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Strong-coupling expansion in the large-N limit

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We study a strong-coupling expansion of a φ^4 quantum field theory with an O(N) internal symmetry. For large values of N we observe that there is a critical point in $2 < D < 4$ dimensions and recover the known results for the two-point and the four-point Green's functions near the critical point, and for the critical exponent.

I. INTRODUCTION

The advent of quantum chromodynamics[1] as a very plausible candidate for a realistic theory of the strong interactions has, in the recent past, provoked increased interest in the development of methods for the solution of strong-coupling field theories. While most successful efforts have been directed toward the study of lattice field theories,[2] the continuum theory presents[5] the advantages of being explicitly causal and Lorentz invariant and therefore is worthy of further studies.

Recently, methods for expanding the functional integral for the generating functional $W(J)$ of several field theories in inverse powers of the coupling constant have been suggested and advanced by several authors.[3-6] We use the method developed independently in Refs. 5 and 6. Although the method has been applied successfully only to the computation of the energy eigenvalues of the anharmonic oscillator in one dimension, the general formalism is particularly simple and therefore attractive.

In this paper we study a φ^4 field theory with an O(N) internal symmetry and develop an expansion of this theory in inverse powers of the coupling constant for the lowest order in $1/N$. The main motivation behind our approach is the success of the $1/N$ expansion in providing a reasonable approximation to several field theory problems. In particular, the renormalization-group functions and the critical indices of the scalar φ^4 theory have been calculated, to zeroth order in $1/N$, in ordinary perturbation theory, a calculation which is valid for any value of g.[7,8] Therefore, those results provide a direct check of the strong-coupling method.

We begin by deriving Feynman rules for the propagator and vertices of the theory. These are obtained by expanding the kinetic energy term in the Lagrangian in a power series in $1/\sqrt{g}$, while the functional integral of the interaction terms is computed exactly. The $1/N$ expansion is next developed and we show that, to zeroth order in $1/N$, the class of diagrams to be summed is tremendously simplified. Vertices and propagators conspire in such a way as to make this expansion a series in powers of $1/\sqrt{gN}$.

The series is then re-arranged to obtain an expression for the mass renormalization term. We find that the theory has a critical point in $2<D<4$ dimensions and compute the critical indices and the renormalized coupling constant. In order to evaluate the vertex functions, which are functional integrals giving the moments of

$$\exp\left[-\left(\sum_1^N \varphi_i^2\right)^2\right],$$

a lattice in D-dimensional space is introduced only as a computational tool and not as a basic ingredient of the theory. A simple cutoff method of regulating the theory is devised which allows us to return to the continuum and eventually obtain results independent of the cutoff.

II. FEYNMAN RULES FOR THE $1/\sqrt{g}$ EXPANSION

Detailed derivations of the expansion in powers of $1/\sqrt{g}$ can be found in Refs. 5 and 6. Here we give only the barest sketch of a derivation, and state the Feynman rules. We give the rules in Euclidean space, although the expansion can also be formulated in Minkowski space, as in Ref. 5.

The model we consider is a scalar theory with a quartic interaction and an O(N) internal symmetry, where N is very large. The Lagrangian is

$$\mathcal{L} = \tfrac{1}{2} \sum_{i=1}^N (\partial_\mu \varphi_i)^2 + \tfrac{1}{2} \sum_{i=1}^N m^2 \varphi_i^2 + g\left(\sum_{i=1}^N \varphi_i^2\right)^2 . \tag{1}$$

The objects to be studied are the connected Green's functions

$$\langle\varphi_i(x_1)\cdots\varphi_j(x_n)\rangle_{\text{conn}}=\frac{\delta}{\delta J_i(x_1)}\cdots\frac{\delta}{\delta J_j(x_n)}\ln\int[d\varphi]\exp\left[-\int\left(\mathcal{L}(\varphi)+\sum_{k=1}^{N}\varphi_kJ_k\right)d^Dx\right]\Bigg|_{J=0}. \quad (2)$$

As usual, we can most easily interpret the path integral by replacing the spacetime integral by a sum over points, each of which has volume Δ. As discussed in Refs. 5 and 6, we make the identification $\Delta=\delta(0)^{-1}$. The $1/\sqrt{g}$ expansion is derived by expressing the kinetic part of the Lagrangian as functional derivatives

$$Z(\vec{J})=\exp\left[\sum_{xy}-\frac{\Delta}{2}\frac{\delta}{\delta\vec{J}(x)}\cdot G^{-1}(x,y)\frac{\delta}{\delta\vec{J}(y)}\right]\int[d\varphi]\exp\left(-\Delta g\sum_x[\vec{\varphi}^2(x)]^2\right)\exp\left(\Delta\sum_x\vec{J}(x)\cdot\vec{\varphi}(x)\right), \quad (3)$$

where $G^{-1}(x,y)=(-\nabla^2+m^2)\delta(x-y)$ and $\vec{\varphi}$ is an N-component vector in internal-symmetry group space. We expand both the first and third exponentials in Eq. (3) in power series and rescale φ by letting $\varphi\rightarrow(\Delta g)^{1/4}\varphi$. The path integral of the second exponential times the factors of φ from the third exponential, which is a product of integrals at each point in spacetime, is to be evaluated exactly. The terms in the product of the two series that contribute when J goes to zero are those terms that have the same number of J's as $\delta/\delta J$'s. We can represent these terms by diagrams in which a line represents $-G^{-1}(x,y)$ and an m-point vertex represents an mth moment of $e^{-(\vec{\varphi}^2)^2}$. Obviously, m (and the number of fields of each type) must be even, so we have two-point, four-point, six-point, etc. vertices in the graphs.

In integrating over the locations of the vertices we must be careful to avoid double counting when two of the vertices coincide. As discussed in detail in Ref. 5, such double counting is avoided by using only the irreducible parts of the moments of $(\vec{\varphi}^2)^2$. That is, we subtract from the numerical factor for an m-point vertex the contributions of all possible ways of dividing the m legs of the vertex into two or more groups. We denote by b_{abc} the moment of $e^{-(\vec{\varphi}^2)^2}$ with a powers of φ_1, b powers of φ_2, etc., where the subscript on φ is the internal-symmetry index. Therefore,

$$b_{abc}\cdots=\frac{\int d\varphi_1\cdots d\varphi_N\,\varphi_1^{\,a}\varphi_2^{\,b}\cdots\exp[-(\sum_{i=1}^{N}\varphi_i^{\,2})^2]}{\int d\varphi_1\cdots d\varphi_N\exp[-(\sum_{i=1}^{N}\varphi_i^{\,2})^2]}. \quad (4)$$

Obviously $b_{abc}\cdots$ is symmetric under all permutations of the indices, and is zero unless all of the indices are even. Notice that the $b_{abc}\cdots$ depend on N. We will examine this dependence later in this paper. The irreducible parts of the moments are denoted by $B_{abc}\cdots$. The simplest examples are

$$B_2=b_2, \quad (5a)$$

$$B_{22}=b_{22}-(B_2)^2, \quad (5b)$$

$$B_4=b_4-3(B_2)^2, \quad (5c)$$

$$B_{222}=b_{222}-3B_{22}B_2-B_2^{\,3}, \quad (5d)$$

$$B_{42}=b_{42}-B_4B_2-6B_{22}B_2-3B_2^{\,3}, \quad (5e)$$

$$B_6=b_6-15B_4B_2-15B_2^{\,3}. \quad (5f)$$

The reason (5b) differs from (5c) is that there is only one way of partitioning $\varphi_1^{\,2}\varphi_2^{\,2}$ to give a nonzero contribution, while there are three ways of dividing four identical fields into pairs. Equations (5) define the B's of each order in terms of the b's of that order and the B's of lower order. In Ref. 5 this recursive definition is solved to express the B's explicitly as functions of the b's. (In the statistical mechanics literature, the B's are known as semi-invariants.)

The Feynman rules for our model are similar to those in Ref. 6, where a φ^4 theory with a single field is discussed. Apart from the trivial modifications of introducing an index on the lines to label different components of the field and conserving the internal-symmetry indices at each vertex, the major difference is in the factors used at the vertices. To compute an unamputated $2n$-point connected Green's function, we draw all diagrams with $2n$ external legs using two-point, four-point, six-point, etc. vertices. For an internal line connecting vertices at x and y there is a factor of $-G^{-1}(x,y)=(\nabla^2-m^2)\delta^D(x-y)$, while for an external line there is simply $\delta(x-y)$. Each line is labeled by an index which runs from one to N. For each m-point vertex there is a factor of

$$\frac{\delta(0)^{-(3/4)m+1}}{(\sqrt{g})^{m/2}}B_{abcd}\cdots, \quad (6)$$

where $a+b+c+d+\cdots=m$ and $\delta(0)^{-1}=\Delta$ is the volume of a point in spacetime, arising from our interpretation of the path integral as a product of ordinary integrals. There is a symmetry factor for each graph equal to one over the number of symmetry operations, such as interchanges of lines, that leave the graph unchanged, exactly as in ordinary perturbation theory. Finally, we integrate over the locations of all the vertices.

III. THE LARGE-N LIMIT

To lowest order in $1/N$ the model we are considering can be easily treated in ordinary perturbation theory.[8] In perturbation theory factors of N arise from summing over the indices on internal loops in the Feynman graphs. One can show that the graphs in the two-point function with the most powers of N at each order in g are proportional to powers of gN, and the leading order in N graphs in the four-point function are proportional to $1/N(gN)^m$. Therefore, one considers the limit $N\to\infty$ with gN fixed, and gN becomes the effective expansion parameter. In this limit it is easy to sum the relevant graphs in the two- and four-point functions. Because we can treat perturbation theory to all orders in g, we can use its results in the limit $N\to\infty$ with gN fixed and large, where the $1/\sqrt{g}$ expansion should be reasonable. We also expect that simplifications similar to those in perturbation theory will occur in the $1/\sqrt{g}$ expansion as $N\to\infty$.

The first thing learned from the large-N perturbation theory is that the effective expansion parameter is gN. In the $1/\sqrt{g}$ expansion we have factors of N from summations over free indices in the graphs and factors of $1\sqrt{g}$ from the vertices, so it is not easy to see how the result could be a function of gN. The answer is that the B coefficients depend on N, and the N dependence of these coefficients conspires with the factors of N from summations over indices to give the same result as perturbation theory. For example, consider the simple graph in Fig. 1. This graph is proportional to $g^{-1}B_{22}N$, where the N comes from summing over the index on the loop. (The term where $i=j$ and the vertex factor is B_4 represents a correction of order $1/N$.) However, it turns out that B_{22} is proportional to N^{-2} for large N, so the graph is actually proportional to $(gN)^{-1}$.

It is not difficult to evaluate the $b_{abc}\ldots$ by doing the integration in Eq. (4) in generalized spherical coordinates. Both the angular integrals and the radial integrals give Γ functions, there are many cancellations, and the doubling theorem can be used to further simplify the result. We find

$$b_{abcd}\cdots=\frac{\Gamma((a+1)/2)\Gamma((b+1)/2)}{\Gamma(1/2)\Gamma(1/2)}\cdots \times 2^{-m/2}\frac{\Gamma(N/4+1/2)}{\Gamma(N/4+1/2+m/4)}, \tag{7}$$

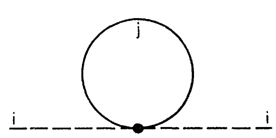

FIG. 1. A simple loop graph. The vertex factor is B_{22}.

where $a, b, c, \ldots$ are even and $m=a+b+c+\cdots$. The factor $\Gamma((a+1)/2)/\Gamma(1/2)$ can be thought of as $(1/2)^{a/2}$ times a combinatoric factor equal to the number of ways of dividing the legs of one type into pairs of legs. For example, if $a=4$ then $2^2\Gamma(5/2)/\Gamma(1/2)=3$, corresponding to the three ways of dividing four legs into two pairs. The final factor in Eq. (7) contains the N dependence and is a function on the total number of lines coming into the vertex. For large N we can make an asymptotic expansion of this factor in powers of $1/N$. For convenience define $y=N/4-1/2$ and $s=m/4$. Then

$$\frac{\Gamma(N/4+1/2)}{\Gamma(N/4+1/2+m/4)}=\frac{\Gamma(y+1)}{\Gamma(y+1+s)}. \tag{8}$$

For large y

$$\frac{\Gamma(y+1)}{\Gamma(y+1+s)}=y^{-s}\left(1+\frac{s^2+s}{y}+\frac{3s^4+2s^3-3s^2-2s}{y^2}+\frac{s^6-s^5-3s^4+s^3+2s^2}{y^3}+\cdots\right)^{-1}. \tag{9}$$

Using this formula, we express the b's as power series in $1/y\approx 1/N$. For example,

$$b_2=\frac{y^{-1/2}}{2}\left(1-\frac{3}{8y}+\frac{25}{128y^2}-\frac{105}{1024y^3}+\cdots\right), \tag{10}$$

$$b_{22}=\tfrac{1}{3}b_4=\frac{y^{-1}}{4}(1-1/y+1/y^2-1/y^3+\cdots). \tag{11}$$

We see from Eq. (9) that each $b_{ab}\ldots$ begins in order $N^{-m/4}$. Fortunately this is not the whole story. When we compute the B_{abc} from the b_{abc} we find that there are cancellations of the leading orders in $1/N$. Each time $m=a+b+c+\cdots$ is increased by 2 there is one more cancellation. This means that B_{22} and B_4 start at order N^{-2} although b_{22} and b_4 start at order N^{-1}. B_{222}, B_{42}, and B_6 have two cancellations and start at order $N^{-7/2}$. In general, $b_{abc}\ldots$ goes as $N^{-m/4}$, while B_{abc} goes as $N^{-3m/4+1}$. We have verified these cancellations through $m=10$, where the first four orders cancel. To leading order in $1/N$ the first few $B_{222}\ldots$ are

$$\begin{aligned}
B_2&=\tfrac{1}{2}N^{-1/2},\\
B_{22}&=-\tfrac{1}{4}N^{-2},\\
B_{222}&=\tfrac{5}{8}N^{-7/2},\\
B_{2222}&=-3N^{-5},\\
B_{22222}&=O(N^{-13/2})\\
&\approx 20N^{-13/2}\ \text{(numerical estimate)}.
\end{aligned} \tag{12}$$

The other B coefficients for $m \leq 10$ can be found by multiplying by the appropriate combinatoric factors.

IV. IMPORTANT DIAGRAMS FOR LARGE N

In the limit $N \to \infty$ we consider only those diagrams at each order in $1/\sqrt{g}$ which have the most factors of N. This means that we consider only diagrams where none of the loops share lines, because if two loops share a line the indices in the loops cannot be summed independently and the graph will be suppressed by a factor of $1/N$ relative to some other graph with the same number of internal lines. Figure 2 shows some of the leading diagrams in the two-point Green's function. The relevant diagrams look similar to those that are considered in ordinary perturbation theory, except that the $1/\sqrt{g}$ expansion contains all $2n$-point vertices. Because we wish the internal-symmetry indices on all of the loops to be independent, the appropriate vertex coefficients are the $B_{222}\cdots$, since requiring more than two of the lines coming into a vertex to have the same index suppresses the graph by factors of N.

Given the leading graphs in the two-point function to some order in $1/\sqrt{g}$, we can generate all of the leading order in $1/N$, next order in $1/\sqrt{g}$ graphs by one of two operations. We can insert a two-point vertex on any line of the graph, including an external line. This introduces a factor of $B_2/\sqrt{g} \approx 1/\sqrt{Ng}$. We can also add a line to the graph, with both ends of the new line on the same previously existing vertex. This adds a loop to the graph and hence a factor of N, but as we have seen, adding two lines to a vertex adds one more cancellation to the B coefficient, giving a factor of $N^{-3/2}/\sqrt{g}$ relative to the previous vertex. Therefore, the factors of N from summations over indices exactly compensate the extra factors of $1/N$

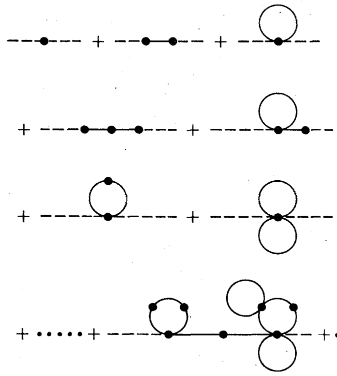

FIG. 2. Some leading order in $1/N$ graphs in the two-point function.

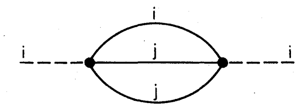

FIG. 3. The simplest nonleading graph in the two-point function.

coming from the cancellations, and the leading graphs to all orders in $1/\sqrt{g}$ in the two-point function depend only on the quantity gN. A similar argument shows that the leading graphs in the four-point function are proportional to $1/N$ times a function of gN, in agreement with the perturbation-theory analysis. We see that the $1/\sqrt{g}$ expansion reproduces in a complicated way results that are quite easy in perturbation theory.

If we wished to go to the next order in $1/N$, the $1/\sqrt{g}$ expansion would be quite complicated. Not only would we have to consider graphs where two loops share a line (see Fig. 3), but we would also have to consider graphs containing vertices of the form $B_{4222}\cdots$, and also consider the next order in $1/N$ in the $B_{222}\cdots$.

V. REGULARIZATION

Before we can evaluate graphs in the $1/\sqrt{g}$ expansion, we must have a method for handling the singular factors in the Feynman rules. Both the vertex factors and the inverse propagator are singular. To calculate in the $1/\sqrt{g}$ expansion we must introduce a smoothed-out δ function, which is equivalent to introducing an explicit cutoff into the theory. There are many ways to do this,[5] the most obvious and intuitive being the introduction of a lattice.[6] However, lattice calculations are unwieldly and Euclidean invariance is not manifest. We prefer to proceed by defining

$$\delta_\Lambda(x) = \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x}\,\theta(\Lambda^2 - p^2)\,, \qquad (13)$$

which becomes the Dirac δ function as $\Lambda \to \infty$. This smoothed δ function has the advantages of manifest Euclidean invariance and ease of manipulation for the problem at hand. With this choice it is easy to evaluate $\delta(0)$:

$$\delta_\Lambda(0) = \int \frac{d^D p}{(2\pi)^D}\,\theta(\Lambda^2 - p^2) = \frac{2\Lambda^D}{D\Gamma(D/2)(4\pi)^{D/2}}\,. \qquad (14)$$

The inverse propagator is then

$$G^{-1}(x) = (-\nabla^2 + m^2)\,\delta_\Lambda(x)$$

$$= \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x}(p^2 + m^2)\,\theta(\Lambda^2 - p^2)\,, \qquad (15)$$

and in momentum space

$$G^{-1}(p)=(p^2+m^2)\,\theta(\Lambda^2-p^2)\,. \tag{16}$$

If we evaluate diagrams in momentum space we will need only to integrate polynomials with a sharp cutoff at $p^2=\Lambda^2$. Of course, in a complicated diagram where several loop monenta run along a line, this could be very messy. However, for the theory we consider the relevant graphs are easy to evaluate.

VI. THE TWO-POINT FUNCTION

We can now analyze the zeroth order in $1/N$ diagrams in the two-point function. As stated earlier, the leading diagrams are those in which no two loops share a line and all of the free indices are different, so that the vertex factors are $B_{222\cdots}$. We define σ as the one-"particle" irreducible (1PI) part of the propagator, as illustrated in Fig. 4. The complete two-point function is then given by the geometric series

$$\begin{aligned}G_2(p^2)&=\sigma-\sigma(p^2+m^2)\,\theta(\Lambda^2-p^2)\sigma\\&\quad+\sigma(p^2+m^2)\,\theta(\Lambda^2-p^2)\sigma(p^2+m^2)\,\theta(\Lambda^2-p^2)\sigma\\&=\frac{1}{p^2+m^2+1/\sigma}\,\theta(\Lambda^2-p^2)+\sigma\,\theta(p^2-\Lambda^2)\,,\end{aligned} \tag{17}$$

so for $p^2<\Lambda^2$ the propagator takes a familiar form with σ equal to one divided by the usual self-energy.

In all of the leading diagrams in σ the two external legs are attached to the same vertex. If the external legs were attached to different vertices, then a cut could be made through the graph cutting an odd number (>1) of internal lines. This means that at least one of these lines would be shared by two loops and the graph would be suppressed. Therefore, σ is independent of the external momentum p^2 and simply gives a mass renormalization. This means that the wave-function renormalization Z_3 is equal to one, as in perturbation theory, and the physical mass μ^2 is

$$\mu^2=m^2+1/\sigma\,. \tag{18}$$

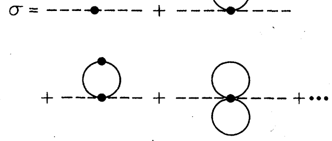

FIG. 4. The simplest graphs in σ, the 1PI part of the propagator.

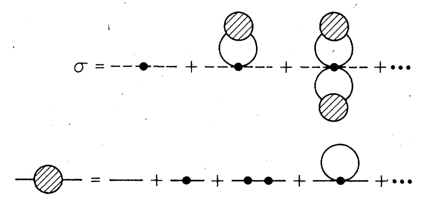

FIG. 5. A rearrangement of the graphs in σ.

We can rearrange the diagrams in σ according to the number of loops attached to the vertex with the external legs, as illustrated in Fig. 5. The blob in Fig. 5 indicates the sum of all possible loops, which is easily seen to be

$$\begin{aligned}\sum(\text{all loops})&\\&=\Big(-(p^2+m^2)\\&\qquad+(p^2+m^2)\frac{1}{p^2+m^2+1/\sigma}(p^2+m^2)\Big)\,\theta(\Lambda^2-p^2)\\&=-\frac{1}{\sigma}\,\frac{p^2+m^2}{p^2+m^2+1/\sigma}\,\theta(\Lambda^2-p^2)\,.\end{aligned} \tag{19}$$

If we denote the vertex coefficient $B_{222\cdots}$ with $2k$ legs by $B_{2(k)}$, then the series for σ is

$$\sigma=\frac{\delta(0)^{-1/2}}{\sqrt{g}}\sum_{k=0}^{\infty}\frac{N^{3k/2}B_{2(k+1)}}{k!}(-\alpha)^k\,, \tag{20}$$

where

$$\alpha=-\tfrac{1}{2}\delta(0)^{-3/2}\frac{1}{\sqrt{gN}}\int^{\Lambda}\frac{1}{\sigma}\,\frac{p^2+m^2}{p^2+m^2+1/\sigma}\,\frac{d^Dp}{(2\pi)^D}\,. \tag{21}$$

The effect of the θ function in $G^{-1}(p^2)$ is just to cut the integral off at $p^2=\Lambda^2$. For the term with k loops there is a symmetry factor of $1/k!\,2^k$, but we have absorbed the 2^k into the definition of α, leaving only the $k!$ in Eq. (20).

Let us digress for a minute to discuss Eq. (20). We introduce a generating function defined by

$$F(\alpha)=\sum_{k=0}^{\infty}\frac{N^{3k/2}b_{2(k)}}{k!}(-\alpha)^k\,, \tag{22}$$

where $b_{2(k)}$ is given by Eq. (7). Taking the logarithm of F replaces the moments $b_{2(k)}$ by the irreducible parts $B_{2(k)}$, and differentiating with respect to $-\alpha N$ is equivalent to inserting two external legs. Therefore, Eq. (20) can be rewritten as

$$\sigma=\frac{\delta(0)^{-1/2}}{\sqrt{gN}}\left(-\frac{d}{d\alpha}\right)\left(\frac{1}{N}\ln F(\alpha)\right). \tag{23}$$

Our assertion that the effective expansion param-

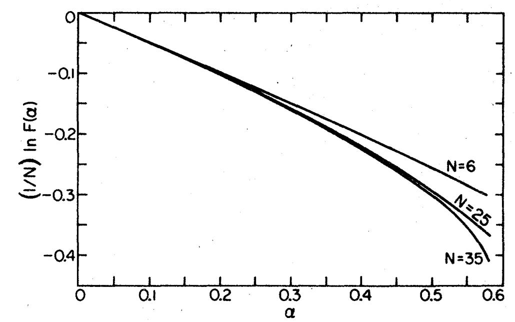

FIG. 6. Graph of $(1/N)\ln F(\alpha)$ for $N=25$ and $N=35$.

eter is $(gN)^{-1/2}$ is equivalent to the assertion that $(1/N)\ln F(\alpha)$ becomes independent of N for N very large. In Fig. 6 we show $(1/N)\ln F(\alpha)$ calculated for $N=6$, 25, and 35. The last two functions are already quite close except near $\alpha\approx 0.6$, where $-\ln F(\alpha)$ appears to be going to infinity [$F(\alpha)$ has a zero]. Finally, we note that $F(\alpha)$ has an integral representation

$$F(\alpha)=(\text{const})(N^{3/2}\alpha)^{-(N-2)/4}\times\int_0^\infty dt\,t^{N/2}e^{-t^4/4}J_{N/2-1}((N^{3/2}\alpha)^{1/2}t) \tag{24}$$

which can be verified by expanding the Bessel function in a power series and integrating term by term. If we are given the bare parameters m^2 and gN, we can use Eqs. (21), (22), and (23) to find (at least numerically) σ and α, and hence the physical mass μ^2.

Although we began our analysis by treating each point in spacetime as independent, we are naturally most interested in the regime where the cutoff becomes infinite, or, equivalently, the correlation length is long relative to Λ^{-1}. That is, if we imagine holding Λ fixed, we wish to study the theory near the critical line where the physical mass μ is infinitesimal and the external momenta are also infinitesimal. Near the critical line the physical mass is related to the bare mass by the critical exponent γ:

$$\mu^2\approx(m^2-m_{\text{crit}}{}^2)^\gamma\,. \tag{25}$$

To calculate γ, note that Eq. (20) expresses σ as a power series in α, which is defined in Eq. (21), and the physical mass is $\mu^2=m^2+1/\sigma$. By taking the reciprocal of the power series in Eq. (20), we get $1/\sigma$ as a power series in α

$$m^2-\mu^2=\sum_{k=0}^{\infty}C_k\left[\int^\Lambda\frac{d^Dp}{(2\pi)^D}\left((p^2+m^2)-\frac{(p^2+m^2)^2}{p^2+\mu^2}\right)\right]^k. \tag{26}$$

Differentiate with respect to μ^2

$$\frac{\partial m^2}{\partial\mu^2}-1=\sum_k kC_k\left[\int^\Lambda\frac{d^Dp}{(2\pi)^D}\left((p^2+m^2)-\frac{(p^2+m^2)^2}{p^2+\mu^2}\right)\right]^{k-1}\left[\frac{\partial m^2}{\partial\mu^2}\int^\Lambda\frac{d^Dp}{(2\pi)^D}\left(1-\frac{2m^2}{p^2+\mu^2}\right)+\int^\Lambda\frac{d^Dp}{(2\pi)^D}\frac{(p^2+m^2)^2}{(p^2+\mu^2)^2}\right]. \tag{27}$$

Now, for $2<D<4$ the last term in Eq. (27) is infrared divergent as $\mu^2\to 0$, while all the other integrals go to a constant. For small μ^2 the last term is proportional to $(\mu^2)^{D/2-2}$ and we have

$$\frac{\partial m^2}{\partial\mu^2}\approx A+B(\mu^2)^{D/2-2}\,. \tag{28}$$

As $\mu^2\to 0$ for $D<4$, A can be neglected and this equation can be integrated to give

$$m^2-m_{\text{crit}}{}^2\approx(\mu^2)^{D/2-1} \tag{29}$$

or

$$\gamma=\frac{1}{D/2-1}\,, \tag{30}$$

which is the well-known result.

VII. THE FOUR-POINT FUNCTION

We next analyze the four-point function and compute the physical coupling. For simplicity, we consider the four-point function with two legs of type i and two legs of type j. That is, the first contribution is B_{22} rather than B_4.

The quantity given by our Feynman rules is the unamputated Green's function

$$G_{22}=\frac{\delta}{\delta J_i(x_1)}\frac{\delta}{\delta J_i(x_2)}\times\frac{\delta}{\delta J_j(x_3)}\frac{\delta}{\delta J_j(x_4)}\ln Z(J)\bigg|_{J=0} \tag{31}$$

The diagrams in G_{22} consist of the one "particle" irreducible piece and four legs, as shown in Fig.

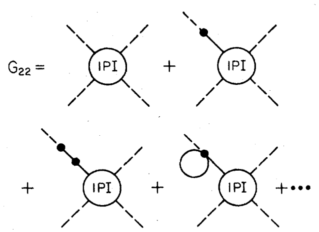

FIG. 7. External legs on the four-point function.

7. The one "particle" reducible part on each leg may be either a bare external line, or an internal line $(-p^2-m^2)$ connected to an external line by any of the diagrams in the full propagator. Thus for each leg there is a factor of

$$1-\frac{p^2+m^2}{p^2+\mu^2}=\frac{\mu^2-m^2}{p^2+\mu^2}\ . \tag{32}$$

To obtain the amputated Green's function Γ_{22} we multiply by $(p^2+\mu^2)$.

Defining τ/N in analogy to σ, as indicated in Fig. 8, we find, to leading order in $1/N$, that the 1PI part of the four-point function is the geometric series of bubbles illustrated by Fig. 9. The shaded blobs in Figs. 8 and 9 have the same meaning as in Fig. 5 and Eq. (19). Then we find, in analogy with Eqs. (20) and (23), that

$$\tau=\frac{\delta(0)^{-2}}{gN}\sum_{k=0}^{\infty}\frac{B_{2(k+2)}N^{2k/2}}{k!}(-\alpha)^k$$
$$=\frac{\delta(0)^{-2}}{gN}\left(-\frac{d}{d\alpha}\right)^2\left(\frac{1}{N}\ln F(\alpha)\right). \tag{33}$$

Summing the series in Fig. 9 and taking into account the external leg factors, we obtain the amputated four-point function at zero external momentum

$$\Gamma_{22}(0)=(\mu^2-m^2)^4\frac{1}{N}\times\frac{\tau}{1-\frac{1}{2}\tau\int\frac{d^Dp}{(2\pi)^D}\left((p^2+m^2)-\frac{(p^2+m^2)^2}{p^2+\mu^2}\right)}\ . \tag{34}$$

A dimensionless renormalized coupling constant can be defined by writing $-\Gamma_{22}(0)$ in units of the physical mass μ^2

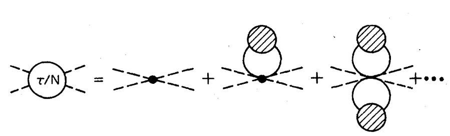

FIG. 8. The definition of τ/N. All four external lines attach to the same vertex.

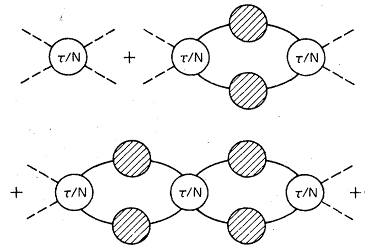

FIG. 9. The 1PI part of G_{22}.

$$Ng_r=-N\Gamma_{22}(0)\mu^{D-4}$$
$$=\frac{(\mu^2-m^2)^4\mu^{D-4}(-\tau)}{1-\frac{1}{2}\tau\int\frac{d^Dp}{(2\pi)^D}\left((p^2+m^2)-\frac{(p^2+m^2)^2}{p^2+\mu^2}\right)^2}\ . \tag{35}$$

τ is negative, so the renormalized coupling is positive. Now if we go to the critical point where $\mu^2/\Lambda^2\to 0$, the last term in the integrand becomes infrared divergent and dominates the integral. We also use $\mu^2\ll m^2$ and the fact that in the important range of the momentum integration $p^2\ll m^2$ to find that near the critical point

$$Ng_r=\mu^{D-4}\left(\frac{1}{2}\int^{\Lambda}\frac{d^Dp}{(2\pi)^D}\frac{1}{(p^2+\mu^2)^2}\right)^{-1}$$
$$=\Gamma(D/2)(4\pi)^{D/2}(4-D), \tag{36}$$

again recovering the well-known result. Finally we note that if we evaluate Γ_{22} with small external momenta, $q_{\text{ext}}{}^2\ll m^2,\Lambda^2$ we find, for the four-point function,

$$N\Gamma_{22}(q_1,q_2,q_3,q_4)$$
$$=-\left(\frac{1}{2}\int^{\Lambda}\frac{d^Dp}{(2\pi)^D}\frac{1}{p^2+\mu^2}\frac{1}{(p+q_1+q_2)^2+\mu^2}\right)^{-1}, \tag{37}$$

where (q_1+q_2) is the momentum flowing through the bubbles in Fig. 9.

VIII. CONCLUSION

We have shown that the zeroth-order terms in a $1/N$ expansion of a $g\varphi^4$ field theory with an O(N) internal symmetry can be expanded in powers of $1/\sqrt{gN}$, and that the resulting expansion gives the same results for the critical index and the renormalized coupling constant in $2<D<4$ dimensions as the usual perturbation expansion of the same theory. As in the usual perturbation theory, we find the quantities related to the critical behavior by summing an infinite series of diagrams, thereby avoiding, for this case, the Padé or Borel approximant techniques usual to strong-coupling expansions. These results show that the $1/\sqrt{g}$ expansion is a useful method for studying theories for large values of the coupling constant and increase our hopes that it can also be applied to more complicated theories without the necessity of defining equivalent lattice field theories. Work in this direction is in progress.

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