

## Instability of Abelian field configurations in Yang-Mills theory

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It is shown that static purely electric or purely magnetic Abelian field configurations in Yang-Mills theory are unstable when the electric or magnetic field strength is too large over too wide a region. The critical parameter is  $gEL^2$  ( $gBL^2$ ), where  $E$  ( $B$ ) is a measure of the field strength and  $L$  is a measure of the distance over which such field strengths extend.

### I. INTRODUCTION AND SUMMARY

It was shown some time ago by Mandula<sup>1</sup> that the Coulomb solution to the Yang-Mills field equations in the presence of a static external source is unstable when  $gQ/4\pi$  exceeds some critical value. Here  $g$  is the gauge coupling constant and  $Q$  is the total external charge. The method used in such a stability analysis is simply to linearize the Yang-Mills equations with respect to the small oscillations around the Coulomb solution and to see whether these small oscillations do or do not have eigenmodes which grow exponentially in time.

In this paper, it is our purpose to show that the instability of Abelian field configurations in Yang-Mills theory is quite a general phenomenon. We call a field configuration Abelian if it is lined up everywhere in the same direction of isospin, or if it can be lined up in this way by a gauge transformation. In Secs. II and III, we will consider respectively purely electric and purely magnetic static Abelian field configurations and show that they are unstable under small perturbations when  $gEL^2$  (or  $gBL^2$  in the magnetic case) exceeds some critical value which depends on the shape of the configuration. Here  $E$  ( $B$ ) is a measure of the electric (magnetic) field and  $L$  is a length characterizing the volume over which such an electric (magnetic) field extends and in which the small perturbations are applied. The instability of these field configurations arises because gluons can be "pulled out of the vacuum" which screen the initial electric or magnetic field by creating opposing color charge and color current densities.

The instability of the Coulomb solution is a particular case of this general phenomenon. One should note in this regard that the critical parameter  $gEL^2$ , in the case of the Coulomb solution, is equivalent to the critical parameter  $gQ/4\pi$ , since  $E = (Q/4\pi)(1/r^2)$  for the Coulomb solution. At any rate, the critical parameter is always the one dimensionless parameter available. Let us recall that  $g$  has dimension of (action)<sup>-1/2</sup>,  $Q$  has dimension of (action)<sup>1/2</sup>, and  $E$  has dimension of

(action)<sup>1/2</sup> × (length)<sup>-2</sup>.

We study the stability of Yang-Mills field configurations independently of the sources that create these configurations, by choosing to consider small oscillations only in the region of space from which the sources are absent. In particular, we make it clear that the instability Mandula found for the Coulomb solution exists independently of the way the external source is treated.

### II. STABILITY ANALYSIS OF ABELIAN ELECTRIC FIELD CONFIGURATIONS

Let us consider an Abelian static electric field configuration which satisfies the Yang-Mills equations in a region  $R$  free of external sources:

$$\vec{E}^a = -\delta^{a3}\vec{\nabla} A_0(\vec{x}), \quad \nabla^2 A_0 = 0 \quad \text{in } R. \quad (2.1)$$

[Here, as in the rest of the paper, we have taken for simplicity the gauge group to be SU(2). The generalization to larger gauge groups is straightforward.] We will show that if such an electric field is too strong over too wide a region, it becomes unstable under small perturbations. Let us thus consider small time-dependent deviations from our background field configuration:

$$A_0^a = \delta^{a3}A_0(\vec{x}), \quad A_i^a(x) = \alpha_i^a(\vec{x}, t), \quad (2.2)$$

where the  $\alpha_i^a(\vec{x}, t)$  vanish on the boundary of the space region  $R$  and are small compared to  $A_0$  and to  $(1/gA_0)\partial_i A_0$ . We have chosen to work in the gauge where the oscillations in the  $A_0$  field vanish. Neglecting terms which are quadratic or higher in the  $\alpha_i^a$  fields, we obtain for the field strengths

$$E_i^a = F_{0i}^a = -\delta^{a3}\partial_i A_0 + \dot{\alpha}_i^a - g\epsilon^{3ab}A_0\alpha_i^b, \quad (2.3)$$

$$B_{ij}^a = F_{ij}^a = \partial_i \alpha_j^a - \partial_j \alpha_i^a + O(\alpha^2). \quad (2.4)$$

We use the dot over a field to indicate that the time derivative has been taken. The linearized Yang-Mills equations are

$$\begin{aligned} (D_\mu F^{\mu 0})^a &= (D_i E_i)^a \\ &= \vec{\nabla} \cdot \dot{\vec{a}}^a - g\epsilon^{3ab} [A_0(\vec{\nabla} \cdot \vec{a}^b) + 2\vec{a}^b \cdot \vec{\nabla} A_0] \\ &= 0, \end{aligned} \quad (2.5a)$$

$$\begin{aligned} (D_\mu F^{\mu j})^a &= -(D_0 E_j)^a + (D_i B_{ij})^a \\ &= -\ddot{a}_j^a + \partial_i \partial_i a_j^a - \partial_j \partial_i a_i^a \\ &\quad + 2g\epsilon^{3ab} A_0 \dot{a}_j^b - g^2(\delta_3^a \delta_b^3 - \delta_b^a)(A_0)^2 a_j^b \\ &= 0. \end{aligned} \quad (2.5b)$$

The oscillations along the 3 direction of isospin just obey the free Maxwell equations:

$$\begin{aligned} \vec{\nabla} \cdot \dot{\vec{a}}^3 &= 0, \\ -\vec{\nabla} \times (\vec{\nabla} \times \vec{a}^3) - \ddot{\vec{a}}^3 &= 0. \end{aligned} \quad (2.6)$$

These oscillations do not see the background field and do not produce any instabilities. This is of course identical to what happens in pure electromagnetism. On the other hand, the oscillations along the 1 and 2 directions of isospin do see the background field. It is useful to introduce the complex field  $\vec{a} = \vec{a}^1 + i\vec{a}^2$ , in terms of which Eqs. (2.5) for  $a=1$  and 2 become

$$\vec{\nabla} \cdot \dot{\vec{a}} + ig[A_0(\vec{\nabla} \cdot \vec{a}) + 2\vec{a} \cdot \vec{\nabla} A_0] = 0, \quad (2.7a)$$

$$-\ddot{\vec{a}} - \vec{\nabla} \times (\vec{\nabla} \times \vec{a}) - 2igA_0 \dot{\vec{a}} + (gA_0)^2 \vec{a} = 0, \quad (2.7b)$$

or, more compactly,

$$2\vec{\nabla} \cdot (D_0 \vec{a}) - D_0(\vec{\nabla} \cdot \vec{a}) = 0, \quad (2.8a)$$

$$D_0 D_0 \vec{a} + \vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = 0, \quad (2.8b)$$

with  $D_0 \vec{a} = \dot{\vec{a}} + igA_0 \vec{a}$ . For an eigenmode of frequency  $\omega$ ,  $\vec{a}(\vec{x}, t) = \vec{a}(\vec{x}) e^{i\omega t}$ , we have

$$(\omega + gA_0) \vec{\nabla} \cdot \vec{a} + 2g\vec{a} \cdot \vec{\nabla} A_0 = 0, \quad (2.9a)$$

$$\omega^2 \vec{a} = \vec{\nabla} \times (\vec{\nabla} \times \vec{a}) - 2gA_0 \omega \vec{a} - (gA_0)^2 \vec{a}. \quad (2.9b)$$

Gauss's law, Eq. (2.9a), is now superfluous since it follows from taking the divergence of the equation of motion (2.9b).

In the Coulomb gauge,  $\vec{\nabla} \cdot \vec{a} = 0$ , we have

$$\omega^2 \vec{a} = -\nabla^2 \vec{a} - 2gA_0 \omega \vec{a} - (gA_0)^2 \vec{a}. \quad (2.10)$$

The first term on the right-hand side of Eq. (2.10) is positive definite but becomes relatively unimportant when the size of the region  $R$ , in which the small oscillations are allowed, is large. The last term on the right-hand side of Eq. (2.10) is negative definite and proportional to the square of the background  $E$  field. Consequently, Eq. (2.10) will in general have runaway solutions—i.e., eigenmodes with imaginary  $\omega$ —when the background  $E$  field is large enough over wide-enough regions.

As an illustrative example, we will consider a constant electric field in a cylindrical box of radius  $R$  and length  $2L$ :

$$A_0(\rho, \phi, z) = -Ez \text{ for } \rho < R \text{ and } -L < z < +L,$$

where  $\rho$ ,  $\phi$ , and  $z$  are the usual cylindrical coordinates. Equation (2.10) becomes in this case

$$\omega^2 \vec{a} = -\nabla^2 \vec{a} + 2gEz\omega \vec{a} - (gEz)^2 \vec{a}. \quad (2.11)$$

Let us use the ansatz

$$\vec{a} = \hat{\phi} \chi(\rho) \psi(z). \quad (2.12)$$

$\chi(\rho)$  must then satisfy the Bessel equation of index 1 and vanish at  $\rho=R$ . Thus

$$\chi_n(\rho) = J_1(k_n \rho), \quad n=1, 2, 3, \dots, \quad (2.13)$$

where  $k_n = \beta_n/R$  and the  $\beta_n$  are the zeros of  $J_1$  ( $\beta_1 = 3.831$ ,  $\beta_2 = 7.002$ ,  $\beta_3 = 10.174$ , ...). For a given  $n$ ,  $\psi(z)$  must be a solution of

$$[\omega^2 - (k_n)^2] \psi(z) = \left[ -\frac{\partial^2}{\partial z^2} + 2gE\omega z - (gEz)^2 \right] \psi(z), \quad (2.14)$$

with the boundary condition  $\psi(z = \pm L) = 0$ . It is useful to redefine  $u = z/L$ ,  $\nu = \omega L$ , and  $\kappa_n = k_n L$ , in terms of which Eq. (2.14) becomes

$$[\nu^2 - (\kappa_n)^2] \psi(u) = \left[ -\frac{\partial^2}{\partial u^2} + 2p\nu u - p^2 u^2 \right] \psi(u), \quad (2.15)$$

where  $p = gEL^2$  and with the boundary condition  $\psi(u = \pm 1) = 0$ . When the parameter  $p$  increases beyond a certain critical value  $P_{\text{crit}}$ , solutions to Eq. (2.15) with negative  $\nu^2$  appear; these are unstable modes. Of course, the  $n=1$  modes will become unstable first.  $P_{\text{crit}}$  is the value of  $p$  for which Eq. (2.15) with  $n=1$ , has its first eigenmode with  $\nu=0$ . Thus  $P_{\text{crit}}$  is determined by the condition that the solution of

$$-(\kappa_1)^2 \psi(u) = \left( -\frac{\partial^2}{\partial u^2} - P_{\text{crit}}^2 u^2 \right) \psi(u) \quad (2.16)$$

has its first zero (node) at  $u = \pm 1$ . The solutions of Eq. (2.16) are the parabolic cylinder functions.

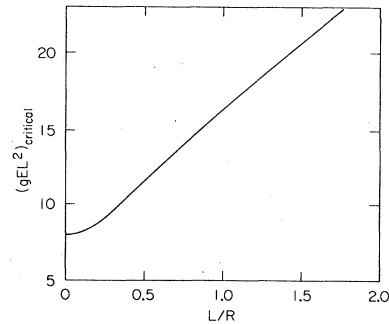


FIG. 1. Plot of the critical value of  $gEL^2$  versus  $R/L$ , for a constant electric field  $E$  in a cylindrical box of radius  $R$  and length  $2L$ .

In Fig. 1,  $P_{\text{crit}}$  is plotted as a function of  $L/R = \kappa_1/3.831$ . We find, of course, that  $P_{\text{crit}}$  increases when  $R$  decreases. When  $R \rightarrow \infty$ ,  $P_{\text{crit}}$  approaches  $\sim 8.0$ .

### III. STABILITY ANALYSIS OF ABELIAN MAGNETIC FIELD CONFIGURATIONS

Let us now consider an Abelian static magnetic field configuration which satisfies the Yang-Mills equations in a region  $R$  free of external sources:

$$B_{ij}^a(\vec{x}) = \delta^{ab}(\partial_i A_j(\vec{x}) - \partial_j A_i(\vec{x})), \quad (3.1)$$

with

$$\partial_i(\partial_i A_j(\vec{x}) - \partial_j A_i(\vec{x})) = 0 \quad \text{in } R. \quad (3.2)$$

We will show that such a  $B$  field configuration becomes unstable under small perturbations, when  $gBL^2$  exceeds a certain critical value. Here  $B$  is a measure of the magnitude of the magnetic field, and  $L$  is a measure of the spatial size of the region over which such a magnetic field extends, and in

which the small perturbations are applied. The distance over which the magnetic field extends in the transverse direction becomes unimportant when it is larger than a few times  $1/\sqrt{gB}$ . The critical value of  $gBl^2$ , where  $l$  is the distance over which the magnetic field extends in the longitudinal direction, is then of order  $\pi^2$ .

Let us thus consider small oscillations around our background field configuration in the  $A_0 = 0$  gauge:

$$A_0 = 0, \quad (3.3)$$

$$A_i^b(\vec{x}, t) = \delta^{b3} A_i(\vec{x}) + a_i^b(\vec{x}, t),$$

where the  $a_i^b$  are small and vanish outside and on the boundary of  $R$ . To first order in  $a_i^b$ , the electric and magnetic fields are

$$E_i^b = \dot{a}_i^b, \quad (3.4)$$

$$B_{ij}^b = \delta^{b3}(\partial_i A_j - \partial_j A_i) + \partial_i a_j^b - \partial_j a_i^b - g\epsilon^{3bc}(A_i a_j^c - A_j a_i^c).$$

The linearized Yang-Mills equations are

$$(\vec{D} \cdot \vec{E})^b = \vec{\nabla} \cdot \dot{\vec{a}}^b - g\epsilon^{3bc} \vec{A} \cdot \dot{\vec{a}}^c = 0, \quad (3.5a)$$

$$(-D_0 E_j + D_i B_{ij})^b = -\ddot{a}_j^b + \partial_i \partial_i a_j^b - \partial_j \partial_i a_i^b - g^2(\delta_c^b - \delta_c^3 \delta_c^3)(A_i A_i a_j^c - A_j A_i a_i^c) + g\epsilon^{3bc} [-(\partial_i A_i) a_j^c + (\partial_i a_i^c) A_j - 2A_i \partial_i a_j^c + 2a_i^c \partial_i A_j - a_i^c \partial_j A_i + A_i \partial_j a_i^c]. \quad (3.5b)$$

For an eigenmode of frequency  $\omega \neq 0$ ,  $\vec{a}^b(\vec{x}, t) = \vec{a}^b(\vec{x}) e^{i\omega t}$ , Gauss's law, Eq. (3.5a), implies

$$\vec{\nabla} \cdot \vec{a}^b - g\epsilon^{3bc} \vec{A} \cdot \vec{a}^c = 0 \quad (3.6)$$

which can be used to simplify Eq. (3.5b) to

$$-\ddot{a}_j^b + \partial_i \partial_i a_j^b - g^2(\delta_c^b - \delta_c^3 \delta_c^3) A_i A_i a_j^c + g\epsilon^{3bc} [-(\partial_i A_i) a_j^c - 2A_i \partial_i a_j^c + 2a_i^c (\partial_i A_j - \partial_j A_i)] = 0. \quad (3.7)$$

The modes along the 3 direction of isospin simply satisfy Maxwell's equations:

$$\begin{aligned} \vec{\nabla} \cdot \vec{a}^3 &= 0, \\ -\ddot{\vec{a}}^3 + \nabla^2 \vec{a}^3 &= 0. \end{aligned} \quad (3.8)$$

They do not see the background  $B$  field and do not produce any instabilities. On the other hand, the modes along the 1 and 2 directions of isospin do see the background field. It is useful to define the complex field

$$\vec{a} = \vec{a}^1 + i\vec{a}^2 \quad (3.9)$$

in terms of which the linearized Yang-Mills equations become

$$\vec{\nabla} \cdot \vec{a} + ig \vec{A} \cdot \vec{a} = 0, \quad (3.10a)$$

$$-\ddot{\vec{a}} + \nabla^2 \vec{a} - g^2(\vec{A})^2 \vec{a} + ig [(\vec{\nabla} \cdot \vec{A}) \vec{a} + 2A_i \partial_i \vec{a} - 2\vec{B} \times \vec{a}] = 0, \quad (3.10b)$$

or more succinctly

$$\vec{D} \cdot \vec{a} = 0, \quad (3.11a)$$

$$\omega^2 \vec{a} = -D_k D_k \vec{a} + 2ig \vec{B} \times \vec{a}, \quad (3.11b)$$

with  $D_k \vec{a} = \partial_k \vec{a} + ig A_k \vec{a}$ . Equation (3.11b) is the equation for a massless charged particle with spin 1 in a magnetic field. The kinetic energy term is positive definite, but the interaction energy of the spin with the magnetic field can have either sign. We will show that, because of the latter, Eq. (3.11b) has in general solutions with negative  $\omega^2$ , i.e., unstable modes.

To illustrate this effect, we will use the example of a constant magnetic field oriented in the  $\hat{z}$  direction of space:

$$\vec{B} = \hat{z}B \text{ and } \vec{A} = \frac{1}{2} \hat{\phi} \rho B, \quad (3.12)$$

where  $z$ ,  $\rho$ , and  $\phi$  are the usual cylindrical coordinates. One obtains

$$\begin{aligned} \omega^2 a_\epsilon &= -D_k D_k a_\epsilon - 2g\epsilon B a_\epsilon \\ &= \left( -\nabla^2 - igB \frac{\partial}{\partial \phi} + \frac{1}{4} g^2 B^2 \rho^2 - 2\epsilon gB \right) a_\epsilon, \end{aligned} \quad (3.13)$$

where  $\epsilon = -1, 0, +1$  and

$$a_0 = a_z, \quad a_{\pm 1} = a_x \pm ia_y. \quad (3.14)$$

Because of the presence of a harmonic-oscillator potential in Eq. (3.13), it is useful to introduce the creation and annihilation operators:

$$C_x = \frac{\partial}{\partial x} + \frac{1}{2} gBx, \quad C_x^\dagger = -\frac{\partial}{\partial x} + \frac{1}{2} gBx, \quad (3.15)$$

$$C_y = \frac{\partial}{\partial y} + \frac{1}{2} gBy, \quad C_y^\dagger = -\frac{\partial}{\partial y} + \frac{1}{2} gBy,$$

which satisfy

$$\begin{aligned} [C_x, C_x^\dagger] &= [C_y, C_y^\dagger] = gB, \\ [C_x, C_y] &= [C_x^\dagger, C_y^\dagger] = 0, \end{aligned} \quad (3.16)$$

and

$$C_u = \frac{1}{\sqrt{2}} (C_x - iC_y), \quad C_u^\dagger = \frac{1}{\sqrt{2}} (C_x^\dagger + iC_y^\dagger), \quad (3.17)$$

$$C_d = \frac{1}{\sqrt{2}} (C_x + iC_y), \quad C_d^\dagger = \frac{1}{\sqrt{2}} (C_x^\dagger - iC_y^\dagger),$$

which satisfy similar commutation relations. We have

$$\begin{aligned} -\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{4} g^2 B^2 \rho^2 &= C_x^\dagger C_x + C_y^\dagger C_y + gB \\ &= C_u^\dagger C_u + C_d^\dagger C_d + gB, \end{aligned} \quad (3.18)$$

and

$$L_z = (-i) \frac{\partial}{\partial \phi} = \frac{1}{gB} (C_u^\dagger C_u - C_d^\dagger C_d). \quad (3.19)$$

Thus, Eq. (3.13) becomes

$$\omega^2 a_\epsilon = \left[ -\frac{\partial^2}{\partial z^2} + 2C_u^\dagger C_u + (1 - 2\epsilon)gB \right] a_\epsilon, \quad (3.20)$$

whose solutions are

$$a_{\epsilon; k, m} = e^{ikz} (C_u^\dagger)^n (C_d^\dagger)^{n-m} e^{-gB\rho^2/4}, \quad (3.21)$$

for  $n=0, 1, 2, 3, \dots$  and  $m=n, n-1, n-2, \dots$  with eigenfrequencies<sup>2</sup>

$$\omega^2 = k^2 + gB(1 - 2\epsilon + 2n) \quad (3.22)$$

and eigenvalues of the  $z$  component of angular momentum

$$L_z a_{\epsilon; k, n, m} = m a_{\epsilon; k, n, m}. \quad (3.23)$$

We thus find that the  $\epsilon = +1$ ,  $n=0$  eigenmodes are unstable for all values of  $m$  and small values of  $k$ . However, so far, we have not yet made use of Gauss's law, Eq. (3.10a), which for our configuration Eq. (3.12) takes the form

$$\begin{aligned} \vec{D} \cdot \vec{a} &= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} + ig \frac{\rho B}{2} (-\sin\phi a_x + \cos\phi a_y) \\ &= \frac{\partial a_0}{\partial z} + \left( \frac{\partial}{\partial x} - \frac{i}{2} gBy \right) \frac{a_{+1} + a_{-1}}{2} \\ &\quad + \left( \frac{\partial}{\partial y} + \frac{i}{2} gBx \right) \frac{a_{+1} - a_{-1}}{2} \\ &= \frac{\partial a_0}{\partial z} + \frac{1}{2} (C_x - iC_y) a_{+1} + \frac{1}{2} (-C_x^\dagger - iC_y^\dagger) a_{-1} \\ &= \frac{\partial a_0}{\partial z} + \frac{1}{\sqrt{2}} C_u a_{+1} - \frac{1}{\sqrt{2}} C_u^\dagger a_{-1} = 0. \end{aligned} \quad (3.24)$$

It is clear that all the unstable modes [Eq. (3.21) with  $\epsilon = +1$  and  $n=0$ ] do satisfy Gauss's law.

If we only allow small oscillations inside a box of length  $l$  along the direction of the magnetic field, we have  $k = p\pi/l$  where  $p$  is a nonzero integer and

$$\omega^2 = \frac{p^2 \pi^2}{l^2} + gB(1 - 2\epsilon + 2n). \quad (3.25)$$

The unstable modes appear as soon as

$$gBl^2 > \pi^2. \quad (3.26)$$

*Note added.* While this work was being written up, we received a report from S.-J. Chang and N. Weiss<sup>3</sup> in which similar calculations are performed.

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<sup>1</sup>J. Mandula, Phys. Lett. **67B**, 175 (1977); M. Magg, *ibid.* **74B**, 246 (1978).

<sup>2</sup>This result has already been stated in N. K. Nielsen and P. Olesen, Nucl. Phys. **B144**, 376 (1978). We

thank J. Sapirstein for pointing out the existence of this paper to us.

<sup>3</sup>S. J. Chang and N. Weiss, preceding paper, Phys. Rev. D **20**, 869 (1969).