Gauge dependence in the Yang-Mills S matrix

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There are formal proofs that S-matrix elements are gauge invariant in gauge field theories. These proofs break down unless infrared singularities of the S matrix are handled with care. In a similar fashion, formal on-shell Ward identities can break down. Dimensional regularization of infrared singularities seems to preserve all formal relations.

I. INTRODUCTION

In gauge field theories, a gauge-fixing term $-\frac{1}{2}F_a(A)^2$ must be added to the Lagrange density before one can calculate amplitudes. This term spoils the gauge invariance of the action, and Green's functions and proper vertices depend upon the F_a one chooses; they are gauge dependent. In contrast to this, one expects rates for physical processes to be independent of F_a. A number of authors[1-6] have solved this problem by showing that gauge-theory S-matrix elements are gauge independent, even though Green's functions are not.

The proofs of the gauge independence of S-matrix elements are formal proofs that require qualifications which I discuss in this paper. Gauge theories without spontaneous breakdown contain massless particles, and the usual S matrix does not exist because of the resulting infrared singularities. This has two consequences. First, one must show that rates for physical processes are infrared finite when the possibility of the radiation of soft quanta is taken into account. This result has been fully established for QED (Abelian gauge theories),[7] but not for Yang-Mills (non-Abelian) gauge theories.[8] I shall have little to say about this subject.

The second consequence of massless particles is that if one wants to discuss the gauge dependence of S-matrix elements individually, one must somehow regulate their infrared singularities. In this paper I show that the proofs of Refs. 1–6 break down for some methods of regularization, including a method (called S_1 below) which is similar to that often used in QED,[9] and sometimes used in Yang-Mills theories.[10] It happens that in QED S_1 is gauge invariant despite the breakdown of the proofs of Ref. 1–6. In fact, for QED there are textbook algebraic proofs (which should be distinguished from Refs. 1–6) showing that S_1 is gauge independent.[11] However, S_1 is gauge *dependent* in Yang-Mills theories. For another method of regulation (called S_2 below) the proofs of Refs. 1–6 are valid, and S_2 is gauge invariant in both QED and Yang-Mills theories. I show in detail how the formal proofs of Refs. 1–6 break down for S_1. Finally, I show that the method of regulating infrared singularities also determines whether or not formal consequences of "on-shell" Ward-Takahashi identities hold. Contradictory results in the literature apparently are due to different treatments of infrared singularities.[10,12]

Let us consider how infrared singularities of S-matrix elements might be regulated. Denote the unrenormalized one-particle-irreducible (1PI) proper vertex function for the process of interest by $\Gamma(p_1 \cdots p_n)$, and let Z_i be the wave-function renormalization constant for the ith particle, normalized at mass k_i^2. Then the renormalized proper vertex functions

$$\Gamma_R=\left(\prod_i Z_i^{1/2}\right)\Gamma(p_1 \cdots p_n) \tag{1.1}$$

are free of ultraviolet singularities when expressed in terms of some off-shell renormalized coupling constant. Also, if there were no massless particles, the S-matrix elements would be a certain on-shell limit of this vertex function,

$$S=\lim_{k_i^2\to m_i^2}\Gamma_R\big|_{p_i^2=k_i^2}, \tag{1.2}$$

where m_i^2 is the mass squared of particle i.

When there are massless particles, the limit in Eq. (1.2) does not exist. I will discuss two alternative S matrices which differ in their methods of dealing with infrared singularities. Perhaps it is worth emphasizing that the two S matrices are equivalent to Eq. (1.2) in theories with no massless particles.

(1) S_1. S_1 is defined by holding $m_i^2-k_i^2$ small but positive in Eq. (1.2), and retaining all terms which are nonvanishing as $m_i^2-k_i^2$ approaches zero. (It may be possible and desirable to let some subset of legs go on-shell without developing singularities.) Note that in S_1 only the external legs are held off-shell because a gauge-meson mass term in the Lagrangian would destroy any prospect of gauge independence. Typical singular factors in S_1 have the form $\ln^n(m_i^2-k_i^2)$, where n increases with the order of perturbation theory.

Because these singularities are so mild, the standard algebraic proof for spinor QED shows that S_1 is the same in the Coulomb, Landau, Feynman, and general R_ξ gauges.[11] However, I will show that S_1 is gauge dependent in Yang-Mills theories.

(2) S_2. With the invention of dimensional regularization in 1971,[13] it became possible to use the space-time dimension D to regulate infrared[14,15] as well as ultraviolet divergences. For dimensions $D>4$, the limit in Eq. (1.2) can be taken, and the infrared divergence of S_2 presents itself as a singularity in D at the physical dimensions $D=4$. (Note that this singularity arises solely from going on-shell because the renormalized off-shell vertices Γ_R are analytic at $D=4$ in every finite order or perturbation theory.) S_2 satisfies the assumptions underlying Refs. 1–6 and is gauge independent in all gauge theories.

The difference between S_1 and S_2 lies in the order in which one approaches $D=4$ and the mass shell

$$S_1=[\lim_{D\to 4}\Gamma_R|_{p_i^2=k_i^2}]_{k_i^2\lessgtr m_i^2}\,, \tag{1.3a}$$

$$S_2=[\lim_{k_i^2\to m_i^2}\Gamma_R|_{p_i^2=k_i^2}]_{D\gtrless 4}\,. \tag{1.3b}$$

S_1 and S_2 should be equally good for calculating physical rates. With S_2, bremsstrahlung contributions are calculated in D dimensions, and are supposed to have a singularity at $D=4$ which cancels that of the primary process.[8] With S_1, I presume that the infrared logarithms and gauge dependence are *both* canceled by bremsstrahlung contributions. This conjecture is based on the belief that physical rates ought not to depend on the method of regulating infrared singularities. (If they do, we are in trouble.) Since rates are gauge independent using S_2, they must be using S_1. In this sense the gauge dependence of S_1 is harmless.

Let us next review the proofs that S-matrix elements are gauge invariant, to see how they break down for S_1. As an example of these proofs I take the demonstration in the review paper of Abers and Lee.[16] Consider a gauge theory generating functional in which gauge fields, scalar fields,... are coupled to sources by adding terms $J_\mu^a A_\mu^a$, $j^a\phi^a,\ldots,$ to the Lagrange density. Connected Green's function are appropriate functional derivatives of the generating functional, evaluated at $0=J_\mu^a=j^a=\cdots$. Now suppose I change the gauge-fixing term from $F_a(A)$ to $F_a(A)+\Delta F_a(A)$. Abers and Lee show that one can calculate Green's functions in the new $(F+\Delta F)$ gauge using old (F) gauge Feynman rules provided the source currents couple to additional, higher powers of the field. The modified couplings are

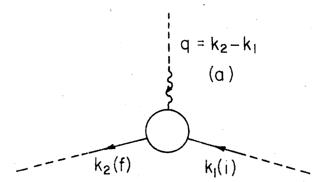

FIG. 1. Gauge-meson–scalar–scalar vertex function $G_\mu^{(3)}$. Dashed lines represent external currents. Indices in parentheses are group indices for the Yang-Mills case. The gauge-fixing term is F.

$$\begin{aligned}&J_\mu^a A_\mu^a\to J_\mu^a(A_\mu^a+\alpha_{\mu\nu\sigma}^{abc}A_\nu^b A_\sigma^c+\cdots)\,,\\ &j^a\phi^a\to j^a(\phi^a+\beta_\mu^{abc}\phi^b A_\mu^c+\cdots)\,.\end{aligned} \tag{1.4}$$

The new couplings α and β are proportional to ΔF_a, and this formalism is valid to first order in ΔF_a. It provides the framework for the argument that S-matrix elements are gauge independent.

As an example of the argument, consider the gauge-meson–scalar–scalar three-point function G_μ. Figure 1 represents G_μ in gauge F. To obtain G_μ in gauge $F+\Delta F$, one must add the diagrams of Fig. 2 to those of Fig. 1. Abers and Lee divide the diagrams of Fig. 2 into those which are one-particle reducible in the leg with the new coupling, and those which are irreducible. Reducible diagrams, like the one in Fig. 3, have a simple interpretation: They correspond to a shift ΔZ_i in the wave-function renormalization constant for the particle in the leg with the new coupling. Next, denote the S matrix defined in terms of the three-point function by S_3,

$$S_3=i\left[\prod_{j=1}^{3}Z_j^{-1/2}(k_j^2-m_j^2)\right]\epsilon_\mu G_\mu^{(3)}|_{p_i^2=k_i^2}\,, \tag{1.5}$$

where ϵ_μ is the gauge-meson polarization vector. To first order in ΔF, the response of S_3 to the gauge change $F\to F+\Delta F$ is

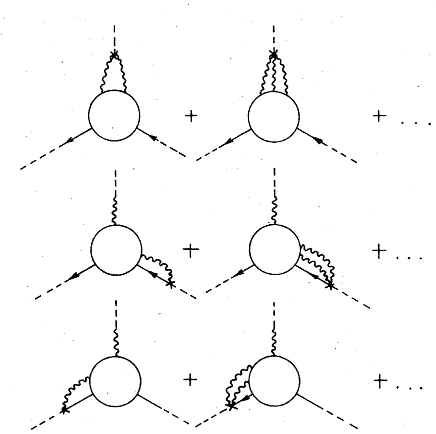

FIG. 2. Diagrams which must be added to Fig. 1 when the gauge-fixing term is $F+\Delta F$. The new current coupling is indicated by a cross.

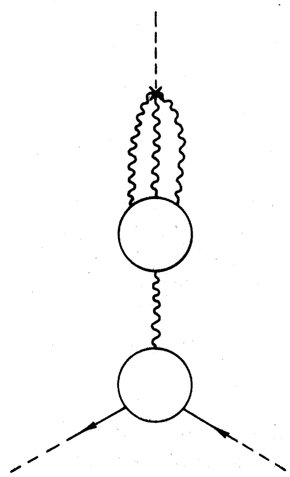

FIG. 3. A diagram which is one-particle reducible in the channel of the new coupling (cross).

$$\Delta S_3 = i\left[\prod_{j=1}^{3} Z_j^{-1/2}(k_j^2 - m_j^2)\right]\epsilon_\mu \Delta G_\mu^{(3)} - \tfrac{1}{2} S_3 \sum_j Z_j^{-1}\Delta Z_j . \tag{1.6}$$

Abers and Lee argue that in the first term of Eq. (1.6) only the one-particle reducible graphs of Fig. 2 contribute because only those have a particle pole in all three channels and survive as one approaches the mass shell. Note that this observation is unaffected by the presence of logarithmic singularities in G_μ and $Z_j^{-1/2}$ because irreducible terms are alleged to be suppressed by a linear zero. By considering propagators one can readily verify that these reducible contributions exactly cancel the second term in Eq. (1.6). One then concludes $\Delta S_3 = 0$; S_3 is gauge independent.

The part of the argument of Abers and Lee which can break down is the assumption that one-particle irreducible contributions to $\Delta G_\mu^{(3)}$ drop out of Eq. (1.6). The fact is that in dimension $D=4$ the coefficients α and β in Eq. (1.4) are so infrared singular that a graph can develop a factor $(m_j^2 - k_i^2)^{-1}$ when the diagram is irreducible. When these extra contributions are included, the cancellation of the two terms in Eq. (1.6) can fail to take place. In this way, S_1 and S_3 can become gauge dependent. On the other hand, for $D>4$ the infrared singularities of α and β are no longer able to generate a factor $(m_i^2 - k_i^2)^{-1}$. As a result, the argument of Abers and Lee stands, and S_2 is gauge independent.

Note that the breakdown of the formal argument for S_1 does not prove that it is gauge dependent; one merely fails to prove that it is gauge independent. As I have already mentioned, there is a separate algebraic proof that S_1 is gauge independent in spinor QED (but not Yang-Mills theories). Because of this fortuitous gauge independence in QED, the issues raised in this paper were not apparent before the rise of Yang-Mills gauge theories.

Apparently, formal Ward-Takahashi (WT) identities relating amplitudes which are "near the mass shell" are subject to the same difficulties as S-matrix elements. For example, the equality $Z_1 = Z_S$ between charge and scalar particle renormalization constants follows from a WT identity.[12] The Z's are infrared divergent when normalized on-shell, and, like the S matrix, can be regulated in various ways. The equality $Z_1 = Z_S$ holds under the procedure of Eq. (1.3b), but not under Eq. (1.3a) in Yang-Mills theories. This explains why Refs. 10 and 12 disagree as to the equality of the two renormalization constants.

In the rest of this paper I will discuss the gauge-meson–scalar–scalar vertex for two gauge theories: scalar electrodynamics and Yang-Mills theories in which the scalar mesons belong to the fundamental representation of the gauge group. I will use the gauge-fixing term of the R_ξ gauge

$$F_a(A) = \sqrt{\xi}\,\frac{\partial A_\mu^a}{\partial x_\mu}. \tag{1.7}$$

The corresponding gauge-meson propagator depends on the gauge parameter ξ,

$$G_{\mu\nu}^{ab} = -\frac{i\delta_{ab}}{q^2}\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\left(1 - \frac{1}{\xi}\right)\right]. \tag{1.8}$$

I will then verify my assertions about gauge dependence by a series of calculations in second- or third-order perturbation theory. In Sec. II I calculate S_1, S_2, Z_1, and Z_S. In Sec. III I calculate ΔS_1, corresponding to a gauge parameter change $\Delta\xi$, using the formalism of Abers and Lee. I find that when irreducible graphs are correctly included, the results agree with what one obtains by differentiating the results of Sec. II. This shows that the breakdown of the formal proofs proceeds in the manner I have stated.

The findings of this paper give added prestige to the dimensional regulation of infrared singularities in gauge theories. Under dimensional regularization, formal relations like the gauge independence of the S matrix and $Z_1 = Z_S$ are valid. Under regulation schemes carried out directly in $D=4$, these relations generally break down.

II. THIRD-ORDER S-MATRIX ELEMENTS

A. Electrodynamics

Scalar electrodynamics is described by the Lagrange density

$$L = -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} + (\partial_\mu \phi^\dagger + ieA_\mu)(\partial_\mu \phi - ieA_\mu) - (m^2 + \delta m^2)\phi^\dagger \phi\,, \qquad (2.1)$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

δm^2 is adjusted so m is the physical scalar-meson mass. In setting up calculations, one must take care not to introduce spurious gauge dependence by the method of regulating ultraviolet and infrared divergences. I regulate ultraviolet divergences dimensionally, calculating in D rather than four space-time dimensions. When I calculate S_1, I follow Eq. (1.3a) and expand around $D=4$ before approaching the mass shell. There is a pole in Γ_R at $D=4$ which could be removed by introducing a renormalized charge, but this step is familiar and irrelevant to the issues of this paper, so I omit it. As a result, S_1 has an "ultraviolet" pole at $D=4$, and S_2 has a mixed "ultraviolet" and "infrared" pole there. For S_1, I let the photon momentum go to zero, holding the scalar-meson momenta off-shell. No infrared singularities appear because the scalars are off-shell. In fact, one can often set the photon momentum to zero before expanding around $D=4$, without affecting the result. This step greatly simplifies the Feynman integrals, and I will indicate where it is allowed.

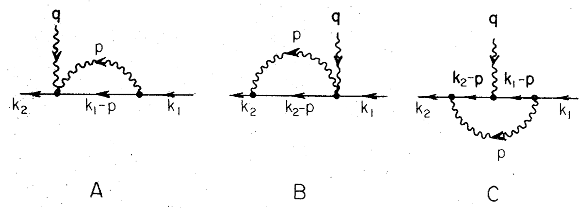

FIG. 4. Diagrams contributing to the proper vertex $\epsilon_\mu V_\mu$ in third-order scalar electrodynamics.

The three graphs shown in Fig. 4 contribute the third-order proper vertex $\epsilon_\mu V_\mu$. First consider the calculation of S_1. It can be shown that in graph C one is allowed to set $q=0$ before expanding around $D=4$. This is done by writing the amplitude in terms of Feynman parameters and noting that for all D the amplitude is independent of q^2 for $|q^2| \ll |k^2 - m^2|$. Amplitudes A and B depend only on $k_1{}^2$ or $k_2{}^2$, so all three graphs can be evaluated at $q=0$ in terms of the integrals

$$J^{\alpha\beta}(k^2, M_1{}^2, M_2{}^2) = \int \frac{d^D p}{(2\pi)^D [(k-p)^2 - M_1{}^2 + i\epsilon]^\alpha [p^2 - M_2{}^2 + i\epsilon]^\beta} \qquad (2.2)$$

and the generalizations $J^{\alpha\beta}_\mu, J^{\alpha\beta}_{\mu\nu}, \ldots$ having factors p_μ, $p_\mu p_\nu, \ldots$ in the integrand. The results are

$$\epsilon_\mu V_{\mu,A} = \epsilon_\mu V_{\mu,B} = -4e^3\left[(k\cdot\epsilon) I^{11} - \left(1 - \frac{1}{\xi}\right) k_\mu \epsilon_\nu I^{12}_{\mu\nu} - \frac{1}{2\xi}\epsilon_\mu I^{11}_\mu\right],$$
$$\epsilon_\mu V_{\mu,C} = 8e^3\left[k^2(\epsilon\cdot k) I^{21} - \left(1-\frac{1}{\xi}\right)(\epsilon\cdot k) k_\mu k_\nu I^{22}_{\mu\nu} - k^2 \epsilon_\mu I^{21}_\mu + \left(1-\frac{1}{\xi}\right)\epsilon_\mu k_\nu k_\sigma I^{22}_{\mu\nu\sigma} - \frac{1}{\xi}(\epsilon\cdot k) k_\mu I^{21}_\mu \right.$$
$$\left. + \frac{1}{\xi}\epsilon_\mu k_\nu I^{21}_{\mu\nu} + \frac{1}{4\xi}(\epsilon\cdot k) I^{20} - \frac{1}{4\xi}\epsilon_\mu I^{20}_\mu\right], \qquad (2.3)$$
$$I^{\alpha\beta} \equiv J^{\alpha\beta}(k^2, m^2, 0)\,.$$

The integrals have been evaluated for $k^2 \sim m^2$ in Appendix B. The complete proper vertex is

$$\epsilon_\mu V_\mu = 2ie(\epsilon\cdot k)\left\{1 + \frac{e^2}{(4\pi)^2}\left[\frac{6}{D-4} - 3\Gamma'(1) + 3\ln\frac{m^2}{4\pi} + 2 + 6\ln(1 - k^2/m^2)\right]\right.$$
$$\left. + \frac{e^2}{(4\pi)^2}\frac{1}{\xi}\left[\frac{-2}{D-4} + \Gamma'(1) - \ln\frac{m^2}{4\pi} - 2 - 2\ln(1-k^2/m^2)\right]\right\}. \qquad (2.4)$$

To calculate S_1 we also must consider the second-order corrections to the propagators. The scalar self-energy part is shown in Fig. 5. The photon loop Σ_A must be taken to be zero in dimensional regularization. The other graph is

$$-i\Sigma_B = -4e^2\left[k^2 I^{11} - \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu I^{12}_{\mu\nu} - \frac{1}{\xi} k_\mu I^{11}_\mu + \frac{1}{\xi} I^{10}\right]. \qquad (2.5)$$

Evaluating the integrals,

$$\Sigma_B = -\frac{e^2}{(4\pi)^2}\left\{m^2\left[\frac{6}{D-4} - 3\Gamma'(1) + 3\ln\frac{m^2}{4\pi} - 7\right]\right.$$
$$\left. + (k^2 - m^2)\left[\frac{6 - 2/\xi}{D-4} + \left(-3 + \frac{1}{\xi}\right)\Gamma'(1) + \left(3 - \frac{1}{\xi}\right)\ln\frac{m^2}{4\pi} - 4 + (6 - 2/\xi)\ln(1 - k^2/m^2)\right]\right\}. \qquad (2.6)$$

The scalar propagator is

$$G_S = \frac{i}{k^2 - m^2 - \Sigma_B - \delta m^2}. \qquad (2.7)$$

I choose $\delta m^2 = -\Sigma_B(k^2 = m^2)$. It is important that this counterterm is gauge independent. Any explicit gauge dependence in the Lagrange density would spoil the formulas developed by Abers and Lee to calculate the response ΔG to a guage change $\Delta\xi$.

The scalar wave-function renormalization constant is

$$Z_S^{-1} = \frac{\partial}{\partial k^2}[k^2 - m^2 - \Sigma_B(k^2) + \Sigma_B(m^2)]$$
$$= 1 + \frac{e^2}{(4\pi)^2}\left[\frac{6 - 2/\xi}{D-4} + \left(-3 + \frac{1}{\xi}\right)\Gamma'(1) + \left(3 - \frac{1}{\xi}\right)\ln\frac{m^2}{4\pi} + 2 - \frac{2}{\xi} + \left(6 - \frac{2}{\xi}\right)\ln(1 - k^2/m^2)\right]. \qquad (2.8)$$

Note that one cannot set $k^2 = m^2$ because of the infrared divergence. As in Eq. (2.4), all terms which do not vanish at $k^2 = m^2$ are carried along. It is useful to note that

$$Z_S \epsilon_\mu V_\mu = 2ie(\epsilon \cdot k). \qquad (2.9)$$

That is, if one ignores the wave-function renormalization of the photon line, all radiative corrections cancel in the S-matrix element. This shows that $Z_1 = Z_S$ in scalar electrodynamics under the limiting procedure of Eq. (1.3a).

The photon polarization part is shown in Fig. 6. The contributions are

$$i\pi_A = e^2(q_\mu q_\nu \bar{I}^{11} - 2q_\mu \bar{I}^{11}_\nu - 2q_\nu \bar{I}^{11}_\mu + 4\bar{I}^{11}_{\mu\nu}),$$
$$i\pi_B = -2e^2 q_{\mu\nu}\bar{I}^{01}, \qquad (2.10)$$
$$\bar{I}^{\alpha\beta} \equiv J^{\alpha\beta}(k^2, m^2, m^2).$$

These integrals are evaluated in Appendix B, with the result

$$\pi_A + \pi_B = (q^2 g_{\mu\nu} - q_\mu q_\nu)\sigma,$$
$$\sigma = \frac{e^2}{(4\pi)^2}\left[\frac{2/3}{D-4} - \tfrac{1}{3}\Gamma'(1) + \tfrac{1}{3}\ln\frac{m^2}{4\pi}\right]. \qquad (2.11)$$

The photon propagator and wave-function renormalization constant are

$$(G_g)_{\mu\nu} = \frac{-i}{q^2(1-\sigma)}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - \frac{iq_\mu q_\nu}{\xi q^2},$$
$$Z_g^{-1} = \frac{\partial}{\partial q^2} q^2(1-\sigma) \qquad (2.12)$$
$$= 1 - \frac{e^2}{(4\pi)^2}\left[\frac{2/3}{D-4} - \tfrac{1}{3}\Gamma'(1) + \tfrac{1}{3}\ln\frac{m^2}{4\pi}\right].$$

Thus the electrodynamic S-matrix element is seen to be independent of ξ in third order

$$S_{1,\,\mathrm{QED}} = Z_S Z_g^{1/2}\epsilon_\mu V_\mu$$
$$= 2ie(\epsilon \cdot k)\left\{1 + \frac{e^2}{(4\pi)^2}\left[\frac{1/3}{D-4} - \tfrac{1}{6}\Gamma'(1) + \tfrac{1}{6}\ln\frac{m^2}{4\pi}\right]\right\}. \qquad (2.13)$$

S_2 is readily calculated by evaluating the integrals of Eqs. (2.3), (2.5), and (2.10) in the limiting prescription of Eq. (1.3b). The formulas of Appendix B give the results

$$\epsilon_\mu V_\mu = 2ie(\epsilon \cdot k) + O(e^5),$$
$$Z_S^{-1} = 1 + O(e^4),$$
$$Z_g^{-1} = 1 - \frac{e^2\Gamma(2 - D/2)}{6(4\pi)^{D/2}}(m^2)^{D/2-2}, \qquad (2.14)$$
$$S_{2,\,\mathrm{QED}} = 2ie(\epsilon \cdot k)\left[1 - \frac{e^2\Gamma(2 - D/2)}{6(4\pi)^{D/2}}(m^2)^{D/2-2}\right].$$

This shows that S_2 is also gauge independent through third order in scalar QED, and in fact, $S_1 = S_2$. There is no second-order contribution to the charge and scalar-meson renormalization

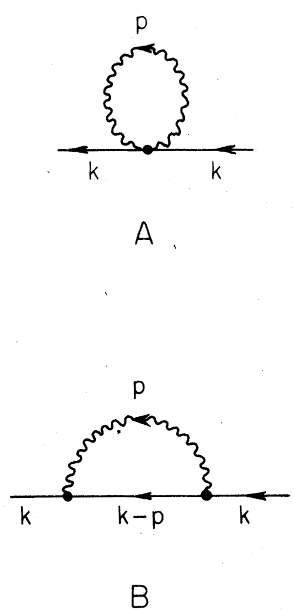

FIG. 5. Diagrams contributing to the scalar-meson self-energy Σ in second order.

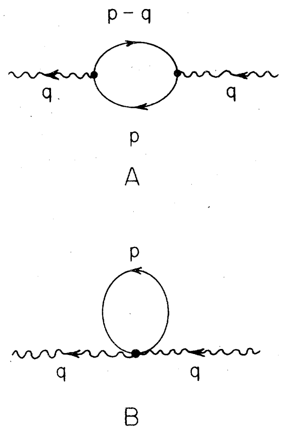

FIG. 6. Diagrams contributing to the photon polarization π in second-order scalar electrodynamics.

constants. Therefore, through second order the Ward identity result $Z_1 = Z_S$ is satisfied in QED under the limiting procedure of Eq. (1.3b).

B. Yang-Mills theory

In Yang-Mills theory the Lagrange density is the same as Eq. (2.1), but with the changes

$$\begin{aligned} &-ie\phi A_\mu \to -igt^a \phi A^a_\mu , \\ &F_{\mu\nu} \to F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, c_{abc} A^b_\mu A^c_\nu , \\ &[\tau^a, \tau^b] = i c_{abc} \tau^c . \end{aligned} \tag{2.15}$$

τ^a is any representation of the gauge group generators, and the last equation defines the structure constants c. All the graphs in QED contribute to the gauge-meson–scalar–scalar proper vertex, but with coefficients depending upon group factors. One of these is the matrix t^a of the fundamental representation of the gauge group, to which the scalar mesons belong. Another is the trace

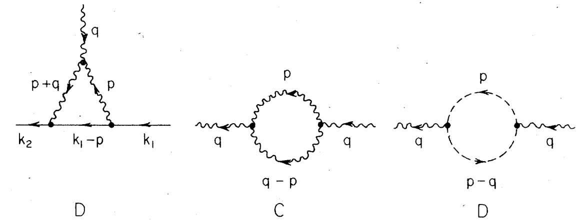

FIG. 7. Additional diagrams in non-Abelian gauge theory. π_D is a ghost loop.

$$T(R)\delta_{ab} = \mathrm{Tr}(t^a t^b) . \tag{2.16}$$

The third is the Casimir eigenvalue C_2

$$C_2 \delta_{ij} = \sum_a (t^a t^a)_{ij} , \tag{2.17}$$

which is needed for the fundamental (R) and adjoint (G) representations. For the group SU(N),

$$T(R) = \tfrac{1}{2},\; C_2(R) = (N^2 - 1)/2N,\; C_2(G) = N . \tag{2.18}$$

The gauge group factors for amplitudes calculated in QED are

$$\begin{aligned} &\epsilon_\mu V_{\mu,A} \text{ and } \epsilon_\mu V_{\mu,B}\text{: } [C_2(R) - \tfrac{1}{4} C_2(G)](t^a)_{fi} , \\ &\epsilon_\mu V_{\mu,C}\text{: } [C_2(R) - \tfrac{1}{2} C_2(G)](t^a)_{fi} , \\ &\Sigma_B\text{: } C_2(R)\delta_{fi} , \\ &\pi_A \text{ and } \pi_B\text{: } T(R)\delta_{ab} . \end{aligned} \tag{2.19}$$

i and f are group indices for scalar mesons, and a and b are group indices for gauge mesons.

There are new graphs in non-Abelian gauge theory shown in Fig. 7. First consider S_1. For the amplitude $\epsilon_\mu V_{\mu,D}$ I found no argument proving that one is allowed to set $q = 0$ before expanding around $D = 4$. One must therefore use the integrals

$$\hat{I}^{\alpha\beta}(k,q) = \int \frac{d^D p}{(2\pi)^D [(k-p)^2 - m^2 + i\epsilon][(p+q)^2 + i\epsilon]^\alpha (p^2 + i\epsilon)^\beta} \tag{2.20}$$

and their generalizations $\hat{I}^{\alpha\beta}_\mu$ etc. However, when I evaluated the amplitude, expanding around $D = 4$ before setting $q = 0$, I found the result is the same as when one sets $q = 0$ first. Therefore the amplitude can be expressed in terms of the simpler integrals $\hat{I}^{\alpha\beta}(k,0) = I^{1,\alpha+\beta}$:

$$\epsilon_\mu V_{\mu,D} = 4g^3 C_2(G)(t^a)_{fi}\left[k^2 \epsilon_\mu I^{12}_\mu - \left(1 - \frac{1}{\xi}\right)\epsilon_\mu k_\nu k_\sigma I^{13}_{\mu\nu\sigma} - \frac{1}{2\xi}\epsilon_\mu k_\nu I^{12}_{\mu\nu} + \frac{1}{2\xi}(k\cdot\epsilon)I^{11} - \frac{1}{\xi}(\epsilon\cdot k)k_\mu I^{12}_\mu\right] . \tag{2.21}$$

Evaluating the integrals,

$$\begin{aligned} \epsilon_\mu V_{\mu,D} = 2ig(\epsilon\cdot k)(t^a)_{fi} C_2(G)\frac{g^2}{(4\pi)^2}\Bigg\{ &\tfrac{9}{4} + \tfrac{3}{2}\ln(1 - k^2/m^2) \\ &+ \frac{1}{\xi}\left[\frac{-3/2}{D-4} + \tfrac{3}{4}\Gamma'(1) - \tfrac{3}{4}\ln\frac{m^2}{4\pi} - \tfrac{1}{2} - \tfrac{3}{2}\ln(1 - k^2/m^2)\right]\Bigg\} . \end{aligned} \tag{2.22}$$

The additional contributions to the gauge-meson polarization part can be written in terms of integrals $\tilde{I}^{\alpha\beta} = J^{\alpha\beta}(q^2, 0, 0)$. The expressions are

$$
\begin{aligned}
i\pi_C = \frac{g^2}{2}\,\delta_{ab}\,C_2(G)\Big\{ &\left(1-\frac{1}{\xi}\right)^2 [(q^2)^2 \tilde{I}^{22}_{\mu\nu} - q^2 q_\mu q_\sigma \tilde{I}^{22}_{\sigma\nu} - q^2 q_\nu q_\sigma \tilde{I}^{22}_{\sigma\mu} + q_\mu q_\nu q_\sigma q_\rho \tilde{I}^{22}_{\sigma\rho}] \\
&+ 2\left(1-\frac{1}{\xi}\right)(\tilde{I}^{11}_{\mu\nu} - 2q_\sigma \tilde{I}^{12}_{\mu\nu\sigma} - q^2 \tilde{I}^{12}_{\mu\nu} - \tilde{I}^{11}_{\mu} q_\nu - q_\mu \tilde{I}^{11}_{\nu} + 3q_\nu q_\sigma \tilde{I}^{12}_{\sigma\mu} \\
&\qquad + 3q_\mu q_\sigma \tilde{I}^{12}_{\sigma\nu} - q_\mu q_\nu \tilde{I}^{11} + 4g_{\mu\nu} q_\sigma \tilde{I}^{11}_{\sigma} - 4g_{\mu\nu} q_\rho q_\sigma \tilde{I}^{12}_{\rho\sigma}) \\
&+ [4g_{\mu\nu} q^2 \tilde{I}^{11} - (6-4D)\tilde{I}^{11}_{\mu\nu} + (3-2D)(q_\mu \tilde{I}^{11}_{\nu} + q_\nu \tilde{I}^{11}_{\mu}) - (6-D) q_\mu q_\nu \tilde{I}^{11}]\Big\}, \\
i\pi_D = -g^2\delta_{ab}\,C_2(G)&(\tilde{I}^{11}_{\mu\nu} - q_\nu \tilde{I}^{11}_{\mu}).
\end{aligned}
\tag{2.23}
$$

These integrals are evaluated in Appendix B:

$$
\begin{aligned}
i(\pi_C + \pi_D) = \frac{ig^2}{(4\pi)^2}\,\delta_{ab}\,C_2(G)(q^2 q_{\mu\nu} - q_\mu q_\nu)\Big\{&\left[\frac{-13/3}{D-4} + \tfrac{97}{36} + \tfrac{13}{6}\Gamma'(1) - \tfrac{13}{6}\ln\left(\frac{-q^2}{4\pi}\right)\right] \\
&+ \frac{1}{\xi}\left[\frac{1}{D-4} + \frac{1}{2} - \tfrac{1}{2}\Gamma'(1) + \tfrac{1}{2}\ln\left(-\frac{q^2}{4\pi}\right)\right] + \frac{1}{4\xi^2}\Big\}.
\end{aligned}
\tag{2.24}
$$

The amplitudes can now be assembled. The proper gauge-meson–scalar–scalar vertex is

$$
\begin{aligned}
\epsilon_\mu V_\mu = 2ig(\epsilon\cdot k)(t^a)_{fi} + 2ig(\epsilon\cdot k)(t^a)_{fi}\,\frac{g^2}{(4\pi)^2}\Big\{ &C_2(R)\left[\frac{6}{D-4} - 3\Gamma'(1) + 3\ln\frac{m^2}{4\pi} + 2 + 6\ln(1-k^2/m^2)\right] \\
&+ \frac{C_2(R)}{\xi}\left[\frac{-2}{D-4} + \Gamma'(1) - \ln\frac{m^2}{4\pi} - 2 - 2\ln(1-k^2/m^2)\right] \\
&+ C_2(G)\left[\frac{-3/2}{D-4} + \tfrac{3}{4}\Gamma'(1) - \tfrac{3}{4}\ln\frac{m^2}{4\pi} - \tfrac{1}{2} - \tfrac{3}{2}\ln(1-k^2/m^2)\right] \\
&+ \frac{C_2(G)}{\xi}\left[\frac{-1/2}{D-4} + \tfrac{1}{4}\Gamma'(1) - \tfrac{1}{4}\ln\frac{m^2}{4\pi} + \tfrac{1}{2} - \tfrac{1}{2}\ln(1-k^2/m^2)\right]\Big\}.
\end{aligned}
\tag{2.25}
$$

Z_S^{-1} is given by Eq. (2.8) except that the coefficient e^2 is replaced by $g^2C_2(R)$. Z_g^{-1} is given by Eq. (2.12), where

$$
\pi_A + \pi_B + \pi_C + \pi_D = (q^2 g_{\mu\nu} - q_\mu q_\nu)\sigma\,. \tag{2.26}
$$

The result is

$$
\begin{aligned}
Z_g^{-1} = 1 - \frac{g^2}{(4\pi)^2}\Big\{ &T(R)\left[\frac{2/3}{D-4} - \tfrac{1}{3}\Gamma'(1) + \tfrac{1}{3}\ln\frac{m^2}{4\pi}\right] + C_2(G)\left[\frac{-13/3}{D-4} + \tfrac{19}{36} + \tfrac{13}{6}\Gamma'(1) - \tfrac{13}{6}\ln\left(\frac{-q^2}{4\pi}\right)\right] \\
&+ \frac{C_2(G)}{\xi}\left[\frac{1}{D-4} + 1 - \tfrac{1}{2}\Gamma'(1) + \tfrac{1}{2}\ln\left(\frac{-q^2}{4\pi}\right)\right] + \frac{C_2(G)}{4\xi^2}\Big\}.
\end{aligned}
\tag{2.27}
$$

The Yang-Mills S-matrix element is now seen to be gauge dependent:

$$
\begin{aligned}
S_{1,\mathrm{YM}} &= Z_S Z_g^{1/2}\epsilon_\mu V_\mu \\
&= 2ig(\epsilon\cdot k)(t^a)_{fi} \\
&\quad + 2ig(\epsilon\cdot k)(t^a)_{fi}\,\frac{g^2}{(4\pi)^2}\Big\{T(R)\left[\frac{1/3}{D-4} - \tfrac{1}{6}\Gamma'(1) + \tfrac{1}{6}\ln\frac{m^2}{4\pi}\right] \\
&\qquad + C_2(G)\left[\frac{-11/3}{D-4} - \tfrac{17}{72} + \tfrac{11}{6}\Gamma'(1) - \tfrac{13}{12}\ln\left(\frac{-q^2}{4\pi}\right) - \tfrac{3}{4}\ln\frac{m^2}{4\pi} - \tfrac{3}{2}\ln(1-k^2/m^2)\right] \\
&\qquad + \frac{C_2(G)}{\xi}\left[1 - \tfrac{1}{4}\ln\frac{m^2}{4\pi} + \tfrac{1}{4}\ln\left(\frac{-q^2}{4\pi}\right) - \tfrac{1}{2}\ln(1-k^2/m^2)\right] + \frac{C_2(G)}{8\xi^2}\Big\}.
\end{aligned}
\tag{2.28}
$$

From this one can calculate the differential response

$$
\Delta S_{1,\mathrm{YM}} = 2ig(\epsilon\cdot k)(t^a)_{fi}\,\frac{g^2}{(4\pi)^2}\,\frac{\Delta\xi}{\xi^2}\,C_2(G)\left[-1 + \tfrac{1}{4}\ln\frac{m^2}{4\pi} - \tfrac{1}{4}\ln\left(\frac{-q^2}{4\pi}\right) + \tfrac{1}{2}\ln(1-k^2/m^2) - \frac{1}{4\xi}\right]. \tag{2.29}
$$

In the next section I will show that the formalism developed by Abers and Lee gives Eq. (2.29) provided all contributions from one-particle irreducible graphs in Fig. 2 are included.

The charge renormalization constant is

$$Z_1^{-1}=\epsilon_\mu V_\mu[2ig(\epsilon\cdot k)(t^a)_{fi}]^{-1}$$
$$=Z_S^{-1}+\frac{g^2C_2(G)}{4(4\pi)^2}\left[\frac{-6-2/\xi}{D-4}+\left(3+\frac{1}{\xi}\right)\Gamma'(1)+\left(-3-\frac{1}{\xi}\right)\ln\frac{m^2}{4\pi}+\left(-2+\frac{2}{\xi}\right)+\left(-6-\frac{2}{\xi}\right)\ln(1-k^2/m^2)\right].\tag{2.30}$$

This shows that $Z_1\neq Z_S$ in Yang-Mills theory under the limiting procedure of Eq. (1.3a).

S_2 can be calculated in Yang-Mills theory using the results of Eqs. (2.19), (2.21), and (2.23):

$$\epsilon_\mu V_\mu=2ig(\epsilon\cdot k)(t^a)_{fi}+O(g^5),$$
$$Z_S^{-1}=Z_1^{-1}=1+O(g^4),$$
$$Z_g^{-1}=1-\frac{g^2\Gamma(2-D/2)}{12(4\pi)^{D/2}}(m^2)^{D/2-2},\tag{2.31}$$
$$S_{2,\mathrm{YM}}=2ig(\epsilon\cdot k)(t^a)_{fi}\times\left[1-\frac{g^2\Gamma(2-D/2)}{12(4\pi)^{D/2}}(m^2)^{D/2-2}\right].$$

The contributions of π_C and π_D vanish under dimensional regularization, that is, at $q=0$ and $D>4$. The proper vertex graph $\epsilon_\mu V_{\mu,D}$ just compensates for the new coefficients multiplying $\epsilon_\mu V_{\mu,A\ldots C}$, and as in QED there is no net third-order correction to $\epsilon_\mu V_\mu$. Evidently $S_{2,\mathrm{YM}}$ is gauge independent, and $Z_1=Z_S$ under the limiting procedure of Eq. (1.3b).

III. DIFFERENTIAL CHANGE OF GAUGE

The formalism developed by Abers and Lee permits one to calculate the response ΔS_1 to a gauge change $\Delta\xi$. In general

$$\Delta S_1=S_1\left[\frac{\Delta Z_S^{1/2}(k_1^2)}{Z_S^{1/2}(k_1^2)}+\frac{\Delta Z_S^{1/2}(k_2^2)}{Z_S^{1/2}(k_2^2)}+\frac{\Delta Z_g^{1/2}(q^2)}{Z_g^{1/2}(q^2)}+\frac{\epsilon_\mu\Delta V_\mu}{\epsilon_\mu V_\mu}\right].\tag{3.1}$$

To third order this becomes

$$\Delta S_1=2ie(\epsilon\cdot k_1)[\tfrac12\Delta Z_S(k_1^2)+\tfrac12\Delta Z_S(k_2^2)+\tfrac12\Delta Z_g(q^2)]+\epsilon_\mu\Delta_\mu.\tag{3.2}$$

In order to use the Abers-Lee formalism, the proper vertex must be related to the three-point function

$$\epsilon_\mu G_\mu^{(3)}=G_S(k_1^2)G_S)(k_2^2)\epsilon_\mu G_g(q^2)_{\mu\nu}V_\nu$$
$$=G_S(k_1^2)G_S(k_2^2)\frac{-i}{q^2(1-\sigma)}\epsilon_\mu V_\mu.\tag{3.3}$$

This leads to the relation between responses

$$\epsilon_\mu\Delta G_\mu^{(3)}=\epsilon_\mu G_\mu^{(3)}\left[\frac{\Delta G_S(k_1^2)}{G_S(k_1^2)}+\frac{\Delta G_S(k_2^2)}{G_S(k_2^2)}+(1-\sigma)\Delta(1-\sigma)^{-1}+\frac{\epsilon_\mu\Delta V_\mu}{\epsilon_\mu V_\mu}\right].\tag{3.4}$$

To third order

$$\epsilon_\mu\Delta V_\mu=[-i(k_1^2-m^2)][-i(k^2-m^2)](iq^2)\epsilon_\mu\Delta G_\mu^{(3)}$$
$$-2e(\epsilon\cdot k_1)(k_1^2-m^2)\Delta G_S(k_1^2)$$
$$-2e(\epsilon\cdot k_1)(k_2^2-m^2)\Delta G_S(k_2^2)$$
$$-2ie(\epsilon\cdot k_1)\Delta\sigma\tag{3.5}$$

and

$$\Delta S_1=[-i(k_1^2-m^2)][-i(k_2^2-m^2)](iq^2)\epsilon_\mu\Delta G_\mu^{(3)}$$
$$+2ie(\epsilon\cdot k_1)[\tfrac12\Delta Z_S(k_1^2)+i(k_1^2-m^2)\Delta G_S(k_1^2)]$$
$$+2ie(\epsilon\cdot k_1)[\tfrac12\Delta Z_S(k_2^2)+i(k_2^2-m^2)\Delta G_S(k_2^2)]$$
$$+2ie(\epsilon\cdot k_1)[\tfrac12\Delta Z_g(q^2)-\Delta\sigma].\tag{3.6}$$

The substitution $e\to g(t^a)_{fi}$ gives the analogous formula for Yang-Mills theory.

A. Electrodynamics

In electrodynamics the only modified coupling of Eq. (1.4) which contributes to the S matrix connects the scalar-meson current to the scalar meson and photon fields; its momentum-space form is given in Appendix A. The factor $1/k^2$ in the modified coupling generates the infrared singularities. The second-order shift in the scalar-meson propagator is given by the diagrams in Fig. 8:

$$\Delta G_S=\frac{i}{k^2-m^2}\frac{ie^2\Delta\xi}{\xi^2}(2k_\mu I_\mu^{12}-I^{11})$$
$$=\frac{i}{k^2-m^2}\frac{\Delta\xi}{\xi^2}\frac{e^2}{(4\pi)^2}\left[\frac{-2}{D-4}+\Gamma'(1)-\ln\frac{m^2}{4\pi}-2\ln(1-k^2/m^2)\right].\tag{3.7}$$

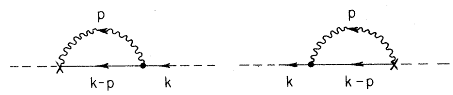

FIG. 8. Diagrams giving the gauge shift in the scalar propagator.

By Eqs. (2.7) and (2.8), to second order

$$\Delta Z_S = \frac{\partial}{\partial k^2}\Delta(\Sigma_B - \delta m^2)$$
$$= -i\,\frac{\partial}{\partial k^2}(k^2 - m^2)^2 \Delta G_S \;. \qquad (3.8)$$

Using Eq. (3.7),

$$\Delta Z_S = \frac{\Delta\xi}{\xi^2}\frac{e^2}{(4\pi)^2}\left[\frac{-2}{D-4} + \Gamma'(1) - \ln\frac{m^2}{4\pi} - 2 - 2\ln(1 - k^2/m^2)\right]. \qquad (3.9)$$

This result can be obtained from Eq. (2.8) by differentiation, showing that the method of Abers and Lee is borne out in practical calculations. There is no shift in Z_g and the transverse photon propagator through second order.

The first term in Eq. (3.6) contributes to S_1 only when there are poles in all three legs of $\epsilon_\mu \Delta G_\mu^{(3)}$. Evidently, the diagrams of Figs. 9(A) and 9(B) contribute and give terms which can be read off Eq. (3.7)

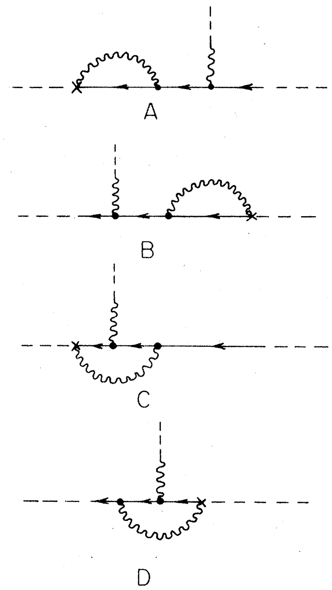

FIG. 9. Diagrams contributing to the gauge shift in $\epsilon_\mu G_\mu^{(3)}$ in electrodynamics.

$$\epsilon_\mu(\Delta G_{\mu,A}^{(3)} + \Delta G_{\mu,B}^{(3)}) = \frac{i}{k_1^2 - m^2}\,\frac{i}{k_2^2 - m^2}\,\frac{-i}{q^2}\,\frac{2ie^3(\epsilon\cdot k_1)}{(4\pi)^2} \times \frac{\Delta\xi}{\xi^2}\left[\frac{-2}{D-4} + \Gamma'(1) - \ln\frac{m^2}{4\pi} - \ln(1 - k_1^2/m^2) - \ln(1 - k_2^2/m^2)\right]. \qquad (3.10)$$

The irreducible diagrams of Figs. 9(C) and 9(D) also contribute:

$$\epsilon_\mu \Delta G_{\mu,C}^{(3)} = \frac{i}{k_1^2 - m^2}\,\frac{-i}{q^2}\,(ie^3)\,\frac{\Delta\xi}{\xi^2}\left[-2(\epsilon\cdot k)k_\mu I_\mu^{22} + (\epsilon\cdot k)I^{21} + 2\epsilon_\mu k_\nu I_{\mu\nu}^{22} - \epsilon_\mu I_\mu^{21}\right]. \qquad (3.11)$$

As in the diagram of Fig. 4(C), we can let $q=0$ before evaluating the diagram at $k_1^2 = k_2^2 = k^2 \neq m^2$. The only term in Eq. (3.11) which develops the pole at $k^2 = m^2$ is I_μ^{22}, giving the result

$$\epsilon_\mu(\Delta G_{\mu,C}^{(3)} + \Delta G_{\mu,D}^{(3)}) = \left(\frac{i}{k^2 - m^2}\right)^2\left(\frac{-i}{q^2}\right) 2ie(k\cdot\epsilon)\,\frac{e^2}{(4\pi)^2}\,\frac{2\Delta\xi}{\xi^2}\;. \qquad (3.12)$$

All other irreducible diagrams which can be constructed do not contribute to Eq. (3.6). Altogether, terms developing poles in all three channels are

$$\epsilon_\mu \Delta G_\mu^{(3)} = \left(\frac{i}{k^2 - m^2}\right)^2\left(\frac{-i}{q^2}\right) 2ie(\epsilon\cdot k)\,\frac{e^2}{(4\pi)^2}\,\frac{\Delta\xi}{\xi^2}\left[\frac{-2}{D-4} + \Gamma'(1) - \ln\frac{m^2}{4\pi} + 2 - 2\ln(1 - k^2/m^2)\right]. \qquad (3.13)$$

Equations (3.6)–(3.8) and (3.12) now give

$$\Delta S_1 = 0\;. \qquad (3.14)$$

The result $\Delta S_1 = 0$ agrees with Sec. II, but it is surprising that an irreducible diagram contributes to ΔS_1. Recall that the argument of Abers and Lee implies that $\Delta S_3 = 0$ when irreducible diagrams do *not* contribute. The point to observe is that $S_1 \neq S_3$ in electrodynamics. The condition for $S_1 = S_3$ is

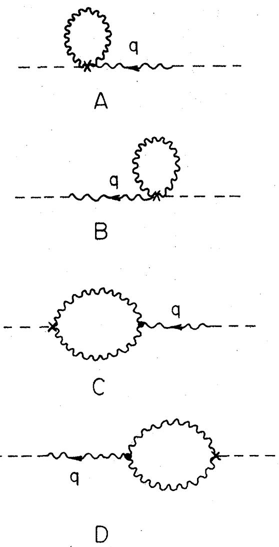

FIG. 10. Diagrams contributing to the gauge shift in the meson propagator in non-Abelian gauge theory.

$$Z_S(k^2)^{-1/2}(k^2-m^2)G_S(k^2) \underset{k^2\to m^2}{\sim} Z_S(k^2)^{1/2}\,. \tag{3.15}$$

This condition is not satisfied at $D=4$ owing to the infrared logarithms in G_S and Z_S. Therefore, S_3 is gauge dependent in electrodynamics, even though S_1 and S_2 are gauge independent.

B. Yang-Mills theory

In Yang-Mills field theory there is an infinite perturbation series of modified couplings to both scalar- and gauge-meson currents. The relevant terms in this series are given in Appendix A.

The calculation of the shift in the gauge-meson propagator $G_g(q)_{\mu\nu}$ is simplified by noting that in the S matrix the propagator is contracted with ϵ_μ, and $\epsilon\cdot q=0$. This means that only propagator shifts proportional to $g_{\mu\nu}$ need be calculated (the diagrams are shown in Fig. 10):

$$\begin{aligned}\Delta G_{g,\mu\nu,A}+\Delta G_{g,\mu\nu,B} &= \delta_{ab}C_2(G)\left(\frac{-i}{q^2}\right)\frac{ig^2\Delta\xi}{\xi^4}\,\tilde I^{12}_{\mu\nu}\\ &= \delta_{ab}C_2(G)g_{\mu\nu}\frac{\Delta\xi}{\xi^2}\frac{q^2}{(4\pi)^2}\left(\frac{-i}{q^2}\right)\left[\frac{1/2}{D-4}-\tfrac{1}{2}-\tfrac{1}{4}\Gamma'(1)+\tfrac{1}{4}\ln\left(\frac{-q^2}{4\pi}\right)\right]+q_\mu q_\nu \text{ terms},\\ \Delta G_{g,\mu\nu,C}+\Delta G_{g,\mu\nu,D} &= \delta_{ab}C_2(G)\left(\frac{-i}{q^2}\right)\frac{ig^2\Delta\xi}{\xi^2}\left[g_{\mu\nu}q^2\tilde I^{21}+\tilde I^{21}_{\mu\nu}-\left(1-\frac{1}{\xi}\right)q^2\tilde I^{22}_{\mu\nu}\right]\\ &= \delta_{ab}C_2(G)g_{\mu\nu}\frac{\Delta\xi}{\xi^2}\frac{g^2}{(4\pi)^2}\left(\frac{-i}{q^2}\right)\left[\frac{-3/2}{D-4}+\tfrac{3}{4}\Gamma'(1)-\tfrac{3}{4}\ln\left(\frac{-q^2}{4\pi}\right)-\frac{1}{2\xi}\right]+q_\mu q_\nu \text{ terms}.\end{aligned} \tag{3.16}$$

From Eq. (2.12),

$$\Delta G_{g,\mu\nu}=\frac{-i\delta_{ab}g_{\mu\nu}\Delta\sigma}{q^2(1-\sigma)^2}+q_\mu q_\nu \text{ terms} \tag{3.17}$$

so to second order

$$\Delta\sigma=C_2(G)\frac{\Delta\xi}{\xi^2}\frac{g^2}{(4\pi)^2}\left[\frac{-1}{D-4}-\tfrac{1}{2}+\tfrac{1}{2}\Gamma'(1)-\tfrac{1}{2}\ln\left(\frac{-q^2}{4\pi}\right)-\frac{1}{2\xi}\right],$$

$$\begin{aligned}\Delta Z_g &= \frac{\partial}{\partial q^2}(q^2\Delta\sigma)\\ &= C_2(G)\frac{\Delta\xi}{\xi^2}\frac{g^2}{(4\pi)^2}\left[\frac{-1}{D-4}-1+\tfrac{1}{2}\Gamma'(1)-\tfrac{1}{2}\ln\left(\frac{-q^2}{4\pi}\right)-\frac{1}{2\xi}\right].\end{aligned} \tag{3.18}$$

The shift Z_g can be verified by differentiating Eq.

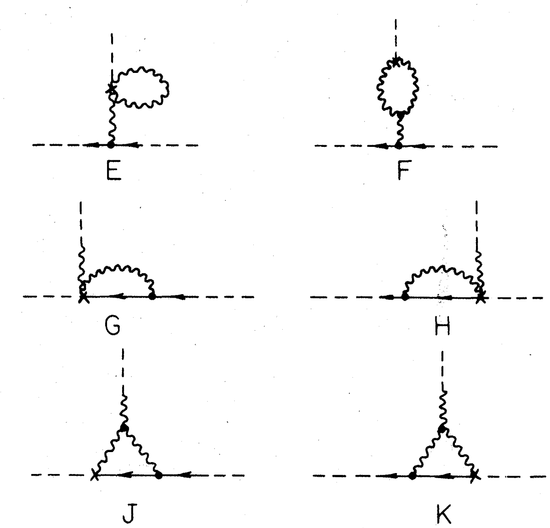

FIG. 11. Additional diagrams contributing to the gauge shift in $\epsilon_\mu G^{(3)}_\mu$ in non-Abelian gauge theory.

(2.27).

The shift in the scalar-meson propagator is given by the diagrams of Fig. 8, with the replacement $e^2 \to C_2(R)g^2$. Since this agrees with Eq. (2.19), ΔG_S and ΔZ_S are correctly given in Yang-Mills theory by the formalism of Abers and Lee.

The response $\epsilon_\mu G_\mu^{(3)}$ of the three-point function is determined by the diagrams of Fig. 9, which also occur in electrodynamics, plus those of Fig. 11. All other irreducible diagrams turn out to give no contribution to Eq. (3.6). Diagrams 9(A) and 9(B) are given by Eq. (3.10) with the replacement $e^3 \to g^3(t^a)_{fi}C_2(R)$, and diagrams 9(C) and 9(D) are given by Eq. (3.12) with the replacement $e^3 \to g^3(t^a)_{fi}\,[C_2(R)-\frac{1}{2}C_2(G)]$. The contributions of diagrams 11(E) and 11(F) can be obtained from Eq. (3.16) by multiplying by $\frac{1}{2}[i/(k^2-m^2)]^2\,2ig \times(\epsilon\cdot k)(t^a)_{fi}$ and omitting $\delta_{ab}g_{\mu\nu}$.

The irreducible diagrams 11(G)–11(K) must be evaluated setting $D=4$ before letting q^2 and then k^2-m^2 become small. I therefore use the integrals of Eq. (2.20):

$$\epsilon_\mu \Delta G^{(3)}_{\mu,G} = \left(\frac{i}{k^2-m^2}\right)\left(\frac{-i}{q^2}\right)\frac{(-ig^3\Delta\xi)}{\xi}(t^a)_{fi}C_2(G)\left[\frac{1}{2\xi}\epsilon_\mu k_\nu \hat{I}^{12}_{\mu\nu} - \frac{1}{4\xi}\epsilon_\mu \hat{I}^{11}_\mu - \frac{1}{2\xi q^2}\epsilon_\mu k_\nu \hat{I}^{11}_{\mu\nu} + \frac{1}{2\xi q^2}\epsilon_\mu \hat{I}^{10}_\mu - \frac{k\cdot q}{2q^2}\epsilon_\mu \hat{I}^{11}_\mu + \frac{\epsilon_\mu q_\nu}{4q^2}\hat{I}^{11}_{\mu\nu} + \frac{1}{2q^2}\left(1-\frac{1}{\xi}\right)\epsilon_\mu k_\nu q_\sigma \hat{I}^{12}_{\mu\nu\sigma} - \frac{1}{2q^2}\left(1-\frac{1}{\xi}\right)\epsilon_\mu q_\nu \hat{I}^{11}_{\mu\nu}\right]. \quad (3.19)$$

The only integral which has the requisite pole at $k^2=m^2$ is $\hat{I}^{12}_{\mu\nu}$. In Appendix B I show that for small q^2

$$\hat{I}^{12}_{\mu\nu} = \frac{i}{(4\pi)^2(m^2-k^2)}\left[\frac{k_\mu k_\nu}{2m^2} - \frac{g_{\mu\nu}}{2} + g_{\mu\nu}\ln\left(\frac{-q^2}{m^2}\right) - \frac{g_{\mu\nu}}{2}\ln(1-k^2/m^2)\right]. \quad (3.20)$$

The resulting shift is

$$\epsilon_\mu \Delta G^{(3)}_{\mu,G} + \epsilon_\mu \Delta G^{(3)}_{\mu,H} = \left(\frac{i}{k^2-m^2}\right)^2\left(\frac{-i}{q^2}\right)2ig(\epsilon\cdot k)(t^a)_{fi}\frac{g^2}{(4\pi)^2} \times \frac{\Delta\xi}{\xi^2}\frac{C_2(G)}{2}\left[\frac{1}{4}\ln\left(\frac{-q^2}{4\pi}\right) - \frac{1}{4}\ln\frac{m^2}{4\pi} - \frac{1}{2}\ln(1-k^2/m)\right]. \quad (3.21)$$

Similar care must be taken in evaluating diagram J. Initially there are 43 terms. A number of these drop out at $q^2=0$, and those that remain can be combined into four having a pole at $k^2=m^2$:

$$\epsilon_\mu \Delta G^{(3)}_{\mu,J} = \frac{i}{k^2-m^2}\left(\frac{-i}{q^2}\right)(-ig^3)\frac{C_2(G)}{2}\frac{\Delta\xi}{\xi^2}\left[\epsilon_\mu k_\nu \hat{I}^{12}_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\epsilon_\mu k_\nu q^2 \hat{I}^{22}_{\mu\nu} + (\epsilon\cdot k)q^2\hat{I}^{12} - (\epsilon\cdot k)\hat{I}^{02}\right]. \quad (3.22)$$

Evaluating these integrals,

$$\epsilon_\mu \Delta G^{(3)}_{\mu,J} + \epsilon_\mu \Delta G^{(3)}_{\mu,K} = \left(\frac{i}{k^2-m^2}\right)^2\left(\frac{-i}{q^2}\right)2ig(\epsilon\cdot k)(t^a)_{fi}\frac{g^2}{(4\pi)^2} \times \frac{\Delta\xi}{\xi^2}\frac{C_2(G)}{8}\left[-3\ln\left(\frac{-q^2}{4\pi}\right) + 3\ln\frac{m^2}{4\pi} + 6\ln(1-k^2/m^2) + 2 - \frac{2}{\xi}\right]. \quad (3.23)$$

The total shift in the three-point function contributing to Eq. (3.6) is

$$\epsilon_\mu \Delta G^{(3)}_\mu = \left(\frac{i}{k^2-m^2}\right)^2\left(\frac{-i}{q^2}\right)2ig(\epsilon\cdot k)(t^a)_{fi}\frac{g^2}{(4\pi)^2}\frac{\Delta\xi}{\xi^2} \times \left\{C_2(R)\left[\frac{-2}{D-4} + \Gamma'(1) - \ln\frac{m^2}{4\pi} + 2 - 2\ln(1-k^2/m^2)\right] + \frac{C_2(G)}{2}\left[\frac{-1}{D-4} + \frac{1}{2}\Gamma'(1) + \frac{1}{2}\ln\frac{m^2}{4\pi} - \ln\left(\frac{-q^2}{4\pi}\right) + \ln(1-k^2/m^2) - 2 - \frac{1}{\xi}\right]\right\}. \quad (3.24)$$

Collecting results, Eq. (3.6) now reproduces Eq. (2.29).

These calculations verify that irreducible diagrams contribute to the Abers-Lee formula for the gauge shift $\Delta S_{1,\mathrm{YM}}$. Because they do, we understand how S_1 can be gauge dependent. The specific value of the irreducible contributions gives the shift $\Delta S_{1,\mathrm{YM}}$ found by direct calculation in Sec. II.

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APPENDIX A: MODIFIED SOURCE COUPLINGS IN THE GAUGE $\xi+\Delta\xi$

According to Abers and Lee, Eq. (22.18), the field coupled to source current J_B is

$$J_B[\phi_B-(\Gamma^C_{BD}\phi_D+\Lambda^C_B)(M_F^{-1})_{CE}\Delta F_E(\phi)]. \tag{A1}$$

ϕ_B is a generic symbol for a field, B is a collection of group and space-time indices, M_F is the Faddeev-Popov determinant, and ΔF is the change in the gauge-fixing function. The group functions Γ and Λ are defined by Abers and Lee.[16]

1. Electrodynamics

The gauge-fixing term is

$$F_E(\phi)=\sqrt{\xi}\int d^4z\,\delta(x-z)\frac{\partial}{\partial z_\mu}A_\mu(z),\; E=\{x\}. \tag{A2}$$

For the electromagnetic current,

$$\Gamma=0,\; \Lambda^C_B=-\frac{1}{g}\delta(z-y)\frac{\partial}{\partial y_\mu},$$
$$C=\{y\},\; B=\{\mu,z\}. \tag{A3}$$

Thus the Faddeev-Popov determinant is

$$(M_F)_{EC}=\frac{\delta F_E}{\delta A_B}\Lambda^C_B=-\frac{\sqrt{\xi}}{g}\delta(x-y)\frac{\partial^2}{\partial y_\mu{}^2},$$
$$(M_F^{-1})_{CE}=-\frac{g}{\sqrt{\xi}}\delta(y-x)\frac{1}{\partial^2/\partial x_\sigma{}^2}. \tag{A4}$$

Since

$$\Delta F_E(\phi)=\frac{\Delta\xi}{2\sqrt{\xi}}\frac{\partial A_\sigma}{\partial x_\sigma}, \tag{A5}$$

the incremental field coupled to the electromagnetic current is

$$-\Lambda^C_B(M_F^{-1})_{CE}\Delta F_E(\phi)=-\frac{\Delta\xi}{2\xi}\frac{\partial}{\partial z_\mu}\frac{1}{\partial^2/\partial z_\tau{}^2}\frac{\partial A_\sigma}{\partial z_\sigma}. \tag{A6}$$

This is a gradient, so it does not contribute to momentum-space Green's functions when photon legs are contracted with a polarization vector. As a result, modification of the electromagnetic source current coupling can be ignored when calculating S-matrix elements.

For the scalar-meson current

$$\Gamma^C_B=-i\delta(z-y),\;\Lambda=0,\; B=\{z\},\; C=\{y\}. \tag{A7}$$

The incremental field coupled to the scalar-meson current J^+ is

$$-\Gamma^C_B\phi_B(M_F^{-1})_{CE}\Delta F_E(\phi)=\frac{-ig\Delta\xi}{2\xi}\phi(z)\frac{1}{\partial^2/\partial z_\tau{}^2}\frac{\partial A_\sigma}{\partial z_\sigma}. \tag{A8}$$

The coupling in momentum space is shown in Fig. 12(A). It can be read off Eq. (A8) by the replacement $\partial/\partial x_\mu\to -ik_\mu$ for an incoming photon, and is

$$\frac{g\Delta\xi}{2\xi}\frac{k_{3\mu}}{k^2}. \tag{A9}$$

2. Yang-Mills theory

The gauge-fixing function is

$$F_E(\phi)=\sqrt{\xi}\sum_d\int d^4z\,\delta(x-z)\delta_{ad}\frac{\partial}{\partial z_\sigma}A^d_\sigma(z),$$
$$E=\{x,a\}. \tag{A10}$$

For the gauge-meson current

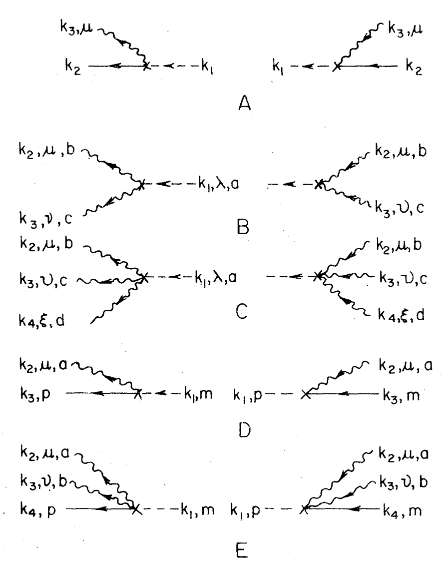

FIG. 12. New current couplings in the gauge $\xi+\Delta\xi$.

$$\Gamma^C_{BD}=g_{\sigma\nu}\delta(z-y)C_{dbc},$$
$$\Lambda^C_B=-\frac{1}{g}\delta_{db}\delta(z-y)\frac{\partial}{\partial y_\sigma}, \tag{A11}$$
$$B=\{z,d,\sigma\},\ C=\{y,b\},\ D=\{y,c,\nu\}.$$

Then

$$(M_F)_{BC}=\frac{\delta F_E}{\delta A_B}(\Gamma^C_{BD}A_D+\Lambda^C_B)$$
$$=\sqrt{\xi}\delta(x-y)\,c_{abc}\frac{\partial}{\partial y_\nu}A^c_\nu(y)-\frac{\sqrt{\xi}}{g}\delta(x-y)\delta_{ab}\frac{\partial^2}{\partial {y_\nu}^2}. \tag{A12}$$

In the first term, $\partial/\partial y_\nu$ operates on both A^c_ν and the function to which M_F is applied. As a preparation to inversion, write this

$$(M_F)_{BC}=\sum_d\int d^4z\left[\frac{-\sqrt{\xi}}{g}\delta(x-z)\frac{\partial^2}{\partial {z_\sigma}^2}\right]\left[\delta_{db}\delta(z-y)-g\,c_{dbc}\delta(z-y)\frac{1}{\partial^2/\partial {y_\tau}^2}\frac{\partial}{\partial y_\nu}A^c_\nu(y)\right]. \tag{A13}$$

The second term is inverted as a power series in g. The terms needed are

$$(M_F{}^{-1})_{CE}=-\frac{g}{\sqrt{\xi}}\delta_{ba}\delta(y-x)\frac{1}{\partial^2/\partial {x_\sigma}^2}-\frac{g^2}{\sqrt{\xi}}\delta(y-x)\,c_{bac}\frac{1}{\partial^2/\partial {x_\sigma}^2}\frac{\partial}{\partial x_\nu}A^c_\nu(x)\frac{1}{\partial^2/\partial {x_\tau}^2}$$
$$-\frac{g^3}{\sqrt{\xi}}\delta(y-x)\,c_{bdc}c_{da\bar{e}}\frac{1}{\partial^2/\partial {x_\sigma}^2}\frac{\partial}{\partial x_\nu}A^c_\nu(x)\frac{1}{\partial^2/\partial {x_\tau}^2}\frac{\partial}{\partial x_{\bar\nu}}A^{\bar e}_{\bar\nu}(x)\frac{1}{\partial^2/\partial {x_\phi}^2}. \tag{A14}$$

Derivatives operate on all terms to the right. The incremental coupling to the electromagnetic current is now obtained from Eq. (A1). The couplings are shown in Figs. 12(B) and 12(C). All gradient terms have been dropped in the following expressions:

$$12\text{B}:\ \pm\frac{ig\Delta\xi}{2\xi}c_{abc}\left(g_{\lambda\mu}\frac{k_{3\nu}}{{k_3}^2}-g_{\lambda\nu}\frac{k_{2\mu}}{{k_2}^2}\right),$$
$$12\text{C}:\ \frac{g^2\Delta\xi}{2\xi}c_{aeb}c_{ecd}g_{\lambda\mu}\left[\frac{(k_3+k_4)_\nu k_{4\xi}}{(k_3+k_4)^2{k_4}^2}-\frac{(k_3+k_4)_\xi k_{3\nu}}{(k_3+k_4)^2{k_3}^2}\right]+\frac{g^2\Delta\xi}{2\xi}c_{aec}c_{edb}g_{\lambda\nu}\left[\frac{(k_2+k_4)_\xi k_{2\mu}}{(k_2+k_4)^2{k_2}^2}-\frac{(k_2+k_4)_\mu k_{4\xi}}{(k_2+k_4)^2{k_4}^2}\right] \tag{A15}$$
$$+\frac{g^2\Delta\xi}{2\xi}c_{aed}c_{ebc}g_{\lambda\xi}\left[\frac{(k_2+k_3)_\mu k_{3\nu}}{(k_2+k_3)^2{k_3}^2}-\frac{(k_2+k_3)_\nu k_{2\mu}}{(k_2+k_3)^2{k_2}^2}\right].$$

The + (−) sign applies to the left (right) coupling.

For the scalar-meson current,

$$\Gamma^C_{BD}=-i\delta(z-y)(t^b)_{pm},$$
$$\Lambda^C_B=0, \tag{A16}$$
$$B=\{z,p\},\ C=\{y,b\},\ D=\{y,m\}.$$

The couplings are shown in Figs. 12(D) and 12(E):

$$12\text{D}:\ \frac{g\Delta\xi}{2\xi}(t^a)_{pm}\frac{k_{2\mu}}{{k_2}^2},$$
$$12\text{E}:\ \pm\frac{ig^2\Delta\xi}{2\xi}c_{abc}(t^a)_{pm}\left[\frac{(k_2+k_3)_\mu k_{3\nu}}{(k_2+k_3)^2{k_3}^2}-\frac{(k_2+k_3)_\nu k_{2\mu}}{(k_2+k_3)^2{k_2}^2}\right]. \tag{A17}$$

APPENDIX B

Integrals are evaluated by using Feynman parametrization to combine denominators. The loop momentum integration can be performed after this is done, leaving the expressions

$$I^{mn}_\alpha=\frac{i(-1)^{m+n}\Gamma(m+n-D/2)}{(m-1)!\,(n-1)!\,(4\pi)^{D/2}}\int_0^1 dx\,x^{D/2-n}(1-x)^{n-1}(m^2-k^2+k^2x)^{D/2-m-n}f_\alpha, \tag{B1}$$

where α stands for a collection of tensor indices, and f_α is a corresponding factor in the integrand. These factors are

$$I^{mn}:\ f=x^{-1};\quad I^{mn}_\mu;\ f_\mu=k_\mu;$$
$$I^{mn}_{\mu\nu}:\ f_{\mu\nu}=k_\mu k_\nu x-\frac{g_{\mu\nu}(m^2-k^2+k^2x)}{2(m+n-1-D/2)}; \tag{B2}$$
$$I^{mn}_{\mu\nu\sigma}:\ f_{\mu\nu\sigma}=k_\mu k_\nu k_\sigma x^2-\frac{(g_{\mu\nu}k_\sigma+g_{\nu\sigma}k_\mu+g_{\sigma\mu}k_\nu)x(m^2-k^2+k^2x)}{2(m+n-1-D/2)}.$$

In order to calculate S_2, these integrals and their derivatives are evaluated at $k^2=m^2$, with the results

$$I^{mn}_\alpha=\frac{i(-1)^{m+n}\Gamma(m+n-A-D/2)\Gamma(D-m-2n+B)(m^2)^{D/2-m-n}F_\alpha}{(m-1)!\,(4\pi)^{D/2}\Gamma(D-m-n+B)}, \tag{B3}$$

where the indices and factors are

I^{mn}: $A=B=0$, $F=1$; I^{mn}_{μ}: $A=0$, $B=1$, $F_\mu = k_\mu$;

$$I^{mn}_{\mu\nu}: \; A=1,\; B=2,\; F_{\mu\nu}=(m+n-1-D/2)k_\mu k_\nu - \tfrac{1}{2}m^2 g_{\mu\nu}\,; \tag{B4}$$

$I^{mn}_{\mu\nu\sigma}$: $A=1$, $B=3$, $F_{\mu\nu\sigma}=(m+n-1-D/2)k_\mu k_\nu k_\sigma - \frac{1}{2}m^2(g_{\mu\nu}k_\sigma + g_{\nu\sigma}k_\mu + g_{\sigma\mu}k_\nu)$.

Also,

$$\frac{\partial}{\partial k^2}(I^{mn}_{\mu\nu\ldots}k_\mu k_\nu \cdots)_{m^2} = \frac{i(-1)^{m+n}\Gamma(m+n+1-B-D/2)\Gamma(D-m-2n-1+B)}{(m-1)!\,(4\pi)^{D/2}\Gamma(D-m-n+B)}(m^2)^{D/2-m-n-1+B}\bar{F}\,, \tag{B5}$$

where B is the index in Eq. (B4), and $\bar{F}$ is

I^{mn}: $\bar{F}=n$;

$$I^{mn}_{\mu}k_\mu: \; \bar{F}=D(1-n/2)+(m+n)(n-1)-n; \tag{B6}$$

$I^{mn}_{\mu\nu}k_\mu k_\nu$: $\bar{F}=(D-m-2n+1)(2m+2n-\frac{5}{2}-D/2)+n(m+n-\frac{1}{2}-D/2)(m+n-1-D/2)$.

When evaluating S_1, infrared-finite integrals can be gotten from Eq. (B3) by expanding around $D=4$. (Integrals having $m+2n-B<4$ are infrared finite at $k^2=m^2$.) Integrals which are ultraviolet convergent but divergent at $k^2=m^2$ can be evaluated by setting $D=4$ in Eq. (B1) and evaluating the integral. The required expressions are

$$\begin{aligned}
I^{21} &= \frac{i}{(4\pi)^2 m^2}\ln(1-k^2/m^2)\,,\\
k_\mu k_\nu I^{22}_{\mu\nu} &= \frac{i}{(4\pi)^2}[-\tfrac{3}{2}-\tfrac{1}{2}\ln(1-k^2/m^2)]\,,\\
I^{22}_{\mu} &= \frac{-ik_\mu}{(4\pi)^2 m^2(k^2-m^2)}\,,\\
I^{12}_{\mu} &= \frac{ik_\mu}{(4\pi)^2 m^2}[1+\ln(1-k^2/m^2)]\,,\\
\epsilon_\mu k_\nu k_\sigma I^{13}_{\mu\nu\sigma} &= \frac{i(\epsilon\cdot k)}{(4\pi)^2}[-\tfrac{1}{8}+\tfrac{1}{4}\ln(1-k^2/m^2)]\,.
\end{aligned} \tag{B7}$$

Contributions which vanish at $k^2=m^2$ have been dropped. There are further integrals which must be calculated through $O(k^2-m^2)$ to evaluate Z_S. These integrals are ultraviolet divergent at $D=4$, so the integrand of Eq. (B1) must be *expanded* around $D=4$ before integration:

$$\begin{aligned}
k^2 I^{11} &= \frac{i}{(4\pi)^2}\left\{m^2\left[\frac{-2}{D-4}+\Gamma'(1)+2-\ln\frac{m^2}{4\pi}\right]+(k^2-m^2)\left[\frac{-2}{D-4}+\Gamma'(1)+2-\ln\frac{m^2}{4\pi}-\ln\left(1-\frac{k^2}{m^2}\right)\right]\right\}\,,\\
k_\mu I^{11}_{\mu} &= \frac{i}{(4\pi)^2}\left\{m^2\left[\frac{-1}{D-4}+\tfrac{1}{2}\Gamma'(1)+\tfrac{1}{2}-\tfrac{1}{2}\ln\frac{m^2}{4\pi}\right]+(k^2-m^2)\left[\frac{-1}{D-4}+\tfrac{1}{2}\Gamma'(1)+1-\tfrac{1}{2}\ln\frac{m^2}{4\pi}\right]\right\}\,,\\
k_\mu k_\nu I^{12}_{\mu\nu} &= \frac{i}{(4\pi)^2}\left\{m^2\left[\frac{-1/2}{D-4}+\tfrac{1}{4}\Gamma'(1)+\tfrac{1}{4}-\tfrac{1}{4}\ln\frac{m^2}{4\pi}\right]+(k^2-m^2)\left[\frac{-1/2}{D-4}+\tfrac{1}{4}\Gamma'(1)+1-\tfrac{1}{4}\ln\frac{m^2}{4\pi}+\tfrac{1}{2}\ln(1-k^2/m^2)\right]\right\}\,.
\end{aligned} \tag{B8}$$

The integrals $\bar{I}^{10}=I^{10}$, $q_\mu q_\nu \bar{I}^{11}$, $q_\mu \bar{I}^{11}_{\nu}$, and $\bar{I}^{11}_{\mu\nu}$ are required to terms of $O(q^2)$ to evaluate the scalar loop contribution to the vector-meson polarization part. This is done by deriving expressions analogous to Eq. (B1), expanding in powers of q, and evaluating the integrals. These integrals have no singularity at $q=0$ for D near 4, so the resulting expressions can be used for S_1 as well as S_2. The results are

$$\begin{aligned}
\bar{I}^{10} &= \frac{-i\Gamma(1-D/2)(m^2)^{D/2-1}}{(4\pi)^{D/2}}\,,\\
q_\mu q_\nu \bar{I}^{11} &= \frac{iq_\mu q_\nu\Gamma(2-D/2)(m^2)^{D/2-2}}{(4\pi)^{D/2}}\,,\\
q_\mu \bar{I}^{11}_{\nu} &= \frac{iq_\mu q_\nu\Gamma(2-D/2)(m^2)^{D/2-2}}{2(4\pi)^{D/2}}\,,\\
\bar{I}^{11}_{\mu\nu} &= \frac{-ig_{\mu\nu}\Gamma(1-D/2)(m^2)^{D/2-1}}{2(4\pi)^{D/2}} - i\left(\frac{q^2}{4}g_{\mu\nu}-q_\mu q_\nu\right)\frac{\Gamma(2-D/2)(m^2)^{D/2-2}}{3(4\pi)^{D/2}}\,.
\end{aligned} \tag{B9}$$

The integrals $\tilde{I}^{11}$, $\tilde{I}^{11}_{\mu}$, $\tilde{I}^{11}_{\mu\nu}$, $\tilde{I}^{12}_{\mu\nu}$, $\tilde{I}^{22}_{\mu\nu}$, and $\tilde{I}^{12}_{\mu\nu\sigma}$ are required to evaluate the gluon and ghost loop contributions to $\Delta G_{g,\mu\nu}$ in Yang-Mills theory. These terms can be obtained by setting $m^2=0$ in Eq. (B1); for $m^2=0$ the integral over x can be evaluated as a β function. The contributions all vanish for S_2, where one sets $q^2=0$ for $D>4$.

The final class of integrals to be considered is the $\hat{I}$ of Eq. (2.20). These integrals require two Feynman parameters. The discussion of all cases is lengthy, and I illustrate the procedure by calculating $\hat{I}^{12}$, which contributes to Eq. (3.22). Introducing Feynman parameters,

$$\hat{I}^{mn}=\frac{i(-1)^{m+n+1}\Gamma(m+n+1-D/2)}{(m-1)!\,(n-1)!\,(4\pi)^{D/2}} \times\int_0^1 dx\int_0^1 du\, x^{m-1}(1-x)^{D/2-m-1}(1-u)^{n-1}[u^2(1-x)k^2+u(m^2-k^2-2xq\cdot k)-xq^2]^{D/2-m-n-1}. \tag{B10}$$

Write the factor $1-u$ as

$$1-u=\frac{2u(1-x)k^2+m^2-k^2-2xq\cdot k}{m^2-k^2-2xq\cdot k}-\frac{u[2(1-x)k^2+m^2-k^2-2xq\cdot k]}{m^2-k^2-2xq\cdot k}. \tag{B11}$$

Inserting this decomposition, the first term can be integrated over u:

$$\hat{I}^{12}=\frac{i\Gamma(3-D/2)}{(4\pi)^{D/2}}\int_0^1\frac{dx(1-x)^{D/2-2}}{m^2-k^2-2xq\cdot k}[(-q^2x)^{D/2-3}-(m^2-k^2x-2xq\cdot k-q^2x)^{D/2-3}]$$
$$-\frac{i\Gamma(4-D/2)}{(4\pi)^{D/2}}\int_0^1\frac{dx(1-x)^{D/2}}{m^2-k^2-2xq\cdot k}[2(1-x)k^2+m^2-k^2-2xq\cdot k]\int_0^1 du\,u[u^2(1-x)k^2+u(m^2-k^2-2xq\cdot k)-xq^2]^{D/2-4}. \tag{B12}$$

At $D=4$ the last integral is at most logarithmically divergent at $q^2=0$, whereas in Eq. (3.22) it is $q^2\hat{I}^{12}$ which appears. Therefore at small q^2 this term can be ignored. Likewise, the second term in the braces can be dropped. Letting $q\to 0$ before $k^2\to m^2$, the significant term in $\hat{I}^{12}$ is

$$\hat{I}^{12}=\frac{i\Gamma(3-D/2)\Gamma(D/2-2)\Gamma(D/2-1)}{(4\pi)^{D/2}(m^2-k^2)\Gamma(D-3)}(-q^2)^{D/2-3}. \tag{B13}$$

Expanding around $D=4$,

$$\hat{I}^{12}=\frac{i}{(4\pi)^2q^2(k^2-m^2)}\left[\frac{2}{D-4}-\Gamma'(1)+\ln\left(\frac{-q^2}{4\pi}\right)\right]. \tag{B14}$$

The other integrals in Eq. (3.22) are $\hat{I}^{02}=I^{12}$ and

$$\hat{I}^{12}_{\mu\nu}=\frac{i}{(4\pi)^2(k^2-m^2)}\left\{g_{\mu\nu}\left[\tfrac{1}{2}+\tfrac{1}{2}\ln(1-k^2/m^2)-\tfrac{1}{4}\ln\left(\frac{-q^2}{4\pi}\right)\right]-\frac{k_\mu k_\nu}{2m^2}\right\},$$
$$\hat{I}^{22}_{\mu\nu}=\frac{ig_{\mu\nu}}{(4\pi)^2q^2(k^2-m^2)}+\text{terms not contributing to (3.22)}. \tag{B15}$$

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