Explicit solutions for the 't Hooft transformation

N. N. Khuri and Oliver A. McBryan*
Rockefeller University, New York, New York 10021
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The 't Hooft transformation is a change of variables of the renormalized coupling constant to a new coupling parameter such that the resulting Gell-Mann–Low (GML) function is a low-order polynomial with coefficients independent of renormalization scheme. Explicit expressions for this transformation are given for $(\phi^4)_d$, $d = 2,3,4$, field theories, QED, and quantum chromodynamics. The solutions are expressed in terms of the original GML function $\beta(g)$ and their properties are given in some detail. In several cases these transformations are shown to be valid up to the first nontrivial zero of $\beta(g)$.

I. INTRODUCTION

In a renormalizable or superrenormalizable field theory one can use different mass-independent renormalization schemes. Each scheme will give a different definition of the renormalized coupling constant g and the corresponding Gell-Mann–Low (GML) function $\beta(g)$ that appears in the renormalization-group equations. Thus if g is the coupling in one scheme with a corresponding $\beta(g)$, another scheme will yield g' with a GML function $\beta'(g')$, such that $g'\equiv G(g)=g+O(g^2)$. The functions β' and β are in general different functions, but the first two coefficients in their perturbation expansion must be identical. For example in $(\phi^4)_2$ theory, if $\beta'(g)=a_1g+a_2g^2+O(g^3)$ and $\beta'(g')=a_1'g'+a_2'g'^2+O(g'^3)$, then $a_1=a_1'$ and $a_2=a_2'$.

't Hooft[1] has suggested that one can exploit this freedom in the choice of g even further and choose a new coupling parameter g_R such that the corresponding $\beta_R(g_R)\equiv a_1g_R+a_2g_R^{\,2}$ (for the above case) and thus has only two terms in its expansion in g_R. The transformation $g\to g_R$, which we shall call the 't Hooft transformation, is defined to satisfy $\beta(g)\partial/\partial g=\beta_R(g_R)\partial/\partial g_R$ so that it preserves the form of the Callan-Symanzik equation when written in terms of g_R.

In perturbation theory the solution to the 't Hooft transformation exists.[1] Namely, given $\beta(g)=\sum a_n g^n$, and writing $g_R\equiv G(g)=g+\sum r_n g^n$, one can determine all the r's in terms of the a's. The following question immediately arises: Given a $\beta(g)$ does a nonsingular invertible $g_R=G(g)$ exist with $\beta_R(g_R)=a_1g_R+a_2g_R^{\,2}$ for some interval $0\le g\le g_1$? If so, how large is the interval $0<g<g_1$ and what are the general properties of $G(g)$ and $\partial G/\partial g$ in this domain?

In this paper we show that the 't Hooft function $G(g)$ can be given explicitly in terms of $\beta(g)$ and we shall rigorously study its properties for several field theories $(\phi^4)_2$, $(\phi^4)_3$, $(\phi^4)_4$, QED, and quantum chromodynamics (QCD).

In addition to the original application of this transformation by 't Hooft[1] there are other applications,[2] but in this paper we shall concentrate only on problems related to the existence of $G(g)$, its properties, and the location of its leading singularities. One of our main objectives is to clarify the relationship of the first nontrivial zero of $\beta(g)$, when such a zero exists, to the zero of $\beta_R(g_R)$ which we call the 't Hooft zero. We also show what happens in cases such as QED and QCD where $\beta_R(\alpha_R)$ has no positive zero.

II. CASE OF $(\phi^4)_2$, $(\phi^4)_3$

We start by treating the case of ϕ^4 field theory in 2 and 3 dimensions. Here $\beta(g)$ has the form, for small g,

$$\beta(g)=a_1g+a_2g^2+O(g^3)\,. \tag{2.1}$$

It is important to recall that in these two cases the signs of a_1 and a_2 are such that

$$a_1<0,\quad a_2>0\,. \tag{2.2}$$

In fact, one usually rescales g such that $a_1=-1$ and $a_2=+1$ as in Ref. 3.

The 't Hooft transformation for this case is defined as follows:

$$g_R\equiv G(g)=g+O(g^2)\,. \tag{2.3}$$

One then defines $\beta_R(g_R)$ as

$$\beta_R(g_R)\equiv\frac{\partial G}{\partial g}\beta(g)\,, \tag{2.4}$$

and looks for a $G(g)$ such that

$$\beta_R(g_R)=a_1g_R+a_2g_R^{\,2}\,. \tag{2.5}$$

Clearly in the domain where G exists and where $\partial G/\partial g>0$ we have

$$\beta_R(g_R)\frac{\partial}{\partial g_R}=\beta(g)\frac{\partial}{\partial g}\,. \tag{2.6}$$

The problem is now, given $\beta(g)$ in some interval

$0<g<g_m$, can one find a $G(g)$ that satisfies Eqs. (2.3), (2.4), and (2.5)? More simply, what we have to do is study the solutions of the first-order nonlinear differential equation,

$$\frac{\partial G(g)}{\partial g}=\frac{1}{\beta(g)}[a_1G(g)+a_2G^2(g)]; \tag{2.7}$$

such that $G(x)=x+O(x^2)$ as $x\to 0$. This equation can be solved exactly in terms of $\beta(g)$. To see this we first write $f(x)\equiv G(x)/x$ and (2.7) becomes

$$\frac{d}{dx}[xf(x)]=k(x)[f(x)+bxf^2(x)], \tag{2.8}$$

with

$$k(x)\equiv a_1x/\beta(x), \tag{2.9}$$

$$b=a_2/a_1. \tag{2.10}$$

Note that $k(x)>0$ for $0<x<g^*$ where g^* is the first IR zero of $\beta(g)$, $\beta(g^*)=0$. Writing

$$f\equiv 1/h \tag{2.11}$$

the differential equation (2.8) reduces to a linear equation

$$h'(x)=-\frac{k(x)-1}{x}h(x)-bk(x). \tag{2.12}$$

Using standard methods and converting back to our original function $G(g)$ we get, using Eq. (2.3),

$$G(g)=\frac{g\exp\left\{\int_0^g dx\left[\frac{a_1}{\beta(x)}-\frac{1}{x}\right]\right\}}{1-bg\exp\left\{\int_0^g dx\left[\frac{a_1}{\beta(x)}-\frac{1}{x}\right]\right\}}. \tag{2.13}$$

We recall that in this case $b=(a_2/a_1)<0$, so the denominator does not vanish for $0<g<g^*$. It is now a matter of simple algebra to calculate dG/dg explicitly also and one gets

$$G'(g)=\frac{\left(\frac{a_1g}{\beta(g)}\right)\exp\left\{\int_0^g dx\left[\frac{a_1}{\beta(x)}-\frac{1}{x}\right]\right\}}{\left(1-bg\exp\left\{\int_0^g dx\left[\frac{a_1}{\beta(x)}-\frac{1}{x}\right]\right\}\right)^2}. \tag{2.14}$$

One should note that from Eq. (2.1) it follows that all the integrals in Eqs. (2.13) and (2.14) above are convergent at $x=0$.

Mathematically, there are now two distinct cases to study: Case (1), where $\beta(g)$ starts out negative near $g=0$, remains finite for some interval $0<g<g^*$ and develops a zero at $g=g^*$, $\beta(g^*)=0$; $\beta(g)<0$ for $0<g<g^*$. We shall assume that the zero is such that $\int_c^g dx/\beta(x)$ diverges as $g\to g^*$. This is the actual physical case for $(\phi^4)_2$ and $(\phi^4)_3$. Case (2) will cover the situation where $\beta(g)$ has no finite zero except at $g=0$, and we shall distinguish between the two possibilities $\int_c^\infty dx/\beta(x)$ finite or divergent.

Given the explicit forms of $G(g)$ and $G'(g)$ it is easy to derive the properties of this function for both cases.

Case (1).

(a) $G(g)=g+O(g^3)$ with no g^2 term for small g.

(b) $G(g)$ exists and is bounded for $0<g<g^*$ and as $g\to g^*$,

$$G(g^*)=-\frac{1}{b}=-\frac{a_1}{a_2}\equiv g_R^*, \tag{2.15}$$

where g_R^* is the 't Hooft zero defined by the nontrivial root of $\beta_R(g_R^*)=a_1g_R^*+a_2g_R^{*2}=0$.

(c) $G'(g)>0$ for $0\leq g<g^*$, and G maps the interval $0\leq g<g^*$ in a one-to-one manner onto $0\leq g_R<g_R^*=-a_1/a_2$.

(d) If $\beta(g)$ has a first zero g^* nearest the origin, then $G(g^*)=g_R^*$, the 't Hooft zero. However, the existence of the 't Hooft zero does not imply the existence of a zero in $\beta(g)$ as we shall see in discussing case (2).

(e) Finally, it is important to study the behavior of $G'(g)$ as $g\to g^*$. To do this we assume that $\beta(g)$ has a simple zero at g^* and write

$$\beta(g)\cong\omega(g-g^*),\quad g\approx g^* \tag{2.16}$$

where $\omega>0$ is the slope of β at the fixed point. It is simple to check from Eq. (2.14) that as $g\to g^*$,

$$G'(g)\cong(g^*-g)^{-a_1/\omega-1}. \tag{2.17}$$

Therefore, in general, $G'(g)$ would vanish or diverge as $g\to g^*$ unless by some accident $\omega=-a_1$. This accident could happen for $(\phi^4)_2$ where the value $\omega=1=-a_1$ is consistent with the best calculations of the slope.[4] However, in $(\phi^4)_3$ the best value for $\omega\cong 0.782$, while $-a_1=1$, which gives a $G'(g)$ that vanishes as $g\to g^*$, even though it vanishes slowly, i.e., $\sim(g^*-g)^{0.27}$. Thus, in general, $G(g)$ will develop a branch point at $g=g^*$.

Before we discuss case (2), we would like to remark on the universality of the slope of $\beta(g)$ at the fixed point. It has been shown[5] that for two different renormalization schemes $\partial\beta/\partial g|_{g=g^*}=\partial\beta'/\partial g'|_{g'=g^{*\prime}}$. However, the proof of this fact depends on the assumption $\partial g'/\partial g\neq 0$ as $g\to g^*$. This assumption is clearly not always true for $\partial g_R/\partial g$ as $g\to g^*$. In fact, we can write

$$\omega_R\equiv\left.\frac{\partial\beta_R}{\partial g_R}\right|_{g_R=g_R^*}=-a_1, \tag{2.18}$$

and thus we can rewrite Eq. (2.17) as $G'(g)\approx(g^*-g)^{(\omega_R/\omega)-1}$. Now one can see that $G'(g)\neq 0$ as $g\to g^*$ can only happen if $\omega_R=\omega$ and the assumption about $G'(g)$ begs the question. However, it might still be interesting to seek a physical explanation for the fact that the quantity $(\omega_R/\omega)-1\approx 0$ for $(\phi^4)_2$ and $(\omega_R/\omega)-1\cong 0.27$ for $(\phi^4)_3$, thus small in both cases. As an approximation $\omega\cong\omega_R$ is only 20% off for $(\phi^4)_3$ and is almost exact for $(\phi^4)_2$. Also, in $(\phi^4)_4$ the approximation $\omega\cong\omega_R$ seems to be good for a certain Borel sum of $\beta(g)$.[6]

Case (2). In this case β remains negative and has no finite zeros for $0<g<\infty$. Then if $\int_c^\infty dx/\beta(x)$ is convergent we get a $G(g)$ which is bounded and monotonically increasing with g. It maps $0<g<\infty$ onto some interval $0<g_R<g_R^{\max}$ such that $g_R^{\max}<g_R^*=-1/b$. The function $G(g)$ increases but never reaches the value of the 't Hooft zero g_R^*. The left-hand side of Eq. (2.4) never vanishes as g varies in $0<g<\infty$.

For the case where $a_1\int_c^\infty dx/\beta(x)\to+\infty$, we again get a monotonically increasing function $G(g)$ for $0<g<\infty$ where now as $g\to\infty$, $G(g)\to g_R^*=-1/b$. The interval $0\leq g<\infty$ is mapped in a one-to-one manner onto $0\leq g_R<g_R^*=-1/b$.

We know that $(\phi^4)_2$, $(\phi^4)_3$ both have a nontrivial IR zero g^* of $\beta(g)$. We only discuss case (2) to make clear what happens to the solution of Eq. (2.13) when β has no zero and to stress the fact that the existence of the 't Hooft zero does not necessarily imply a zero in $\beta(g)$. However, if $\beta(g)$ is known to have a first zero at $g=g^*$ that zero must lead to the 't Hooft zero under the transformation $g_R^*=G(g^*)$.

Finally, for completeness, we mention what happens when $\beta(g)$ develops a finite zero such that $\int_c^g dx/\beta(x)$ converges as $g\to g^*$. In this case $G(g)$ will be bounded and monotonically increasing for $0\leq g\leq g^*$ with $G(g^*)<-1/b$, and the value of the 't Hooft zero is not reached. However, now it is evident from Eq. (2.14) that $G'(g)\sim[\beta(g)]^{-1}$ as $g\to g^*$ and will diverge in such a way that the right-hand side of Eq. (2.4) remains finite as $g\to g^*$. Although this is an unphysical case for $(\phi^4)_2$ and $(\phi^4)_3$ a branch-point-type behavior in β cannot be ruled out in general in other field theories and can be made consistent with the renormalization group.[7]

III. CASE OF $(\phi^4)_4$

In four dimensions the GML function has a power-series expansion of the form

$$\beta(g)=a_2g^2+a_3g^3+O(g^4)\,, \tag{3.1}$$

where

$$a_2=\tfrac{3}{2},\ a_3=-\tfrac{17}{6}\,. \tag{3.2}$$

Proceeding as before we define

$$g_R\equiv G(g)=g+r_2g^2+O(g^3) \tag{3.3}$$

and

$$\beta_R(g_R)=\frac{\partial G}{\partial g}\beta(g)\,, \tag{3.4}$$

with

$$\beta_R(g_R)=a_2g_R{}^2+a_3g_R{}^3\,. \tag{3.5}$$

The coefficient of g^2 in (3.3) denoted by r_2 will turn out to be a free parameter and appear as an integration constant below.

As before, we seek solutions of the differential equation

$$\frac{dG}{dg}=\frac{a_2}{\beta(g)}(G^2+bG^3),\ b=\frac{a_3}{a_2}<0\,. \tag{3.6}$$

Unfortunately, unlike the previous case, we cannot write out the solution explicitly for this case. However, we can do almost as well by getting an implicit form for the solution.

Omitting the algebra we get

$$b\ln\left(\frac{1}{G(g)}+b\right)-\frac{1}{G(g)}=\int_0^g dx\left(\frac{a_2}{\beta(x)}-\frac{1}{x^2}+\frac{b}{x}\right)-\frac{1}{g}-b\ln g+r_2\,, \tag{3.7}$$

where r_2 is the constant that appears in Eq. (3.3). It is easy to check that G defined by (3.7) is the solution of Eq. (3.6) with initial conditions specified by Eq. (3.3).

Although Eq. (3.7) is not explicit, it is more than adequate for determining the relevant properties of $G(g)$ starting from $g=0$. To do this we must keep in mind that in this case $b=a_3/a_2<0$.

Again we have to consider two cases separately: case (1), where $\beta(g)$ starts positive for $g>0$ and develops an ultraviolet (UV) zero at $g=g_\infty$, $\beta(g_\infty)=0$, $-\beta'(g_\infty)<\infty$; case (2), where $\beta(g)$ has no zeros for finite $g=0$ and is positive for $0<g<\infty$.

Case (1). $G(g)\approx g$ for small $g>0$ and increases monotonically, $G'(g)>0$ in the interval $0<g<g_\infty$. As $g\to g_\infty$ from below the right-hand side of Eq. (3.7) diverges, this can occur only if the argument of the logarithm on the left vanishes and we get as $g\to g_\infty$ from below

$$G(g_\infty)=-\frac{1}{b}=-\frac{a_2}{a_3}\equiv g_R^\infty\,. \tag{3.8}$$

Thus again $G(g)$ maps the interval $0\leq g\leq g_\infty$ in a one-to-one manner onto $0\leq g_R\leq g_R^\infty$, where g_R^∞ is the zero of the 't Hooft β_R, $\beta_R(g_R^\infty)=a_2g_R^{\infty 2}+a_3g_R^{\infty 3}=0$.

However, again as $g\to g_\infty$, $G(g)$ will in general develop a branch point. The nature of this branch point will depend on the slope of β at $g=g_\infty$. As before, we take

$$\beta(g)\cong\omega(g-g_\infty),\ \ g\approx g_\infty),\ \ \omega<0\,. \tag{3.9}$$

Then from Eq. (3.7) we get as $g\to g_\infty$ from below

$$G(g)+1/b\cong(g_\infty-g)^{a_2/\omega b},\quad g\approx g_\infty\,. \tag{3.10}$$

The quantity $a_2/\omega b=a_2^2/\omega a_3>0$ for $(\phi^4)_4$, $\omega<0$, $a_3<0$. It is instructive to express Eq. (3.10) in terms of the slope of the 't Hooft function $\beta_R(g_R)$. We define

$$\omega_R=\frac{\partial\beta_R}{\partial g_R}\bigg|_{g_R=g_R^\infty}=a_2^2/a_3,\quad \omega_R<0\,. \tag{3.11}$$

This gives us

$$G(g)\cong-1/b+(g_\infty-g)^{\omega_R/\omega},\quad g\to g_\infty \tag{3.12}$$

and

$$G'(g)\approx\frac{\omega_R}{\omega}(g_\infty-g)^{\omega_R/\omega-1},\quad g\to g_\infty\,. \tag{3.13}$$

As in the $(\phi^4)_2$ and $(\phi^4)_3$ cases, $G'(g)$ will in general vanish or tend to infinity as $g\to g_\infty$. Only if $\omega_R=\omega$ will $G'(g)$ be nonzero as $g\to g_\infty$. The 't Hooft transformation by itself obviously gives no relation between ω and ω_R. However, if from some additional physical input one can show that $G'(g)$ vanishes slowly as $g\to g_\infty$, then $(\omega_R/\omega)-1\ll1$ and one gets $\omega_R\approx\omega$. For the specific Borel summation method used in Ref. 2 the function $\beta_P(g)$ and its corresponding $G_P(g)$ seem to satisfy this property and this explains the approximate agreement between ω_P and ω_R in that case to within 20%.

Case (2). Here $\beta(g)$ has no finite zeros. $G(g)$ increases monotonically in $0\le g<\infty$. The behavior as $g\to\infty$ depends on whether $\int_c^\infty dx/\beta(x)$ converges or diverges. In the first case $\lim_{g\to\infty}G(g)=g_R^{\max}$ and $g_R^{\max}<(-1/b)$, thus $0\le g\le\infty$ is mapped onto $0\le g_R\le g_R^{\max}$. In the second case $\lim_{g\to\infty}G(g)=-1/b=g_R^\infty$, and $0\le g<\infty$ is mapped onto $0\le g_R<g_R^\infty$.

Finally, we should note that in case (1) we only discussed the consequences of a simple zero in $\beta(g)$ at $g=g_\infty$. In general, one could study other possibilities including the situation where the zero at $g=g_\infty$ is a branch point and $\int_c^g dx/\beta(x)$ is convergent as $g\to g_\infty$. For the sake of brevity we limited ourselves to cases (1) and (2).

IV. QED

This case differs from the ϕ^4 cases considered before because the relative sign in the first two terms of the Gell-Mann–Low function is positive not negative. This makes a significant difference in the behavior of the 't Hooft transformation functions.

The renormalization-group function $\beta(\alpha)$ is given up to third order in α by De Rafael and Rosner,[8]

$$\beta(\alpha)=\frac{2}{3\pi}\alpha+\frac{1}{2\pi^2}\alpha^2-\frac{121}{144\pi^3}\alpha^3+O(\alpha^4)\,. \tag{4.1}$$

However, this is not the function that should be transformed by the 't Hooft transformation, since in the Callan-Symanzik equation for QED[9] the operator $\alpha\beta(\alpha)\partial/\partial\alpha$ appears. One should thus look for a 't Hooft transformation for $\bar\beta\equiv\alpha\beta(\alpha)$ which transforms $\bar\beta$ to a two-term expansion. We shall do this below but first it is instructive to consider the transformation of β as given by (4.1).

We seek a transformation $\alpha\to\alpha_R=G(\alpha)$ such that

$$\beta_R(\alpha_R)=a_1\alpha_R+a_2\alpha_R^2,\quad a_1=\frac{2}{3\pi},\ a_2=\frac{1}{2\pi^2}\,. \tag{4.2}$$

The equation we have to solve is

$$\frac{\partial G(\alpha)}{\partial\alpha}=\frac{1}{\beta(\alpha)}[a_1G(\alpha)+a_2G^2(\alpha)]\,, \tag{4.3}$$

with $G(\alpha)=\alpha+O(\alpha^2)$. This is the same as Eq. (2.7) in the $(\phi^4)_2$ case with g replaced by α. The solution will be the same as in Eq. (2.13), but now $b=a_2/a_1>0$, and the denominator can vanish. The interesting case to study is again that where a first zero develops at $\alpha=\alpha_0$, $\beta(\alpha_0)=0$, such that $\int_c^\alpha dx/\beta(x)$ diverges as $\alpha\to\alpha_0$. Then it follows from Eq. (2.13) that $G(\alpha)$ will diverge at some $\alpha=\alpha_1$ with $\alpha_1<\alpha_0$, and given by

$$1=\frac{a_2}{a_1}\alpha_1\exp\left\{\int_0^{\alpha_1}dy\left[\frac{a_1}{\beta(y)}-\frac{1}{y}\right]\right\}\,. \tag{4.4}$$

It is fairly easy to check that for the above case a solution of Eq. (4.4) always exists with $\alpha_1<\alpha_0$. As $\alpha\to\alpha_1$, $G(\alpha)$ will develop a simple pole, $G(\alpha)\approx(\alpha_1-\alpha)^{-1}$, $\alpha\approx\alpha_1$. The domain $0<\alpha<\alpha_1$ is mapped in an invertible manner onto $0<\alpha_R<\infty$ but the zero of $\beta(\alpha)$ is outside this domain. [There is nothing, however, to prevent one from studying the solution in the region $\alpha>\alpha_1$. The second branch of the function $G(\alpha)$ will map the domain $\alpha_1<\alpha<\alpha_0$ onto $-\infty<\alpha_R<-a_2/a_1$. $\beta_R(\alpha_R)$ in this case has no positive zeros, but it has a negative zero at $\alpha_R^0=-a_2/a_1<0$.]

The behavior in the case where $\int_c^\alpha dx/\beta(x)$ converges as $\alpha\to\alpha_0$, $\beta(\alpha_0)=0$, is also worth noting here. In this case Eq. (4.4) may *not* have a solution for $0<\alpha_1<\alpha_0$. The resulting $G(\alpha)$ will vary in some interval $0\le\alpha_R\le\alpha_R^{\max}$ for $0\le\alpha\le\alpha_0$. As $\alpha\to\alpha_0$, $G'(\alpha)\approx[\beta(\alpha)]^{-1}$ and will diverge at the zero. Nevertheless, the whole interval $0\le\alpha\le\alpha_0$ is mapped in a nonsingular manner.

The physically relevant 't Hooft transformation to consider, however, is the one for the function $\alpha\beta(\alpha)\equiv\bar\beta(\alpha)$ since this is the combination that appears in the Callan-Symanzik equation. We want a transformation such that

$$\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}=\bar\beta(\alpha)\frac{\partial}{\partial\alpha}\equiv\bar\beta_R(\alpha_R)\frac{\partial}{\partial\alpha_R}\,, \tag{4.5}$$

where

$$\bar{\beta}_R(\alpha_R) = a_2\alpha_R^{\,2} + a_3\alpha_R^{\,3}\,, \tag{4.6}$$

with $a_2 = 2/3\pi$, $a_3 = 1/2\pi^2$, and $a_3/a_2 \equiv b > 0$. The transformation we want $\alpha_R = \bar{G}(\alpha) = \alpha + O(\alpha^2)$ is now defined by the differential equation

$$\frac{\partial \bar{G}(\alpha)}{\partial \alpha} = \frac{1}{\bar{\beta}(\alpha)}\left[a_2\bar{G}^2(\alpha) + a_3\bar{G}^3(\alpha)\right]. \tag{4.7}$$

This is the same as Eq. (3.6) for $(\phi^4)_4$, except here $b \equiv a_3/a_2 > 0$. The solution is as before

$$b\ln\left(\frac{1}{\bar{G}(\alpha)} + b\right) - \frac{1}{\bar{G}(\alpha)} = \int_0^\alpha dx\left(\frac{a_2}{\bar{\beta}(x)} - \frac{1}{x^2} + \frac{b}{x}\right) - \frac{1}{\alpha} - b\ln\alpha + r_2\,. \tag{4.8}$$

Again the interesting case to study is the case where $\bar{\beta}(\alpha)$ develops a zero at some $\alpha = \alpha_0$, α_0 being the zero nearest the origin, and $\int_c^\alpha dx/\bar{\beta}(x)$ diverges as $\alpha \to \alpha_0$.

The solution $\bar{G}(x)$ starts at zero and increases monotonically and $\bar{G}(x)$ will diverge as $x \to \alpha_1 < \alpha_0$, where α_1 will be given by

$$b\ln b \equiv \int_0^{\alpha_1} dx\left(\frac{a_2}{\beta(x)} - \frac{1}{x^2} + \frac{b}{x}\right) - \frac{1}{\alpha_1} - b\ln\alpha_1 + r_2\,. \tag{4.9}$$

If $\bar{\beta}(\alpha_0) = 0$, and $\int_c^{\alpha_0} dx/\beta(x)$ diverges, this equation always has a solution for some $\alpha_1 < \alpha_0$. To see this, one has to notice that the right-hand side of (4.8) increases monotonically from $-\infty \to +\infty$ as α varies in the interval $0 \le \alpha \le \alpha_0$. It is easy to check that as $\alpha \to \alpha_1$, $\bar{G}(\alpha) \approx (\alpha_1 - \alpha)^{-1/2}$ and develops a branch point at $\alpha = \alpha_1$.

Hence, because of the relative sign of the first two coefficients, the result in QED is not as useful as in the $(\phi^4)_4$ case and we do not get a nonsingular mapping in a domain large enough to reach the first zero of $\bar{\beta}(\alpha)$, if such a zero exists.

The case where $\bar{\beta}(\alpha)$ has no finite zeros can similarly be studied and the result will depend on whether $\int_c^\infty dx/\bar{\beta}(x)$ is finite or diverges.

Actually one can do better in QED if one generalizes the 't Hooft idea and transforms to a new variable α_R that gives a $\bar{\beta}_R$ with three terms, namely,

$$\bar{\beta}_R(\alpha_R) \equiv a_2\alpha_R^{\,2} + a_3\alpha_R^{\,3} + a_4\alpha_R^{\,4}\,. \tag{4.10}$$

The coefficient a_4 willin this case depend on the renormalization scheme of the original $\bar{\beta}(\alpha)$. This is a distinct change in the original motivation of 't Hooft which was to express the GML function in a way that would be independent of a renormalization scheme. However, it is still a useful transformation to consider.

The main thing to notice is that in the scheme of Ref. 8, the first three coefficients as given in Eq. (4.1) give a polynomial which has two nonvanishing real roots, one positive and one negative.

The problem is to study the differential equation

$$\frac{\partial H}{\partial \alpha} = \frac{1}{\bar{\beta}(\alpha)}\left(a_2H^2 + a_3H^3 + a_4H^4\right). \tag{4.11}$$

The solution with initial conditions $H(x) = x + r_2x^2 + O(x^3)$ will be given by

$$\frac{\lambda_1^{\,2}}{\lambda_1 - \lambda_2}\ln\left(\frac{1}{H(\alpha)} + \lambda_1\right) + \frac{\lambda_2^{\,2}}{\lambda_2 - \lambda_1}\ln\left(\frac{1}{H(\alpha)} + \lambda_2\right) - \frac{1}{H(\alpha)} = \int_0^\alpha dx\left(\frac{a_2}{\bar{\beta}(x)} - \frac{1}{x^2} + \frac{a_3}{a_2x}\right) - \frac{1}{\alpha} - \frac{a_3}{a_2}\ln\alpha + r_2\,, \tag{4.12}$$

where $(-\lambda_1)^{-1}$ and $(-\lambda_2)^{-1}$ are the two real roots of the polynomial $1 + (a_3/a_2)x + (a_4/a_2)x^2$, and with a_4 as given by Ref. 8, $\lambda_1 > 0$ and $\lambda_2 < 0$,

$$\lambda_1\lambda_2 = a_4/a_2, \quad \lambda_1 + \lambda_2 = a_3/a_2\,. \tag{4.13}$$

One can now check the properties of $H(x)$ from Eq. (4.12). We do this first for the interesting case where $\bar{\beta}(\alpha_0) = 0$ for some $\alpha = \alpha_0$, α_0 being the zero nearest the origin. Then again $H(\alpha)$ starts at zero for $\alpha \approx 0$ and increases monotonically until $1/H$ becomes equal to $-\lambda_2$, where $(\lambda_2)^{-1}$ is the negative root of the polynomial in (4.10). We are assuming of course that $\int_c^\alpha dx/\bar{\beta}(x)$ diverges as $\alpha \to \alpha_0$ from below we get

$$\lim_{\alpha\to\alpha_0} H(\alpha) = -\frac{1}{\lambda_2} = \alpha_R^0, \quad \lambda_2 < 0 \tag{4.14}$$

where $(-\lambda_2)^{-1}$ can be calculated from Eq. (4.13) and is a zero of $\bar{\beta}_R(\alpha_R)$.

In QED there is a respectable conjecture that if $\bar{\beta}(\alpha_0) = 0$ this zero is an essential zero.[9] If such is the case then $\partial H/\partial\alpha$ as $\alpha \to \alpha_0$ will also develop an essential zero. Otherwise, if $\bar{\beta}$ has a simple zero at $\alpha = \alpha_0$ the properties of $\partial H/\partial\alpha$ as $\alpha \to \alpha_0$ will be the same as in the previous section.

What we have gained by adding an additional term in Eq. (4.11) is that now the full domain up to the first zero $0 \le \alpha < \alpha_0$ is mapped in a nonsingular manner onto $0 \le \alpha_R < \alpha_R^0$, where α_R^0 is now the positive zero of the three-term $\bar{\beta}_R(\alpha_R)$ given in Eq. (4.10). The properties of $H(\alpha)$ in the case where $\bar{\beta}$ has no finite zeros can be deduced as before.

V. QUANTUM CHROMODYNAMICS

This case is similar to QED. Asymptotic freedom here does not make a difference, the relevant property is the relative sign of the first two coefficients of $\beta(\alpha)$. Writing $g^2 = \alpha$ one has[1]

$$\beta(\alpha) = a_2\alpha^2 + a_3\alpha^3 + O(\alpha^4) , \tag{5.1}$$

where for the standard case

$$a_2 < 0,\ a_3 > 0, \text{ and } b \equiv a_3/a_2 > 0 . \tag{5.2}$$

In the notation of Ref. 1

$$a_2 = (8\pi^2)^{-1}(\tfrac{2}{3}N_f - 11) , \quad a_3 = -(8\pi^2)^{-1}(51 - \tfrac{19}{3}N_f) , \tag{5.3}$$

where N_f is the number of flavors, and we consider the case

$$N_f < \tfrac{153}{19} \quad (N_f \leq 8) , \tag{5.4}$$

so that we not only have asymptotic freedom but

$$b = a_3/a_2 > 0 .$$

For N_f such that $\frac{33}{2} > N_f > \frac{153}{19}$ we still have asymptotic freedom but $b = a_3/a_2 < 0$. We shall briefly only discuss the standard case, $N_f \leq 8$.

This is already covered by the case of QED given in Eq. (4.7). Note that the ratio $a_2/\beta(x)$ is always positive in the neighborhood of the origin regardless of asymptotic freedom.

One obtains $\alpha_R = G(\alpha)$ given essentially by Eq. (4.8) and $\beta_R(\alpha_R) = a_2{\alpha_R}^2 + a_3{\alpha_R}^3$. In QCD it is at least hoped that $\beta(\alpha)$ has no finite zeros. We discuss this possibility first. Then, as before, $G(\alpha)$ will increase monotonically as α increases from zero. It could diverge at some point, $\alpha = \alpha_1$, given by

$$b \ln b = \int_0^{\alpha_1} dx\left(\frac{a_2}{\beta(x)} - \frac{1}{x^2} + \frac{b}{x}\right) - \frac{1}{\alpha_1} - b \ln\alpha_1 + r_2 , \tag{5.5}$$

where r_2 is an arbitrary integration constant. Thus, if $\int_c^\infty dx/\beta(x)$ is *divergent*, one can always find a solution of (5.5) for any finite r_2, for some positive α_1. The region in which the 't Hooft transformation is nonsingular is $0 \leq \alpha < \alpha_1$, and as before as $\alpha \to \alpha_1$, $G(x) \approx (\alpha_1 - \alpha)^{-1/2}$.

On the other hand, if $\int_c^\infty dx/\beta(x)$, $c > 0$, converges then one can always choose $|r_2|$ large enough so that Eq. (5.5) has no solution. Then one obtains a 't Hooft mapping which is nonsingular for the whole interval $0 \leq \alpha < \infty$.

The case in QCD where $\beta(\alpha)$ develops a zero for some $\alpha = \alpha_0$ can of course be easily handled and the results are almost identical with the discussion in the previous section following Eq. (4.8).

Finally, for $\frac{153}{19} < N_f < \frac{33}{2}$ we have $b < 0$, and a situation analogous to the $(\phi^4)_4$ case.

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*Permanent address: Department of Mathematics, Cornell University, Ithaca, N. Y. 14853.

[1]G. 't Hooft, Erice Lectures, 1977 (unpublished).

[2]One can use some of the identities obtained via the 't Hooft transformation to check the self-consistency of approximate Borel summation methods such as the one recently given by the first-named author. See N. N. Khuri, Phys. Lett. 82B, 83 (1979).

[3]G. A. Baker, Jr., B. G. Nickel, and D. I. Meiron, Phys. Rev. B 17, 1365 (1978).

[4]J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. 39, 95 (1977).

[5]David J. Gross, in *Methods in Field Theory*, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976), pp. 177–179.

[6]N. N. Khuri, Rockefeller University Report No. C00-2232B-169, 1979 (unpublished). The assumption $G'(g) \neq 0$ at $g = g_\infty$ was made in that paper. Without that assumption the identification of ω with ω_R cannot be made. However, for the specific Borel sums considered $\omega_R \approx \omega_P$, to about 15%, so $G'(g)$ vanishes slowly as $g \to g_\infty$. One of us (N.N.K.) would like to thank C. Itzykson and M. Creutz for helpful comments on this point.

[7]See for example R. Oehme and W. Zimmermann, Phys. Lett. 79B, 314 (1978).

[8]E. De Rafael and J. L. Rosner, Ann. Phys. (N.Y.) 82, 369 (1974).

[9]S. L. Adler, Phys. Rev. D 5, 3021 (1972).