosen to be sufficiently large that $f(\vec{r})$ vanish at all the zeros of ϕ, and Λ is given by

$$\begin{aligned} \Lambda(\vec{r}) &= \int d^3r'\,\langle\vec{r}|(D_j{}^2 + \phi^2)^{-1}|\vec{r}'\rangle\,\Delta(\vec{r}')\,, \\ \Delta(\vec{r}) &= v[x_i D_j{}^2(f\hat{\phi}) + 2D_i(f\hat{\phi})]\,. \end{aligned} \tag{C3}$$

[Note that $\Delta(\vec{r})$ vanishes for $r<R$ and decreases as e^{-vr} at spatial infinity.] This is indeed a rotation mode, i.e., it corresponds to a naive rotation combined with a gauge transformation. Furthermore, it is easy to verify, using Eq. (2.12), that the background gauge condition, Eq. (3.6), is satisfied. It only remains to demonstrate that the mode is square-integrable.

In doing so we will find it convenient to apply gauge transformations to the unperturbed solution. (Note that this is quite different from adding to δA_μ a term corresponding to an infinitesimal gauge transformation.) Under such transformations δA_μ transforms as an isotopic vector, as may be readily verified from Eq. (C1), so $(\delta A_\mu)^2$ is invariant. Therefore we may first choose a gauge in which it is manifest that $(\delta A_\mu)^2$ is everywhere integrable and then choose another in which it is clear that $(\delta A_\mu)^2$ falls sufficiently rapidly at spatial infinity.

We begin by transforming the solution so that it satisfies the axial gauge conditions

$$\begin{aligned} &A_3 = 0\,, \\ &A_2 = 0\,, \quad z = z_0\,, \\ &A_1 = 0\,, \quad y = y_0\,, \quad z = z_0\,, \end{aligned} \tag{C4}$$

which imply that A_i and ϕ are given by line integrals of the field strength and are thus continuous. Since Λ is now the solution of an elliptic differential equation with continuous coefficients, it and its first derivative are continuous,[19] as is $D_\mu\Lambda$. The first two terms in δA_μ are also continuous, since the field strengths are, so we conclude that $(\delta A_\mu)^2$ is everywhere integrable.

Next we transform the solution so that it behaves asymptotically like the Prasad-Sommerfield solution, i.e., so that $\hat{\phi}^a = \hat{r}^a$ and A_i falls like $1/r$. In this gauge it is clear that $D_\mu\Lambda$ decreases asymptotically like $1/r^2$, as does the sum of the first two terms in δA_μ. Therefore δA_μ will be square-integrable.

[1] G. 't Hooft, Nucl. Phys. B79, 276 (1974).
[2] A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].
[3] B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975).
[4] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).
[5] E. B. Bogomol'nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)].
[6] S. Coleman, S. Parke, A. Neveu, and C. M. Sommerfield, Phys. Rev. D 15, 544 (1977).
[7] N. S. Manton, Nucl. Phys. B126, 525 (1977).
[8] C. J. Callias, Commun. Math. Phys. 62, 213 (1978).
[9] Greek indices run from 0 to 3 or 1 to 4, according to the context, while Latin indices run from 1 to 3.
[10] L. S. Brown, R. D. Carlitz, and C. Lee, Phys. Rev. D 16, 417 (1977).
[11] E. Mottola, Phys. Lett. 79B, 242 (1978); Phys. Rev. D 19, 3170 (1979).
[12] See Appendix A for a discussion of the M^2 dependence of $\mathcal{G}(M^2)$.
[13] S. L. Adler, Phys. Rev. D 19, 2997 (1979).
[14] S. Adler, Phys. Rev. D (to be published).
[15] For a different approach to the multimonopole problem, see W. Nahm, CERN Report No. TH-2642 (unpublished).
[16] C. Bernard, A. Guth, and E. Weinberg, Phys. Rev. D 17, 1053 (1978).
[17] E. Weinberg, Phys. Rev. D 19, 3008 (1979).
[18] Requiring that $\hat{\phi}$ be constant over all space will introduce string singularities. These may be avoided by imposing this requirement only outside a sphere containing all the zeros of ϕ and only for a solid angle of less than 4π. Since the solid angle in which $\hat{\phi}$ is chosen constant is arbitrary, one obtains the same results.
[19] L. Bers, F. John, and M. Schechter, *Partial Differential Equations* (American Mathematical Society, Providence, R.I., 1964), p. 240.