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Instability of constant Yang-Mills fields

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Both a constant (non-Abelian) electric field and a constant magnetic field are solutions to the sourceless Yang-Mills field equations in Minkowski space. We discuss the stability of these solutions under small fluctuations in the potentials. We conclude that a magnetic field B, which is constant over a region of size L, has instabilities which grow exponentially in time provided $gBL^2 \gtrsim 1$. A constant electric field is unstable. A fluctuation in such a field will accelerate continuously in the direction of the field.

I. INTRODUCTION

In the past few years the theory of a classical Yang-Mills field interacting with static external sources has been shown to possess interesting behavior.[1-5] This theory is defined for any gauge group G [which we shall usually take to be SU(2)] by a potential function $A_\mu^a(\vec{x},t)$ in the adjoint representation of G, where $\mu = 0,\ldots,3$ are space-time indices and $a=1,\ldots,n$ [where n is the order of G; $n=3$ for SU(2)] are the group indices. A_μ^a is analogous to the covariant vector potential in electrodynamics. From A_μ^a one defines a field strength tensor

$$F_{\mu\nu}^a(\vec{x},t) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \,, \tag{1.1}$$

where f^{abc} are the structure constants of G. [$f^{abc} = \epsilon^{abc}$ for $G=\text{SU}(2)$.] The Yang-Mills equations are the analogs of Maxwell's equations and are given by

$$D_\mu F_{\mu\nu}^a = \partial_\mu F_{\mu\nu}^a + g f_{abc} A_\mu^b F_{\mu\nu}^c = J_\nu^a \,, \tag{1.2}$$

where J_μ^a is an external current. For a static source (Ref. 6) $J_\mu^a(\vec{x},t) = \delta_{\mu 0}\rho^a(\vec{x},t)$, Eq. (1.2) becomes

$$\begin{aligned} D_i E_i^a &= \rho^a \,, \\ D_0 E_j^a &= D_i F_{ij}^a \,, \end{aligned} \tag{1.3}$$

where

$$E_i^a = F_{0i}^a \,, \quad (D_\mu \phi)^a = \partial_\mu \phi^a + g f^{abc} A_\mu^b \phi^c \,, \tag{1.4}$$

and $i=1,2,3$ refers to the spatial components of a vector.

Equation (1.2) has the property of being gauge invariant. If $U(\vec{x},t)$ is an element of G in the adjoint representation [U is an $n\times n$ matrix; 3×3 for SU(2)] and if

$$A_\mu^a(\vec{x},t) \to U_{ab} A_\mu^b + \frac{1}{2g} f^{abc} (\partial_\mu U)_{bd} U_{dc}^{-1} \tag{1.5a}$$

and

$$J_\mu^a(\vec{x},t) \to U_{ab} J_\mu^b \,, \tag{1.5b}$$

then Eqs. (1.2) and (1.3) are invariant. The conserved energy of a configuration is given by

$$H = \int d^3x \, \tfrac{1}{2}(E^2 + B^2) \,, \tag{1.6}$$

where

$$E^2 = E_i^a E_i^a \,, \quad B^2 = \tfrac{1}{2} F_{ij}^a F_{ij}^a \,.$$

Suppose $\rho^a(\vec{x},t) = \delta^{a1} q(\vec{x})$ so that the sources are lined up in the same isotopic (or gauge group) direction. [For SU(2) this usually can be accomplished by a gauge transformation. This point, as well as its generalization to other groups, is discussed in Ref. 3]. Then with the ansatz $A_\mu^a(\vec{x},t) = \delta_{a1} C_\mu(\vec{x},t)$ all the nonlinearities in (1.3) disappear and Eq. (1.3) reduces to the usual Maxwell equations of electrodynamics. As a result such a source admits fields which are simply a Coulomb field plus an arbitrary radiation field.

Mandula[1] has studied the stability of these "Coulomb solutions" under small fluctuations in the Coulomb field. He considered a thin spherical shell of isotopic charge for the gauge group SU(2) and asked whether small fluctuations in the Coulomb field have any modes which grow exponentially with time, indicating an instability in the Coulomb solution. He found that for $gQ < \frac{3}{2}$ (where Q is the strength of the source) there were no unstable modes, whereas for $gQ > \frac{3}{2}$ there were unstable modes which tend to screen the isotopic charge of the source. Further work on these instabilities has been done by Maag.[2] Sikivie and Weiss[3] have found large classes of solutions to (1.3) with lower energy than the Coulomb solution and with the isotopic charge of the source completely screened. Some of these solutions exist for all values of gQ and others only if $gQ > (gQ)_{\text{crit}}$ which depends on the shape of the source.

In this paper we study the question of stability of source-free Yang-Mills fields by considering separately a constant magnetic field $B_i^a = \frac{1}{2}\epsilon_{ijk} F_{jk}^a$ and a constant electric field $F_{0i}^a = E_i^a$. These fields will be constant in space and in time and they are solutions to the field equations if A_μ^a is properly chosen. We consider these configurations $A_\mu^a(\vec{x},t)$

and imagine small fluctuations about A^a_μ, $A^a_\mu \to A^a_\mu + \delta A^a_\mu$. The field equations (1.2) (with $J_\mu = 0$) are linearized in δA and we find the normal modes of oscillations which satisfy $\delta A(\vec{x}, t) = \delta A(\vec{x}) e^{-i\omega t}$. If ω is real for all modes, then δA will remain small for all time. The field will then remain essentially unchanged to order δA. This is, of course, the case in classical electrodynamics. However, if one or more modes have complex ω, then δA will grow exponentially in time. Eventually, nonlinearities will change its character drastically.

In Sec. II, we write down the equations satisfied by a small fluctuation in the presence of a background classical field. We also introduce our notation. In Sec. III we discuss the stability of a constant magnetic field B. We find that the equations of motion for a small fluctuation are identical to those of an ordinary charged particle with a magnetic moment moving under the influence of a magnetic field. In particular, we find that the lowest orbital modes are unstable. These unstable modes are localized and have a size $\sim(gB)^{-1/2}$. We can summarize our findings crudely by saying that if a fluctuation has a size L then it will be unstable if and only if $gBL^2 \gtrsim 1$. As a consequence, any field configuration for which B can be regarded as constant over a region of size $L \gtrsim 1/\sqrt{gB}$ will have unstable modes.

In Sec. IV the stability of a constant electric field $E^a_i = E\delta_{a3}\delta_{i1}$ is explored. The results are strikingly different from those for the constant B field. We find that all the normal modes in the one and two isotopic directions have a velocity $(\nabla\delta A)$ which increases to infinity as $|x| \to \infty$, where x is the direction of the electric field. The physical interpretation is that a localized instability carries isotopic charge. It will thus be accelerated by the E field in the $\pm x$ direction. Eventually its velocity becomes so large $(\nabla\delta A \sim E)$ that nonlinearities will appear and the field at $t \gg 0$ will no longer be approximately a constant E field. In Sec. V we discuss our results.

II. THE MODEL

In this paper we study the instabilities for two classes of simple classical solutions—the constant B field and the constant E field. In this section we shall derive the equations for small field fluctuations and introduce our notation.

We consider a classical Yang-Mills system described by

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} , \tag{2.1}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu . \tag{2.2}$$

We can generalize our consideration to other gauge groups by replacing ϵ^{abc} by f^{abc}, the structure constant of the group. In this paper we shall concentrate on the SU(2) gauge group.

In Eqs. (2.1), (2.2), and hereafter, superscripts a, b, c stand for gauge group (referred to as isotopic spin) indices, and subscripts $\mu, \nu, \ldots$, stand for Lorentz indices. Repeated indices are summed over, and

$$A_\mu B_\mu \equiv -A_0 B_0 + A_i B_i , \quad i = 1, 2, 3 . \tag{2.3}$$

The action principle gives rise to the field equation

$$(D_\mu F_{\mu\nu})^a \equiv (\delta^{ac}\partial_\mu + g\epsilon^{abc} A^b_\mu) F^c_{\mu\nu} = 0 . \tag{2.4}$$

Equations (2.2) and (2.4) are the Yang-Mills equations.

We now consider a small variation to a given classical Yang-Mills solution

$$A'^a_\mu = A^a_\mu + \delta A^a_\mu , \tag{2.5a}$$

$$F'^a_{\mu\nu} = F^a_{\mu\nu} + \delta F^a_{\mu\nu} . \tag{2.5b}$$

It is straightforward to see that to order δA

$$\delta F^a_{\mu\nu} = (D_\mu \delta A_\nu)^a - (D_\nu \delta A_\mu)^a . \tag{2.6}$$

The Yang-Mills equations (2.4) become

$$(D_\mu \delta F_{\mu\nu})^a + g\epsilon^{abc} F^c_{\mu\nu} \delta A^b_\mu = 0 , \tag{2.7}$$

where

$$D^{ac}_\mu \equiv \delta^{ac}\partial_\mu + g\epsilon^{abc} A^b_\mu .$$

We shall see that the presence of the second term in (2.7) is crucial for the instability of the classical Yang-Mills solutions discussed in the next section. Equations (2.6) and (2.7) are linear in δF and δA. We can thus study the stability condition by examining individual normal modes.

III. CONSTANT MAGNETIC FIELD

In this section we consider a constant magnetic field $B^a_i = \delta_{i3}\delta_{a3} B$ $(E^a_i = 0)$ which can be "derived" from the potential

$$A^a_i = \delta_{a3} \tfrac{1}{2} B(-y\delta_{i1} + x\delta_{i2}) , \tag{3.1a}$$

$$A^a_0 = 0 . \tag{3.1b}$$

The potential A^a_μ and the field tensor $F^a_{\mu\nu} = (E^a, B^a)$ form a solution to the Yang-Mills field equations. We study the stability of this solution by applying a small disturbance δA^a_i to (3.1) and linearizing the field equations in δA^a_i as in Sec. II. We shall work in the gauge $A_0 = 0$ so that

$$\delta A^a_0 = 0 . \tag{3.2}$$

The unperturbed electric and magnetic field strength are

$$F^a_{0i} \equiv E^a_i = 0 , \tag{3.3a}$$

$$F^a_{ij}=\epsilon_{ij3}B \quad (i,j=1,2,3)\,, \tag{3.3b}$$

and the field fluctuations are

$$\delta E^a_i = \partial_0 \delta A^a_i\,, \tag{3.4a}$$

$$\delta F^a_{ij} = \partial_i \delta A^a_j - \partial_j \delta A^a_i + \tfrac{1}{2} g B \epsilon^{a3b}[\delta A^b_j(-y\delta_{i1}+x\delta_{i2}) - \delta A^b_i(-y\delta_{j1}+x\delta_{j2})] + O(\delta A^2)\,. \tag{3.4b}$$

In the $A_0=0$ gauge the field equations are [see Eqs. (2.4) and (2.5)]

$$\partial_i E^a_i + g(A_i \times E_i)^a = 0\,, \tag{3.5a}$$

$$\frac{dE^a_i}{dt} = (D_i F_{ij})^a = \partial_i F^a_{ij} + g(A_i \times F_{ij})^a\,, \tag{3.5b}$$

where the symbol $\times$ denotes a vector product in isotopic spin space.

We shall start with Gauss's law Eq. (3.5a) which, in our case, is equivalent to (2.7) with $\nu=0$:

$$\partial_i \partial_0 \delta A^a_i + \tfrac{1}{2} g B[\hat{3}\times(-y\partial_0\delta\vec{A}_1 + x\partial_0\delta\vec{A}_2)]^a = 0\,, \tag{3.6}$$

where $\hat{3}^a \equiv \delta_{a3}$ is a unit vector in the third isotopic direction. Since (3.1) has no explicit time dependence and since our equations will be linear in δA, we shall look for the normal modes of oscillation by setting $\partial_0 \to -i\omega$ so that (3.6) becomes

$$\partial_i \delta\vec{A}_i = -\tfrac{1}{2} g B \hat{3}\times(-y\delta\vec{A}_1 + x\delta\vec{A}_2)\,, \tag{3.7}$$

where we have used the vector notation to denote the isotopic spin. Equation (3.7) is a simpler form of Gauss's law (3.5a).

We now turn to the time evolution equation (3.5b) which is equivalent to (2.7) with $\nu=j$. Using (3.3) and (3.4) and substituting (3.7) for $\partial_i \delta A_i$, we find

$$\partial_0{}^2 \delta\vec{A}_j = \nabla^2 \delta\vec{A}_j + 2gB\hat{3}\times(\delta_{j1}\delta\vec{A}_2 - \delta_{j2}\delta\vec{A}_1) + gB\hat{3}\times(-y\partial_1\delta\vec{A}_j + x\partial_2\delta\vec{A}_j) + \frac{g^2B^2(x^2+y^2)}{4}\hat{3}\times(\hat{3}\times\delta\vec{A}_j)\,. \tag{3.8}$$

In cylindrical coordinates,

$$x=\rho\cos\theta\,, \quad y=\rho\sin\theta\,, \tag{3.9}$$

we have

$$x^2+y^2=\rho^2\,, \quad x\partial_2 - y\partial_1 = \partial_\theta\,,$$

and

$$\nabla^2 = \nabla_\perp{}^2 + \frac{\partial^2}{\partial z^2} \tag{3.10}$$

with

$$\nabla_\perp{}^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\,. \tag{3.11}$$

In the Fourier-transformed space of x_0 and x_3 we have

$$\partial_0 \to -i\omega\,, \quad \partial_3 \to ik_3\,,$$

and consequently,

$$(-\omega^2+k_3{}^2)\delta\vec{A}_j = \nabla_\perp{}^2\delta\vec{A}_j + 2gB\hat{3}\times(\delta_{j1}\delta\vec{A}_2 - \delta_{j2}\delta\vec{A}_1) + gB\hat{3}\times(\partial_\theta\delta\vec{A}_j) + \frac{g^2B^2\rho^2}{4}\hat{3}\times(\hat{3}\times\delta\vec{A}_j)\,. \tag{3.12}$$

In the following we shall consider various components of (3.12) separately.

A. $a=3$ and j arbitrary

The third isotopic component of (3.12) is

$$(-\omega^2+k_3{}^2)\delta A^{(3)}_j = \nabla_\perp{}^2\delta A^3_j = -k_\perp{}^2\delta A^3_j\,, \tag{3.13}$$

which gives rise to the eigenfrequency

$$\omega^2 = k_\perp{}^2 + k_3{}^2\,. \tag{3.14}$$

Thus, the normal modes corresponding to $\delta A^3_j \neq 0$ are all real and stable, and small fluctuations do not blow up in time.

B. $a=1,2$ and $j=3$

The $j=3$ component of (3.12) with $a=1,2$ is

$$(-\omega^2+k_3{}^2)\delta A^1_3 = \nabla_\perp{}^2\delta A^1_3 - gB\partial_\theta\delta A^2_3 - \tfrac{1}{4}g^2B^2\rho^2\delta A^1_3\,, \tag{3.15a}$$

$$(-\omega^2+k_3{}^2)\delta A^2_3 = \nabla_\perp{}^2\delta A^2_3 + gB\partial_\theta\delta A^1_3 - \tfrac{1}{4}g^2B^2\rho^2\delta A^2_3\,. \tag{3.15b}$$

This can be diagonalized by defining

$$\delta A^{(\pm)}_3 = \delta A^{(1)}_3 \pm i\delta A^{(2)}_3 \tag{3.16}$$

so that

$$(-\omega^2+k_3{}^2)\delta A^{(\pm)}_3 = \nabla_\perp{}^2\delta A^{(\pm)}_3 \pm gBi\partial_\theta\delta A^{(\pm)}_3 - \tfrac{1}{4}g^2B^2\rho^2\delta A^{(\pm)}_3\,. \tag{3.17}$$

In the state of a given angular momentum $L_3=m$, we can replace ∂_θ by im, obtaining

$$\nabla_\perp{}^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} \tag{3.18}$$

and

$$\left(-\nabla_\perp{}^2 + \frac{g^2B^2\rho^2}{4}\right)\delta A^{(\pm)}_3 = (\omega^2 - k_3{}^2 \mp gBm)\delta A^{(\pm)}_3\,. \tag{3.19}$$

This is the equation for a two-dimensional harmonic oscillator. Physically, Eqs. (3.17) and (3.19) describe precisely the motion of a charged particle with spin $S_3=0$ in a constant magnetic field.[7] In either case, for $\delta A^{(\pm)}_3$ to be bounded, we must have

$$(\omega^2 - k_3^2 \mp gBm) = (2n + |m| + 1)gB, \quad n = 0,1,2,\ldots. \tag{3.20}$$

Thus,

$$\omega^2 = k_3^2 + (2n+1)gB + (|m| \pm m)gB \geq gB > 0. \tag{3.21}$$

As a result, all normal modes with $\delta A_3^{(1,2)} \neq 0$ are stable since ω is always real for them.

C. $a = 1,2$ and $j = 1,2$

Finally we turn to Eq. (3.12) for $j=1,2$ and $a=1,2$. The algebra is similar to that for δA_3. We define

$$\delta A_\pm^a = \delta A_1^a \pm i\delta A_2^a \tag{3.22a}$$

and

$$\delta A_j^\pm = \delta A_j^1 \pm i\delta A_j^2. \tag{3.22b}$$

After some algebra, we find

$$(-\nabla_\perp^2 + \tfrac{1}{4}g^2B^2\rho^2)\delta A_+^\pm = (\omega^2 - k_3^2 \mp mgB \pm 2gB)\delta A_+^\pm, \tag{3.23a}$$

$$(-\nabla_\perp^2 + \tfrac{1}{4}g^2B^2\rho^2)\delta A_-^\pm = (\omega^2 - k_3^2 \mp mgB \mp 2gB)\delta A_-^\pm. \tag{3.23b}$$

Equations (3.23) have the same structure as those of charged particles with spin $S_3 = \pm 1$ and magnetic moment $2g$ moving in a constant magnetic field.[7] The eigenvalue conditions for the Schrödinger equations for $\delta A_\pm^+$ and $\delta A_\pm^-$ are, respectively,

$$\omega^2 = k_3^2 + 2ngB + gB(1 \mp 2) + gB(|m| + m) \tag{3.24a}$$

and

$$\omega^2 = k_3^2 + 2ngB + gB(1 \pm 2) + gB(|m| - m). \tag{3.24b}$$

Thus, if we choose $n=0$, and if we choose only $\delta A_+^+ \neq 0$ for $m<0$, or only $\delta A_-^- \neq 0$ for $m>0$, then

$$\omega^2 = k_3^2 - gB. \tag{3.25}$$

Hence, ω^2 is negative if $k_3^2 < gB$. Thus, for $k_3^2 < gB$, we have unstable modes. The spatial dependence of the wave function is

$$\delta A = \text{const} \times e^{im\theta}\rho^{|m|}e^{-gB\rho^2/4}. \tag{3.26}$$

For $m=0$ the unstable-mode wave function has a size $L \sim (gB)^{-1/2}$. For $m \neq 0$ the unstable-mode wave function forms a ring of radius $\rho \sim (2|m|/gB)^{1/2}$ and a width $\sim (gB)^{-1/2}$.

To be sure we really have a solution to the field equations we need to check if Gauss's law (3.7) is satisfied by the unstable mode. We should also be certain that the field strengths grow exponentially with time to ensure that our instability is not gauge dependent. For the case $m=0$, $k_3=0$, the unstable mode is

$$\delta A_1^a = \epsilon e^{-gB\rho^2/4}e^{\sqrt{gB}\,t}\delta_{a1}, \tag{3.27a}$$

$$\delta A_2^a = -\epsilon e^{-gB\rho^2/4}e^{\sqrt{gB}\,t}\delta_{a2}, \tag{3.27b}$$

$$\delta A_3^a = 0. \tag{3.27c}$$

Here ϵ is an arbitrary small parameter and $e^{-gB\rho^2/4}$ is the ground state $m=0$ solution. It is easy to check that (3.27) obeys Gauss's law (3.7). The result can be generalized to other $n=0$ solutions with $m \neq 0$ and $k_3 \neq 0$.

The electric and magnetic fields are

$$E_1^a = \epsilon\sqrt{gB}\,e^{-gB\rho^2/4}e^{\sqrt{gB}\,t}\delta_{a1}, \tag{3.28a}$$

$$E_2^a = -\epsilon\sqrt{gB}\,e^{-gB\rho^2/4}e^{\sqrt{gB}\,t}\delta_{a2}, \tag{3.28b}$$

$$E_3^a = 0, \tag{3.28c}$$

$$B_1^a = B_2^a = 0, \tag{3.29a}$$

$$B_3^a = \delta_{a3}(B - g\epsilon^2 e^{-gB\rho^2/2}e^{2\sqrt{gB}\,t}). \tag{3.29b}$$

Notice that the magnetic field for this mode is unchanged to order ϵ. It is easy to check that Eqs. (3.27)–(3.29) obey the Yang-Mills field equations. Note that E^2 is a gauge-invariant quantity which grows exponentially in time. Thus, the instability is present in any gauge. The energy density $H = \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$ is given by

$$H = \tfrac{1}{2}B^2 + O(\epsilon^4). \tag{3.30}$$

This exponential growth cannot go on indefinitely since eventually $\vec{B}^2$ will reach zero. The nonlinearities must stop the exponential growth of the potentials.

The physical meaning of the $m \neq 0$ states is simple. The field configuration $B =$ constant is translationally invariant. The unstable state with $n=0$, $m=0$, corresponds to a localized exponential growth at the origin. Obviously, there is other localized exponential growth located at different points (x,y). The $m \neq 0$ states are those localized states expressed in terms of an angular momentum basis.

IV. CONSTANT ELECTRIC FIELD

Consider a constant electric field $E_i^a = \delta_{a3}\delta_{i1}E$ in the x direction which can be derived from the potential:

$$A_0^a = -Ex\delta_{a3}, \tag{4.1a}$$

$$A_i^a = 0. \tag{4.1b}$$

We shall work in the gauge $A_3^a = 0$ and let $\mu, \nu, \ldots = 0,\ldots,3$; $\alpha, \beta, \ldots = 0,1,2$; $i, j, \ldots = 1,2,3$. We consider a fluctuation δA_μ in the potential which satisfies

$$\delta A_3 = 0 \tag{4.2}$$

due to our gauge condition.

The field equations $D_\mu F_{\mu\nu} = 0$ can be divided into $\nu = 3$ and $\nu = \alpha$ so that

$$(D_\alpha F_{\alpha 3})^a = \partial_\alpha F^a_{3\alpha} + g(A_\alpha \times F_{3\alpha})^a = 0 \,, \tag{4.3a}$$

$$\partial_3 F^a_{3\alpha} = -D_\beta F^a_{\beta\alpha} \,, \tag{4.3b}$$

where $A_\alpha B_\alpha \equiv -A_0 B_0 + A_1 B_1 + A_2 B_2$.

Equation (4.3a) is the analog of Gauss's law and when linearized implies

$$\partial_3 \partial_\alpha \delta \vec{A}_\alpha = -gEx\hat{3} \times \partial_3 \delta \vec{A}_0 \,. \tag{4.4}$$

Since (4.1) is independent of z we will be able to replace $\partial_3 \to ik_3$ for a normal mode to find

$$\partial_\alpha \delta \vec{A}_\alpha = -gEx\hat{3} \times \delta \vec{A}_0 \,. \tag{4.5}$$

We now proceed to solve Eq. (4.3b). Using (4.2) and (4.5), and after some algebra, we find

$$\begin{aligned} -\partial_3{}^2 \delta \vec{A} = {} & (-\partial_0{}^2 + \partial_1{}^2 + \partial_2{}^2)\delta \vec{A}_\alpha \\ & + 2gE\,\hat{3} \times (\delta_{\beta 1} \delta \vec{A}_0 + \delta_{\beta 0} \delta \vec{A}_1) \\ & + 2gEx(\hat{3} \times \partial_0 \delta \vec{A}_\beta) \\ & - g^2 E^2 x^2 \hat{3} \times (\hat{3} \times \delta \vec{A}_\beta) \,. \end{aligned} \tag{4.6}$$

As in Sec. III, we replace $\partial_3 \to ik_3$, $\partial_2 \to ik_2$, and $\partial_0 \to -i\omega$, and with $k^2 = k_2{}^2 + k_3{}^2$ we have

$$\begin{aligned} (k_\perp{}^2 - \omega^2)\delta \vec{A}_\alpha = {} & \partial_1{}^2 \delta \vec{A}_\alpha + 2gE\,\hat{3} \times (\delta_{\beta 1} \delta \vec{A}_0 + \delta_{\beta 0} \delta \vec{A}_1) \\ & + 2i\omega g E x(\hat{3} \times \delta \vec{A}_\beta) \\ & - g^2 E^2 x^2 (\hat{3} \times (\hat{3} \times \delta \vec{A}_\beta)) \,. \end{aligned} \tag{4.7}$$

As for the constant B field, the $\delta A^{(3)}_\alpha$ component has no unstable modes. To analyze the $\delta A^{(1,2)}_\alpha$ components we define

$$\delta A_2^{(\pm)} \equiv \delta A_2^{(1)} \pm i \delta A_2^{(2)} \,, \tag{4.8a}$$

$$\delta A^a_\pm \equiv \delta A^a_0 \pm \delta A^a_1 \,, \tag{4.8b}$$

and

$$\delta A^{(\pm)}_\pm \equiv \delta A^{(1)}_\pm \pm i \delta A^{(2)}_\pm \,. \tag{4.8c}$$

Then we have

$$[-\partial_1{}^2 - (gEx \mp \omega)^2]\delta A_2^{(\pm)} = -k_\perp{}^2 \delta A_2^{(\pm)} \,, \tag{4.9a}$$

$$[-\partial_1{}^2 - (gEx - \omega)^2]\delta A_+^{(\pm)} = (-k_\perp{}^2 \pm 2igE)\delta A_+^{(\pm)} \,, \tag{4.9b}$$

$$[-\partial_1{}^2 - (gEx + \omega)^2]\delta A_-^{(\pm)} = (-k_\perp{}^2 \mp 2igE)\delta A_-^{(\pm)} \,. \tag{4.9c}$$

Equation (4.9) depends explicitly on ω rather than on ω^2 alone. We should thus consider real and imaginary ω separately. In general we divide ω into its real and imaginary part $\omega = \omega_R + i\omega_I$ and obtain

$$[-\partial_1{}^2 - (gEx \mp \omega_R \mp i\omega_I)^2]\delta A_2^{(\pm)} = -k_\perp{}^2 \delta A_2^{(\pm)} \,, \tag{4.10a}$$

$$[-\partial_1{}^2 - (gEx - \omega_R - i\omega_I)^2]\delta A_+^{(\pm)} = (-k_\perp{}^2 \pm 2igE)\delta A_+^{(\pm)} \,, \tag{4.10b}$$

$$[-\partial_1{}^2 - (gEx + \omega_R + i\omega_I)^2]\delta A_-^{(\pm)} = (-k_\perp{}^2 \mp 2igE)\delta A_-^{(\pm)} \,. \tag{4.10c}$$

The key observation is that if ω is real ($\omega_I = 0$), Eq. (4.9) [or (4.10)] is a Schrödinger equation with an *inverted* harmonic-oscillator potential. Thus it has no solutions which vanish at $x = \pm\infty$. This observation has an interesting physical interpretation. A fluctuation δA in a constant electric field carries a certain isotopic charge. The background electric field will act on this charge causing it to accelerate to $x = \pm\infty$. This in itself is a form of instability under small fluctuations. Although ω is real and δA does not grow in time, it will spread up in x. Eventually $\partial \delta A / \partial x$ will be large owing to the potential $V(x) \propto -x^2$ and nonlinearities will start to come into play.

We encounter a similar instability in QED. Under the influence of a strong electric field, electron-positron pairs can be spontaneously created via quantum tunneling. Once produced, the electron and positron will move under the influence of the E field and be accelerated to ultrarelativistic speeds. The final kinetic energy of the electron and the positron are limited only by the size of the system. The initial quantum tunneling effect is needed to overcome the rest masses of the pair. In the present system the classical Yang-Mills fields are massless. Thus, no quantum tunneling is needed to initiate the instability. A small classical disturbance can become unstable under the constant E field.

V. DISCUSSION

A. Magnetic moment

In Sec. III, we showed that δA obeys the same equation as a charged particle of magnetic moment $2g$ moving under the influence of a magnetic field.[7] In this section, we wish to verify that δA indeed carries a magnetic moment $2g$. Let us look at the equations of motion for δA^a_μ in the presence of a background field A^a_μ. They are given by (2.6) and (2.7):

$$\delta F^a_{\mu\nu} = (D_\mu \delta A_\nu)^a - (D_\nu \delta A_\mu)^a \,, \tag{2.6}$$

$$(D_\mu \delta F_{\mu\nu})^a + g\epsilon^{abc} F^c_{\mu\nu} \delta A^b_\mu = 0 \,, \tag{2.7}$$

where

$$(D_\mu)^{ac} \equiv \delta^{ac} \partial_\mu + g\epsilon^{abc} A^b_\mu \,. \tag{5.1}$$

Substituting (2.6) into (2.7), we have

$$[D_\mu(D_\mu \delta A_\nu - D_\nu \delta A_\mu)]^a + g\epsilon^{abc} F^c_{\mu\nu} \delta A^b_\mu = 0 . \qquad (5.2)$$

Changing the order $D_\mu D_\nu$ in (5.1), and making use of

$$[D_\mu, D_\nu]^{ac} \equiv D^{ab}_\mu D^{bc}_\nu - D^{ab}_\nu D^{bc}_\mu = g\epsilon^{abc} F^b_{\mu\nu} , \qquad (5.3)$$

we have

$$[D_\mu{}^2 \delta A_\nu - D_\nu(D_\mu \delta A_\mu)]^a + 2g\epsilon^{abc} F^b_{\nu\mu} \delta A^c_\mu = 0 . \quad (5.4)$$

The result will be simple if A^a_μ is time independent. In this case we can Fourier transform (5.4) in time by setting $\partial_0 \to -i\omega$. In the temporal gauge $\delta A^a_0 = 0$, and by the use of Gauss's law, we find

$$(D_i \delta A_i)^a \equiv \partial_i \delta A^a_i + g\epsilon^{abc} A^b_i \delta A^c_i = 0 , \qquad (5.5)$$

so that

$$D_\mu \delta A_\mu = 0 . \qquad (5.6)$$

Equation (5.4) then becomes

$$(D_\mu{}^2 \delta A_\nu)^a + 2g\epsilon^{abc} F^b_{\nu\mu} \delta A^c_\mu = 0 . \qquad (5.7)$$

We can rewrite (5.7) as

$$(D_\mu{}^2 \delta A_\nu)^a + 2ig\epsilon^{abc} \tfrac{1}{2} F^b_{\alpha\beta} (S_{\alpha\beta})_{\nu\mu} \delta A^c_\mu = 0 , \qquad (5.8)$$

where

$$i(S_{\alpha\beta})_{\nu\mu} \equiv -i(g_{\alpha\nu} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\nu}) \qquad (5.9)$$

is the spin angular momentum operator for a vector particle.

Suppose we have a constant magnetic field (as in Sec. III) so that the only nonvanishing field variables are $F^{(3)}_{12} = B \neq 0$ and $A^{(3)}_i$. If we define

$$\delta A_i \equiv (\delta A^{(1)}_i + i\delta A^{(2)}_i)/\sqrt{2} , \qquad (5.10a)$$

$$\delta A^\dagger_i \equiv (\delta A^{(1)}_i - i\delta A^{(2)}_i)/\sqrt{2} , \qquad (5.10b)$$

we can reduce (5.8) to

$$(\partial_\mu + igA^{(3)}_\mu)^2 \delta A_i - 2gB_3 (S_3)_{ij} \delta A_j = 0 , \qquad (5.11a)$$

$$(\partial_\mu - igA^{(3)}_\mu)^2 \delta A^\dagger_i - 2gB_3 (S_3)_{ij} \delta A^\dagger_j = 0 , \qquad (5.11b)$$

with $S_3 \equiv S_{12}$. From (5.11) we see immediately that δA_j has a magnetic moment $2g$.

Another way of obtaining this result is to work directly on the Lagrange function $\mathfrak{L}$. We shall have to keep second-order variations in F. Consider the Lagrange function (2.1) and the field variables

$$A'^a_\mu = A^a_\mu + \delta A^a_\mu , \qquad (5.12)$$

$$F'^a_{\mu\nu} = F^a_{\mu\nu} + \delta^{(1)} F^a_{\mu\nu} + \delta^{(2)} F^a_{\mu\nu} , \qquad (5.13)$$

where $\delta^{(1)} F^a_{\mu\nu}$ is given in (2.6), and

$$\delta^{(2)} F^a_{\mu\nu} = g\epsilon^{abc} \delta A^b_\mu \delta A^c_\nu . \qquad (5.14)$$

Now consider the perturbation in $\mathfrak{L}$ due to (5.12). Owing to the action principle, the first-order perturbation in $\mathfrak{L}$ will not contribute to the action. The second-order perturbation is

$$\begin{aligned}\delta^{(2)}\mathfrak{L} &= -\tfrac{1}{4}(\delta^{(1)} F_{\mu\nu})^2 - \tfrac{1}{2}(\delta^{(2)} F_{\mu\nu}) F_{\mu\nu} \\ &= -\tfrac{1}{2}(D_\mu \delta A_\nu)^2 + \tfrac{1}{2} D_\mu \delta A_\nu D_\nu \delta A_\mu \\ &\quad - \tfrac{1}{2} g\epsilon^{abc} F^a_{\mu\nu} \delta A^b_\mu \delta A^c_\nu . \end{aligned} \qquad (5.15)$$

Moving D_μ in the second term from δA_ν to δA_μ and ignoring a total divergence in the action, we have

$$\begin{aligned}\delta^{(2)}\mathfrak{L} &= -\tfrac{1}{2}(D_\mu \delta A_\nu)^2 - \tfrac{1}{2}\delta A_\nu D_\nu (D_\mu \delta A_\mu) \\ &\quad - \tfrac{1}{2}\delta A_\nu [D_\mu, D_\nu] \delta A_\mu - \tfrac{1}{2} g\epsilon^{abc} F^a_{\mu\nu} \delta A^b_\mu \delta A^c_\nu \\ &= -\tfrac{1}{2}(D_\mu \delta A_\nu)^2 - g\epsilon^{abc} F^a_{\mu\nu} \delta A^b_\mu \delta A^c_\nu , \end{aligned} \qquad (5.16)$$

where we have made use of (5.3) and (5.6) to simplify the result. Using the definition of $S_{\mu\nu}$ in (5.9), we find

$$\delta^{(2)}\mathfrak{L} = -\tfrac{1}{2}(D_\mu \delta A_\nu)^2 - ig\tfrac{1}{2}\epsilon^{abc} F^a_{\mu\nu} \delta A^b_\alpha (S_{\mu\nu})_{\alpha\beta} \delta A^c_\beta . \qquad (5.17)$$

For a static magnetic field in the third isospin direction, we can write (5.17) as

$$\begin{aligned}\delta^{(2)}\mathfrak{L} &= -(\partial_\mu - igA^{(3)}_\mu)\delta A^\dagger_i (\partial_\mu + igA^{(3)}_\mu) \delta A_i \\ &\quad - 2gB^{(3)}_i \delta A^\dagger_\alpha (S_i)_{\alpha\beta} \delta A_\beta . \end{aligned} \qquad (5.18)$$

The first term in (5.18) is the usual Lagrangian for a charged particle in a field $A^{(3)}_\mu$. Obviously, the second term in (5.18) corresponds to a magnetic moment coupling $-\mu \vec{S} \cdot \vec{B}$ with $\mu = 2g$.

Knowing $\mu = 2g$, it is now easy to understand the instability. The lowest orbital magnetic moment is g, and is always opposite to the B field so that it increases the (energy)^2 by gB. If the spin magnetic moment points in the same direction as the B field, it will lower the (energy)^2 by $2gB$. Hence, we have

$$\omega^2 = k_3{}^2 + gB - 2gB = k_3{}^2 - gB .$$

Instability occurs at $k_3{}^2 - gB < 0$.

B. The size of the fluctuation region

In Sec. III we showed that the spatial dependence of the instability modes (at $k_3 = 0$) is given by

$$\delta A = \text{const} \times e^{im\theta} \rho^{|m|} e^{-gB\rho^2/4} . \qquad (3.26)$$

Thus, the magnitude of δA is

$$|\delta A| = \rho^{|m|} e^{-gB\rho^2/4} = \exp[f(\rho)] , \qquad (5.19)$$

where

$$f(\rho) \equiv -\tfrac{1}{4} gB\rho^2 + |m| \ln\rho . \qquad (5.20)$$

For $m \neq 0$, $|\delta A|$ vanishes both at $\rho = 0$ and $\rho = \infty$. It peaks at the zero of

$$f'(\rho) = -\tfrac{1}{2} gB\rho + |m|/\rho = 0 ,$$

or at

$$\rho = \rho_m \equiv (2|m|/gB)^{1/2} . \qquad (5.21)$$

In the neighborhood of $\rho=\rho_m$,

$$f(\rho)=f(\rho_m)+\tfrac{1}{2}f''(\rho_m)(\rho-\rho_m)^2+\cdots$$
$$=f(\rho_m)-\tfrac{1}{2}gB(\rho-\rho_m)^2+\cdots . \quad (5.22)$$

Hence, we have

$$|\delta A|\simeq|\delta A(\rho_m)|\exp[-\tfrac{1}{2}gB(\rho-\rho_m)^2] . \quad (5.23)$$

Thus, (δA) forms a ring peaked at $\rho=\rho_m$, and its value drops to zero rapidly for both $\rho>\rho_m$ and $\rho<\rho_m$.

Now let us count how many unstable modes exist inside a region of radius $\rho\gg(1/gB)^{1/2}$. We only need to include those modes whose peak radius ρ_m is smaller than ρ, i.e., we only include those unstable modes which obey

$$\rho_m=(2|m|/gB)^{1/2}<\rho . \quad (5.24)$$

Thus, the number of unstable modes is

$$N=(2|m|)_{\max}=gB\rho^2 . \quad (5.25)$$

Equation (5.25) tells us that the number of unstable modes increases linearly with the area $\pi\rho^2$. (Recall that we have kept $k_3=0$, and hence, have not yet taken the longitudinal space into account.) The area per unit unstable mode is

$$\frac{\pi\rho^2}{2|m|}=\frac{\pi}{gB} . \quad (5.26)$$

When we take the longitudinal space variation into account and keep $k_3\neq 0$ modes as well, we find that a typical unstable mode also requires a longitudinal dimension of $L=\pi/\sqrt{gB}$. This result indicates that if $B\neq 0$ only in a small region of size L, and if L is smaller than the required size $\sim 1/\sqrt{gB}$, then no unstable mode can exist. In a sense, the parameter $\sqrt{gB}L$ plays a role similar to that of the Reynolds number in a fluid. The instability occurs only when $\sqrt{gB}L$ is large compared to 1. It is interesting to see how the above idea can be applied qualitatively to the Coulomb field of a magnetic source. Consider the Coulomb field due to a magnetic source,

$$B=\frac{Q}{r^2} .$$

Since a Coulomb field has no intrinsic scale, it is natural to choose $L=r$. Thus the Reynolds number becomes

$$\sqrt{gB}r=\sqrt{gQ} ,$$

which is independent of r. Hence, we expect a stable configuration for $gQ\ll 1$, and an unstable configuration for $gQ\gg 1$; we expect that the critical Q is $gQ\simeq O(1)$. This is analogous to what Mandula found in the case of an electric source. However, our method is too crude to give the exact transition point.

C. Concluding remarks

The instabilities in constant field configurations have implications to other physical situations as well. For instance, we can study qualitatively the stability condition for a plane wave of amplitude B (or E) and wavelength λ. If B and λ are large, the B field changes slowly. We then have effectively a constant B field within a region of size $L=\lambda$. Thus, for $\sqrt{gB}\lambda\gg 1$ we expect the system to be unstable. We can also apply the above considerations to bag and tubelike configurations which are the suggested structures of hadrons.

The instabilities may also have important implications to the nature of the ground state of quantum chromodynamics (QCD). Owing to quantum fluctuations, the physical ground state contains all kinds of field configurations. The relative amplitude for the existence of a given field configuration is measured by the exponential of its action e^{-S}. Thus, large-scale but small-B fluctuations are possible if $B^2L^4\sim 1$. For sufficiently large L the effective running coupling constant g can be very large, and hence the Reynolds number $\sqrt{gB}L$ may be much larger than 1. Such fluctuations will be unstable, and could give rise to a disordered vacuum. Hence, if we look at the physical ground state on a large space and time scale, the ground state may appear as an ocean of turbulence.

There are many important problems to be solved. From our work we found that the instabilities are characterized by a number which is similar to the Reynolds number in fluid mechanics. It would be interesting to know whether other phenomena in fluids, such as the onset of turbulence, and their associated scaling properties also have analogs in non-Abelian gauge theories. Finally, we would like to know whether the instability associated with B field has a quantum-mechanical interpretation. The instability associated with E field certainly exists in the quantized theory.

Note added in proof: Nielsen and Olesen have demonstrated the existence of an unstable mode in a Yang-Mills field theory associated with asymptotic freedom. However, their motivation and approach are different from ours. See Ref. 8 for details. After submitting this paper, we were informed that similar results were also obtained by Sikivie.[9]

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