Interaction between classical and quantum systems: A new approach to quantum measurement. II. Theoretical considerations

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We develop further a new approach to the quantum measurement process. In this approach, which we proposed in a recent paper, the apparatus is treated as a classical system; however, the classical apparatus is directly coupled to the quantum system. A principle of integrity, which requires that the observables of the classical apparatus retain their classical integrity, is introduced. We examine the constraints which this principle places upon the coupling between the apparatus and the quantum system. To illustrate our approach we use a model loosely based on the Stern-Gerlach experiment. For this model we exhibit a coupling which satisfies the principle of integrity.

I. INTRODUCTION

In a previous paper,[1] to which we refer as I, we proposed a new approach to the description of the quantum measurement process. The unusual feature of this approach is to treat the apparatus as a classical system, to be described by classical physics. The system to be observed is, however, a quantum system. In I we concentrated on setting up the formalism for such a description, and on introducing the tools necessary in the detailed description of the approach. The proposal which we forwarded in I, and which we further develop in this paper, is an attempt to construct an alternative theory of measurement in quantum mechanics. It does not yet have the status of a theory, as its logical consistency is yet to be proved. If that is shown, then a decision as to which theory is correct can only be made on the basis of agreement with experiment. In the present paper we pursue theoretical questions concerning the interaction of quantum and classical systems: how they should be constrained, and whether there are any simple examples to illustrate the approach.

Before we discuss such questions, we shall review briefly the results of I. The main thrust of that paper was in setting up a formalism which would allow the direct interaction of a classical system with a quantum system. The main tool used was, in effect, a new way to view classical systems. It was described how one could envisage a classical system embedded in a much-larger quantum structure (insofar as the dynamical variables are noncommuting), and yet the observable part of this larger quantum-mechanical system would mimic exactly in its behavior the original classical system.

Let us denote the dynamical variables of the classical system by $(q_1, \ldots, q_n)$ and $(p_1, \ldots, p_n)$. The Hamiltonian is a function of these phase-space points, $H(q,p)$. Their development in time is given by Hamilton's equations

$$\dot{q}_i = \frac{\partial H(q,p)}{\partial p_i}, \qquad \dot{p}_i = \frac{\partial H(q,p)}{\partial q_i} . \tag{1.1}$$

Specifying initial conditions $q_i(t=0)=q_i^0$ and $p_i(t=0)=p_i^0$ then determines fully the future time development of the system.

We can describe the quantum system in which we find the above classical system embedded as follows. This description involves the following ingredients:

(1) The use of operators acting on a state space as the dynamical variables; for this quantum system we write them as

$$\omega = \{\omega^1, \ldots, \omega^{2n}\} \equiv \{q_1, \ldots, q_n; p_1, \ldots, p_n\} . \tag{1.2}$$

(2) The introduction of operators conjugate to the ω^μ with respect to commutation: Operators π^ν such that

$$[\omega^\mu, \pi^\nu] \equiv \omega^\mu \pi^\nu - \pi^\nu \omega^\mu = i\delta^{\mu\nu} . \tag{1.3}$$

A representation of the π operators is

$$\pi^\nu = -i \frac{\partial}{\partial \omega^\nu} . \tag{1.4}$$

(3) The Hamiltonian operator

$$\mathcal{H} = -\sum_j \left[\frac{\partial H(q,p)}{\partial q_j} \pi_j^p - \frac{\partial H(q,p)}{\partial p_j} \pi_j^q \right] \equiv \frac{\partial H(\omega)}{\partial \omega^\nu} \epsilon^{\mu\nu} \pi^\mu . \tag{1.5}$$

The Heisenberg picture is used to discuss the time development of the system. Using the Hamiltonian operator defined in (1.5), the equations of motion of the ω^μ operators

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$$\dot{\omega}^{\mu}(t) = -i[\omega^{\mu}(t), \mathcal{H}] \tag{1.6}$$

mimic exactly the classical equations (1.1).

(4) The analog of the initial values of the usual classical description is the specification of the state of the quantum system in the Heisenberg picture. This is made more precise by the choice of state

$$|\psi\rangle = |\omega_0\rangle = |q^0, p^0\rangle , \tag{1.7}$$

where the initial values seen earlier are the eigenvalues of the ω^{μ} on this state.

(5) We distinguish between the observable and unobservable parts of the system by invoking a superselection principle: The set of operators $\{\omega^1, \ldots, \omega^{2n}\}$ are superselecting operators. The immediate consequences of this principle are that the conjugate π^{ν} operators are unobservable, and that the algebra of observables generated by the ω^{μ} is commutative. We note here that our use of superselection does not follow the conventional usage.[2] For example, the Hamiltonian (1.5) is not an observable. As a result no superselection rule applies. This usage of superselection has been discussed in I in more detail, where some simple illustrative examples were discussed.

The observable sector of the resulting quantum theory exactly mimics the simple classical system first discussed. Our proposal was to use this model to couple together a classical apparatus and a quantum system. First we construct the "quantum-enlarged" apparatus system, in the manner described above. The enlarged system is then coupled to the quantum system under investigation.

Let us denote the quantum variables by $\{\xi\}$, and the undisturbed quantum Hamiltonian by $X(\eta)$. Then the Hamiltonian operator for the coupled "enlarged apparatus" and quantum system is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} , \tag{1.8}$$

where

$$\mathcal{H}_0 = \frac{\partial H(\omega)}{\partial \omega^{\nu}} \epsilon^{\mu\nu} \pi^{\mu} + X(\eta) \tag{1.9a}$$

and

$$\mathcal{H}_{\text{int}} = \Phi(\omega, \pi; \xi; t) . \tag{1.9b}$$

We should point out that we are restricting our attention, at this stage of the program, to closed systems about which we have maximum knowledge allowed by theory. Our quantum-mechanical systems will then be described by elementary quantum mechanics so that, for example, the time development will be effected by a unitary transformation, and state vectors rather than density matrices are employed.

Finally in I we addressed the question as to what restrictions should be placed on the interaction Hamiltonian (1.9b). There were two general requirements:

(1) A measurement is achieved if unambiguous information concerning the values of certain variables of the quantum system being examined can be "stored" in the variables of the classical apparatus.

(2) After the interaction has occurred the apparatus must be "classical" in some sense.

The second requirement is rather vague as stated. As we saw in I the classical nature of the state of the apparatus is not retained when interactions with quantum systems are allowed. Thus we clearly cannot require that the classical nature of the state be retained. However, the remaining classical property is a statement about the classical observables, which form a commuting set. We proposed that the requirement (2) above be applied to this property of the apparatus system. This proposal was formulated in both weak and strong forms as follows:

Weak form. After the interaction has ceased the apparatus observables should retain their classical integrity. While the interaction is taking place no such restriction is enforced.

Strong form. The apparatus observables should retain their classical integrity at all times.

We called this requirement the "principle of integrity." Requiring that the interactions satisfy this principle is the weakest requirement which we can impose if we wish the apparatus to be classical, in any sense, after interacting with the quantum system.

In this paper we wish to examine how such a principle constrains the coupling (1.9b) of the apparatus and quantum system. We shall, however, restrict our attention to the strong form of the principle and to time-independent interactions. In Sec. II we shall derive a set of integrity criteria which can be used to check if the principle is satisfied by a given interaction. In Sec. III we introduce a model which will be used in Sec. IV to illustrate the use of the integrity criteria: The resulting model, which can be viewed as a variant of the Stern-Gerlach experiment, is an example of an interaction which satisfies the principle of integrity. In Sec. V we shall discuss the relevance of a π-independent coupling in the Hamiltonian, referring in particular to the model of Sec. III.

II. THE INTEGRITY CRITERIA

The strong form of the principle of integrity, as stated in the Introduction, requires that the observables of the classical system retain their classical integrity when interactions with quantum systems, as in (1.8) and (1.9), are allowed. In this section we examine the constraints which the strong principle of integrity places on the coupling between classical and quantum systems. To examine whether or not a given interaction satisfies the principle, it is convenient to use the Heisenberg picture, as in that formulation it is the dynamical variables which develop in time.

Within the context of the approach reviewed in the Introduction, the classical nature of the apparatus observables can be characterized by both of the following properties:

(i) $\omega^\mu(t)$ are observable for all times t.
(ii) $\omega^\mu(t)$ and $\omega^\nu(t')$ are compatible operators for all times t and t'.

The first property tells us that the "trajectories" of the apparatus observables are observable for all times. The second tells that we can measure the different trajectories without disturbing the measurable aspects of the system. These two properties are not independent of one another, however. The first requires that the commutator

$$[\omega^\mu(t), \omega^\nu(0)] \tag{2.1}$$

vanish for all times t, and for all μ, ν. To satisfy (ii) the commutator

$$[\omega^\mu(t), \omega^\nu(t')] \tag{2.2}$$

must vanish, for which it suffices to consider commutators of the form (2.1) because

$$[\omega^\mu(t), \omega^\nu(t')] = e^{i\mathcal{H}t'}[\omega^\mu(t-t'), \omega^\nu(0)]e^{-i\mathcal{H}t'}. \tag{2.3}$$

We note that for the uncoupled classical system properties (i) and (ii) are automatically satisfied because the Hamiltonian (1.5) is at most linear in the unobservable π^μ.[1] For a general time-independent interaction

$$\mathcal{H}_{\text{int}} = \Phi(\omega, \pi; \xi) \tag{2.4}$$

this result no longer follows. Clearly, if the coupling function Φ is quadratic (or higher) in the unobservables π^μ the apparatus observables will not be characterized by (i) and (ii). If Φ is linear in π^μ it may occur that (i) and (ii) are retained even in the presence of some interactions. Thus we can restrict our attention to interactions of the form

$$\mathcal{H}_{\text{int}} = \phi^\mu(\omega; \eta')\pi^\mu + h(\omega; \hat{\epsilon}) \ , \tag{2.5}$$

where $\{\eta'\}$ and $\{\hat{\epsilon}\}$ are subsets of the quantum variables. This form, however, is not sufficient to guarantee that the apparatus observables retain their classical integrity. Both the primary coupling functions ϕ^μ and the secondary coupling function h depend on unspecified quantum variables. It remains to find what further restrictions, if any, on the functional form of these coupling functions can be deduced by requiring the principle of integrity to be satisfied. Or, if that fails, we need to derive criteria which can be used to check different models.

For our purposes (i) and (ii) are not in a very usable form. From the interaction Hamiltonian (2.5) we see that $\dot{\omega}^\mu(t)$ depends on the primary coupling function ϕ^μ. Thus we should formulate (i) and (ii) in terms of the time derivatives of $\omega^\mu(t)$. We consider the following:

(iii) $$\left[\frac{d^m}{dt^m}\omega^\mu(t), \frac{d^n}{dt^n}\omega^\nu(t)\right] = 0 \text{ for all } m, n \geq 0 \text{ and all times } t \ .$$

This condition is easily seen to follow from either (i) or (ii). Thus it is a necessary condition for the apparatus to remain classical. If $\omega^\mu(t)$ are everywhere regular it can also be seen that (i) and (ii) follow from (iii). In that case (iii) is equivalent to (i) and (ii). However, it may be that $\{\omega^\mu(t)\}$ are not all regular functions, so that they do not have power-series expansions valid everywhere. Then it is clear that (iii) is a necessary, but not sufficient, condition that the apparatus remain classical.

We can abstract from (iii) the following equations:

(a) $$\left[\frac{d^m}{dt^m}\omega^\mu(t), \ \frac{d^n}{dt^n}\omega^\nu(t)\right] = 0 \text{ for } m, n > 0 \ ,$$

(b) $$\left[\omega^\mu(t), \frac{d^n}{dt^n}\omega^\nu(t)\right] = 0 \qquad \text{for } n > 0 \ ,$$

(c) $$[\omega^\mu(t), \omega^\nu(t)] = 0 \ .$$

Equation (c) is automatically satisfied, as the unitary time development preserves commutation relations. From (a) we see that either all of the time derivatives vanish, or they belong to a commutative algebra. The first alternative cannot happen even for the uncoupled system. Then (b) tells us that $\omega^\mu(t)$ itself must also belong to this same commutative algebra.

We now turn to the coupling between apparatus and quantum system, as in (2.5), to see what we can learn by applying these restrictions. We can write the Hamiltonian in the form

$$\begin{aligned}\mathcal{H} &= \frac{\partial H(\omega)}{\partial \omega^\mu}\epsilon^{\nu\mu}\pi^\nu + X(\eta) + \phi^\mu(\omega, \eta')\pi^\mu + h(\omega, \hat{\epsilon}) \\ &= F^\mu(\omega, \eta')\pi^\mu + G(\omega, \epsilon) \ ,\end{aligned} \tag{2.6}$$

where $\{\epsilon\} = \{\hat{\epsilon}\} \cup \{\eta\}$. The time derivatives of $\omega^\mu(t)$ then take the form

$$\dot{\omega}^\mu(t) = -i[\omega^\mu(t), \mathcal{H}] = -F^\mu(\omega(t), \eta'(t)) \ , \tag{2.7a}$$

$$\ddot{\omega}^\mu(t) = i[F^\mu, \mathcal{H}] = i[F^\mu, F^\nu\pi^\nu + G] \ , \tag{2.7b}$$

$$\dddot{\omega}^{\mu}(t)=[[F^{\mu},\mathcal{H}],\mathcal{H}]=[[F^{\mu},\mathcal{H}],F^{\nu}\pi^{\nu}+G] \ , \qquad (2.7c)$$

$$\frac{d^m}{dt^m}\,\omega^{\mu}(t)=-(-i)^{m-1}[\cdots[[F^{\mu},\mathcal{H}],\mathcal{H}]\cdots,\ F^{\mu}\pi^{\nu}+G] \ . \qquad (2.7d)$$

The first restriction which we derive, using (2.7a) and (a), is that the F^{μ} belong to a commutative algebra for all μ. However, from the decomposition

$$[F^{\mu},F^{\nu}]=\left[\frac{\partial H(\omega)}{\partial\omega^{\rho}}\epsilon^{\mu\rho},\ \phi^{\nu}\right]+\left[\phi^{\mu},\ \frac{\partial H(\omega)}{\partial\omega^{\lambda}}\epsilon^{\nu\lambda}\right]+[\phi^{\mu},\phi^{\nu}] \qquad (2.8)$$

and the fact that

$$[\omega^{\mu}(0),\eta'(0)]=[\omega^{\mu}(t),\eta'(t)]=0 \ , \qquad (2.9)$$

we have equivalently the condition that the set of ϕ^{μ} be commutative. This condition merely tells us that the operators ϕ^{μ} must commute among themselves. It does not give us any information about the functional form of ϕ^{μ}. Thus, $\{\eta'\}$ may be a commuting set, in which case ϕ^{μ} automatically satisfy the requirement, or it is a noncommuting set such that the ϕ^{μ} commute with each other.

This is not the only condition that we need impose on the coupling functions. We include in Appendix A a derivation of the other conditions. Before stating them explicitly we introduce some shorthand notation. We denote by $\{\rho\}$ the maximal (i.e., largest) algebraically independent subset of the operators $\{\phi^{\mu}\}$. In forming algebraic combinations we allow the coefficients to depend on the ω variables. In other words, any function of the ϕ^{μ} can be expressed as a function of $\{\rho\}$ and $\{\omega\}$.

Sometimes we need (and are able to find), from the algebra of dynamical variables describing the quantum-mechanical system, an extension of this commuting set by additional operators which we shall call ρ'. The ρ' are algebraically independent of the ρ but commute with them. The extension of $\{\rho\}$ by inclusion of the ρ' will be denoted by $\{\rho\}\cup\{\rho'\}$.

We can now state our results in the form of necessary and sufficient conditions that condition (iii) be obeyed. They are as follows:

(A) $\{\phi^{\mu}\}$ forms a commuting set.

(B) $F^{\nu}[\rho_m,\pi^{\nu}]+[\rho_m,G]$ commute with ϕ^{μ} and each other for all m. We denote the resultant commuting set by $\mathfrak{C}$.

(C) $F^{\nu}[\rho'_m,\pi^{\nu}]+[\rho'_m,G]$ commute with each other and each element of $\mathfrak{C}$ for those ρ'_m which occur in the expansions of

$$[\rho_m,\mathcal{H}],\ [[\rho_m,\mathcal{H}],\mathcal{H}],\ \text{etc.}$$

(A), (B), and (C) are also the necessary and sufficient conditions that the apparatus remain classical if the interaction is such that $\omega^{\mu}(t)$ is regular (analytic) for all times t. In that case we refer to (A), (B), and (C) as the integrity criteria.[3]

However, it may occur[4] that the interaction explicitly precludes the regularity of the apparatus observables. In such a case (A), (B), and (C) are necessary, but not sufficient, conditions that the apparatus remain classical. We must go back to condition (b), and examine $[\omega^{\mu}(t),\ (d/dt')\omega^{\nu}(t')]$, or equivalently $[\omega^{\mu}(t),\ \phi^{\nu}(t')]$ for $t=0$, for example. Then we must have $[\omega^{\nu}(0),\ \phi^{\nu}(t')]=0$ for all times t'. In the nonregular cases, the criteria (A), (B), and (C) must be supplemented in this fashion to yield the integrity criteria.

We must now ask what these conditions tell us about the functional form of ϕ^{μ} and G. To begin with we see that if the ϕ^{μ} depend on at most a commuting set of quantum variables, then $[\rho_i,\pi^{\nu}]$ already belongs to the algebra $\mathfrak{C}$. Furthermore, in this case we can assume ρ'_i also depends on a commuting set of quantum variables. Then conditions (B) and (C) reduce to conditions on the allowed $G=X(\eta)+h(\omega,\hat{\epsilon})$. If ϕ^{μ} depends in a nontrivial manner on noncommuting quantum variables, then we must check that conditions (B) and (C) are satisfied in each case.

We can understand the ϕ^{μ} depending on a commuting set of quantum variables from the measurement viewpoint. The information stored in the classical variables after the interaction has occurred will be about these operators. On the other hand, if the ϕ^{μ} depend on noncommuting variables the information stored will be of the $\{\phi^{\mu}\}$ rather than the set $\{\eta'\}$. Since the $\{\phi^{\mu}\}$ are not operators of the quantum system, we are led to the conclusion that, whereas an interaction has taken place, the effect on the apparatus cannot be identified as a measurement on the original quantum system.

To illustrate the merits of the criteria derived in this section, we will next turn to a simple interaction between a classical system and a quantum system, and see what restrictions on the coupling functions these criteria give us.

III. A SIMPLE MODEL

In the preceding section, we examined some general aspects of the interaction between a quantum-mechanical system and a classical apparatus. In particular, we derived a set of criteria which could be used to see if an interaction satisfied the principle of integrity, i.e., if the classical observables retain their integrity when interacting with a quantum system. Clearly, it is not *a priori* obvious that any interactions would preserve the classical integrity of the apparatus observables. In this section we introduce an apparatus and quantum system which interact together in the manner we have discussed. In Sec. IV we will apply the integrity criteria to the model.

We shall examine a particularly simple type of quantum system. Its dynamical variables have discrete spectra only. Furthermore, it is chosen to be inert. It is a quantum spin system, with the three spin projections S_1, S_2, and S_3 and the total spin squared $S^2 = S_1{}^2 + S_2{}^2 + S_3{}^2$ as the operators of the system. These operators satisfy the usual commutation relations

$$[S_i, S_j] = i\epsilon_{ijk} S_k \ , \qquad [S^2, S_i] = 0 \ . \tag{3.1}$$

Since the system is to be inert, the Hamiltonian vanishes—the system does not change in time if left undisturbed. The state of the system is specified by choosing a particular eigenstate of, for example, S^2 and S_1, which constitute a complete commuting set of operators of the system.

We choose for our classical apparatus a simple system, namely a (classical) particle freely moving in three dimensions. The quantum enlarged description of this system was given in I. The dynamical variables for the apparatus will be

$$\{\omega\} = \{q_i, p_j\} \quad \text{and} \quad \{\pi\} = \{\pi_i^q, \pi_j^p\} \ , \tag{3.2}$$

the conjugate unobservable operators. The ω^μ are superselecting operators, and they generate the algebra of observables for the apparatus. The state of the apparatus can be specified to be an eigenstate of the observables, which all commute with each other. Finally the time development of the apparatus system is given by the Hamiltonian

$$\mathcal{H} = \frac{1}{m} \vec{p} \cdot \vec{\pi}^q \ , \tag{3.3}$$

which is derived from the usual classical free-particle Hamiltonian $(1/2m)p^2$ using the prescription (1.5).

In order to describe an interaction between the apparatus and the spin system, we envisage the quantum spin system being carried along as internal degrees of freedom by the (electrically neutral) particle whose translatory degrees of freedom are classical. [For example, the internal quantum degrees of freedom may give rise to a magnetic moment for the particle.] The interaction is induced by causing the classical particle to pass through an inhomogeneous magnetic field, as in the Stern-Gerlach experiment.[5]

The primary coupling functions ϕ^μ are specified by requiring that the apparatus observables satisfy the correct classical equations of motion. At the purely classical level the potential energy of the particle when in the external magnetic field is

$$\vec{\mu} \cdot \vec{B} \ , \tag{3.4}$$

where $\vec{\mu} = \gamma \vec{S}$ is the magnetic moment of the particle. The force exerted on the particle is

$$\vec{F} = -\vec{\nabla}(\vec{\mu} \cdot \vec{B}) = -\gamma S_i \vec{\nabla} B_i \ . \tag{3.5}$$

Then the correct equations for the classical observables result, with the choice

$$\phi_i^q = 0, \quad \phi_i^p = F_i \tag{3.6}$$

yielding the Hamiltonian

$$\mathcal{H} = \frac{1}{m} \vec{p} \cdot \vec{\pi}^q - \gamma S_i (\vec{\nabla} B_i) \cdot \vec{\pi}^p \ . \tag{3.7}$$

This form for the Hamiltonian can be derived from the classical Hamiltonian

$$\frac{1}{2m} p^2 + \vec{\mu} \cdot \vec{B} \tag{3.8}$$

by using the prescription (1.5).

For purposes of illustration (3.7) is sufficient for use as an interaction Hamiltonian. However, we shall, in Sec. V, discuss the role of the secondary coupling function $h(\omega, \hat{\epsilon})$ which is chosen to vanish in (3.7). For this later discussion, we include the possibility of a nonvanishing secondary coupling

$$\mathcal{H} = \frac{1}{m} \vec{p} \cdot \vec{\pi}^q - \gamma S_i (\vec{\nabla} \cdot B_i) \cdot \vec{\pi}^p + h(\omega, \vec{S}) \ , \tag{3.9}$$

where, for example, we could choose

$$h(\omega, \vec{S}) = -\gamma \vec{B} \cdot \vec{S} \ . \tag{3.10}$$

This choice does not affect the time development of the observables, but it does play a role in the time development of the apparatus unobservables.

We shall denote the model with the Hamiltonian (3.9) as model 1 and the model with the Hamiltonian (3.7) as model 2. In the following section we examine each of these in turn to test the use of the integrity criteria.

IV. ILLUSTRATION OF THE USE OF THE INTEGRITY CRITERIA

The integrity criteria derived in Sec. II can be used to check whether or not the strong principle of integrity is satisfied by a given interaction. In this section we shall apply these criteria to the models presented in the preceding section. We wish to supply answers to the following questions: Does an interaction exist which will satisfy the principle, and can it be realized? Does the resulting interaction couple noncommuting quantum variables to the apparatus in a nontrivial manner?

Within the context of the models of Sec. III we allow for the most general coupling by letting $\vec{B}(q)$ be an arbitrary inhomogeneous magnetic field. As a result of this arbitrariness the coupling functions (3.6) and (3.10) are functions of a noncommuting set of quantum variables. We will restrict our attention mainly to magnetic fields which are regular everywhere, although we shall comment on the nonregular case also.

We first direct our attention towards model 1, which uses the slightly more complicated Hamiltonian operator (3.9).

Criterion (A) states that the primary coupling functions must form a commuting set. From (3.5) and (3.6) we can write the nonzero primary coupling functions as

$$\phi_i^p = \vec{S}\cdot\vec{f}_i, \quad \text{where } \vec{f}_i = -\gamma \frac{\partial}{\partial q_i}\vec{B} \ . \tag{4.1}$$

The commutator of two such coupling functions has the structure

$$[\phi_i^p, \phi_j^p] = [\vec{S}\cdot\vec{f}_i, \vec{S}\cdot\vec{f}_j] = i\vec{S}\cdot(\vec{f}_i \times \vec{f}_j) \ , \tag{4.2}$$

and so the ϕ_i^p form a commuting set if and only if

$$\vec{f}_i \times \vec{f}_j = 0 \ , \tag{4.3}$$

i.e., $\vec{f}_i$ and $\vec{f}_j$ must be parallel.

Criterion (B) requires the vanishing of the commutator

$$[[\phi_i^p, \mathcal{H}], \phi_j^p] \ . \tag{4.4}$$

The vanishing of this commutator leads directly to the following important result: In each disjoint region of support of $\vec{B}$, into which the particle will travel, the magnetic field must take the form

$$B(q)\vec{n} \ , \tag{4.5}$$

where the unit vector $\vec{n}$ is constant in that region of support. This form for the magnetic field means that in each region the magnitude of the magnetic field may vary, but its direction must be fixed. The magnetic field can only change direction by going through zero values. The derivation of this result is given in Appendix B.

In different support regions, however, $\vec{n}$ could point in different directions. If we restrict our attention to regular functions, for which the zeros are isolated, then the magnetic field can have only one distinct region of support. For such magnetic fields, criterion (B) can only be satisfied if they are of the form (4.5), where $\vec{n}$ is a uniquely determined constant vector.

With this very restrictive form for the magnetic field the remaining integrity criterion (C) is automatically satisfied. Furthermore, we see that the coupling functions are now functions of

$$S_n = \vec{S}\cdot\vec{n} \tag{4.6}$$

alone, and this is a trivial case of a commuting set of quantum variables.

In this section, so far, we have shown that an interaction exists, in principle, between the classical particle and the quantum spin system which allows the apparatus observables to retain their classical integrity. The question now arises as to whether this interaction can be realized physically.

The experiment we are discussing is modeled after the Stern-Gerlach experiment.[5] In that experiment the magnetic field $\vec{B}$ results from a static magnet. Thus it obeys Maxwell's equations of the following form:

$$\vec{\nabla}\cdot\vec{B} = 0 \ , \tag{4.7a}$$

$$\vec{\nabla}\times\vec{B} = 0 \ . \tag{4.7b}$$

Using (4.7a) and (4.5) gives us

$$\vec{n}\cdot\vec{\nabla}B = 0 \ , \tag{4.8}$$

i.e., $\vec{n}$ is perpendicular to $\vec{\nabla}B(q)$. Using the second of Eqs. (4.7) with (4.5) gives us

$$\vec{n}\times\vec{\nabla}B = 0 \ , \tag{4.9}$$

i.e., $\vec{n}$ is parallel to $\vec{\nabla}B(q)$. Thus the only solution, if $\vec{n}$ is a uniquely determined direction, is

$$\vec{\nabla}B(q) = 0 \ , \tag{4.10}$$

i.e., the unidirectional magnetic field is homogeneous. However, the magnets used in the Stern-Gerlach experiment are specifically chosen so that the magnetic field is inhomogeneous. We conclude that the form (4.5) for the magnetic field cannot be valid in a description of the Stern-Gerlach experiment.[6]

What is possible, however, is to realize a magnetic field of the form (4.5) by means of a nonzero current $\vec{J}(q)$. Consider a current of the form

$$\vec{J} = (J_1(q_1, q_2), J_2(q_1, q_2), 0) \ , \tag{4.11}$$

where the continuity equation is

$$\vec{\nabla}\cdot\vec{J} = 0 \ , \tag{4.12}$$

i.e., the charge density is zero. Then consider the function

$$f(q_1, q_2) = \int_0^{q_2} J_1(q_1, q_2')dq_2' - \int_0^{q_1} J_2(q_1', q_2)dq_1' \tag{4.13}$$

and the magnetic field

$$\vec{B} = f(q_1, q_2)\vec{n} \ , \tag{4.14}$$

where $\vec{n}$ is a unit vector in the q_3 direction. Maxwell's equations now read

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{4.15a}$$

and

$$\vec{\nabla}\times\vec{B} = \vec{J} \ . \tag{4.15b}$$

The choice of (4.14) guarantees that (4.15a) is satisfied. Use of the continuity equation and (4.13) likewise shows that Eq. (4.15b) is satisfied.

Thus, it is possible to construct, in principle if not in practice, a magnetic field which is of the form (4.5). Such a magnetic field will induce interactions between the classical particle and the quantum spin system which allow the apparatus observables to retain their integrity, as required by the strong form of the principle of integrity.

The derivation of the first result of this section, namely that the magnetic field should take the form $B(q)\vec{n}$, where $\vec{n}$ is a uniquely determined constant unit vector, depends crucially on the assumption that we are dealing with regular functions, which have valid power-series expansions everywhere. However, if we allow magnetic fields $\vec{B}(q)$ which are not regular, the result does not immediately follow. As discussed in Sec. II, the criteria (A), (B), and (C) must, in that case, be supplemented to yield the integrity criteria. The extra criterion is enough to ensure that, in this case also, the result (4.5) holds.[7] This result will be illustrated in a separate paper,[4] in which particular experiments will be examined.

The analysis of this section has so far dealt with model 1 only. We now direct our attention towards model 2, which makes use of the simpler Hamiltonian operator (3.7). In this case the restrictions derived on the form of the magnetic field will be slightly less severe.

As indicated earlier, criterion (A) leads to the result (4.3). However criterion (B) leads to the following restriction on the inhomogeneous magnetic field: In each disjoint region of support of $\vec{B}$, into which the particle will travel, the *magnetic field gradients* must take the form

$$\frac{\partial}{\partial q_i}\vec{B}(q) = f_i(q)\vec{n} \ , \tag{4.16}$$

where the unit vector $\vec{n}$ is constant in that region of support. This result is also derived in Appendix B. As before, if we restrict our attention to regular functions there is only one region of support. The expression (4.16) can be written as a direct restriction on the form of the magnetic field $\vec{B}(q)$, rather than the magnetic field gradients as

$$\vec{B}(q) = \vec{B}(0) + F(q)\vec{n} \ , \tag{4.17}$$

where $F(0) = 0$.

We see at once that (4.17) is less restrictive than (4.5), as expected, since the Hamiltonian for model 2 is less complicated than that for model 1. We see that the effect of the secondary coupling function (3.10) in the Hamiltonian for model 2 is to force $\vec{B}(0)$ to be parallel to $\vec{n}$, rather than being an arbitrary direction. Nevertheless, the form (4.17) does not allow the Stern-Gerlach interaction for the same reasons as before. The realization of the result (4.5) by means of the magnetic field (4.14) is also a particular realization of (4.17), for which $\vec{B}(0)$ is parallel to $\vec{n}$.

V. THE SECONDARY COUPLING FUNCTION

In Sec. II when we introduced an interaction term in the total Hamiltonian to describe the coupling of the classical apparatus to the quantum system, we made use of both primary and secondary coupling functions, $\phi^\mu(\omega,\eta)$ and $h(\omega,\hat{\epsilon})$, respectively. The role of the primary coupling function is evident—it is responsible for the direct coupling between the quantum system and the apparatus observables $\{\omega\}$. The role played by the secondary coupling is not so transparent. In this section we shall discuss briefly its role.

As we have already seen, if the primary coupling functions ϕ^μ depend on a commuting set of quantum variables, the secondary coupling function must be chosen so that, in the language of Sec. II, and of Appendix A,

$$\text{(i) } [\rho, X(\eta) + h(\omega,\hat{\epsilon})] \in \mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\})$$

and (5.1)

$$\text{(ii) } [\rho', X(\eta) + h(\omega,\hat{\epsilon})] \in \mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\}) \ ,$$

where the $\{\rho'\}$ and the $\{\rho\}$ and $\{\omega\}$ form a commuting set.

Given a particular quantum system Hamiltonian $X(\eta)$, we are free to choose $h(\omega,\hat{\epsilon})$ subject to (i) and (ii), if such a function exists. If it does not, the coupling will not allow the apparatus to remain classical.

When ϕ^μ depends on noncommuting quantum variables, the restrictions on $h(\omega,\hat{\epsilon})$ are even more strict, as we must then have

$$\text{(i}'\text{) } F^\nu[\rho,\pi^\nu] + [\rho, X(\eta) + h(\omega,\hat{\epsilon})] \in \mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\})$$

and (5.2)

$$\text{(ii}'\text{) } F^\nu[\rho',\pi^\nu] + [\rho', X(\eta) + h(\omega,\hat{\epsilon})] \in \mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\}) \ .$$

In Sec. IV we applied the integrity criteria to two versions of a simple model. In model 1 we chose a secondary coupling function with physical significance, namely the potential energy of a classical particle, with a magnetic moment, moving in an external magnetic field. As we noted in Sec. III, this choice of secondary coupling function was not essential for the logical consistency of that example. Model 2, for example, does not have a secondary coupling function and it is adequate for our purposes, as we shall see in this section.

We wish now to discuss another role of the secondary coupling function. This role is connected with ensuring the form invariance of the total Hamiltonian under unitary "gauge" transformations $U(\omega,\epsilon)$. Let us consider the effect of such a unitary transformation on our theory. To recall, our Hamiltonian is

$$\mathcal{H}=\frac{\partial H(\omega)}{\partial\omega^\mu}\epsilon^{\nu\mu}\pi^\nu+X(\eta)+\phi^\mu(\omega,\eta')\pi^\mu+h(\omega,\hat{\epsilon})\ . \tag{5.3}$$

The dynamical variables transform as follows:

$$\begin{aligned}&\omega\to\omega'=U(\omega,\epsilon)\omega U^{-1}(\omega,\epsilon)=\omega\ ,\\&\pi\to\pi'=U(\omega,\epsilon)\pi U^{-1}(\omega,\epsilon)=\pi+U[\pi,U^{-1}]\ ,\\&\xi_\alpha\to\xi'_\alpha=U(\omega,\epsilon)\xi_\alpha U^{-1}(\omega,\epsilon)=\xi_\alpha+f_\alpha(\omega,\epsilon)\ .\end{aligned}\tag{5.4}$$

The Hamiltonian operator is easily shown to transform as follows:

$$\begin{aligned}\mathcal{H}\to\mathcal{H}'=&\frac{\partial H(\omega)}{\partial\omega^\mu}\epsilon^{\nu\mu}\pi^\mu+X(\eta)\\&+\psi^\mu(\omega,\xi)\pi^\mu+\hat{h}(\omega,\xi)\ ,\end{aligned}\tag{5.5}$$

where

$$\begin{aligned}\hat{h}(\omega,\xi)=&h(\omega,\hat{\xi})+Z(\omega,\xi)+\frac{\partial H(\omega)}{\partial\omega^\mu}\epsilon^{\nu\mu}U[\pi^\nu,U^{-1}]\\&+Y(\omega,\xi)+\psi^\mu(\omega,\xi)U[\pi^\mu,U^{-1}]\ ,\end{aligned}\tag{5.6}$$

if

$$\begin{aligned}&\phi^\mu(\omega,\eta')\to\psi^\mu(\omega,\xi)\ ,\\&X(\eta)\to X(\eta)+Y(\omega,\xi)\ ,\\&h(\omega,\hat{\epsilon})\to h(\omega,\hat{\epsilon})+Z(\omega,\xi)\ .\end{aligned}\tag{5.7}$$

Thus it is the secondary coupling function which absorbs the extra terms generated by the transformation, thereby exhibiting what we mean by the form invariance.

Clearly, if we were to formulate the problem without a secondary coupling function, we could in this manner generate such a coupling by making a unitary transformation of the above type. Similarly, if we were to formulate the problem initially with an h coupling which depended on only a commuting subset of the quantum variables by means of a unitary transformation, we could generate an h coupling depending on noncommuting quantum variables. Of course, the primary coupling functions will also change in the process, as shown in (5.7).

There are two interesting questions which one can pose. Can we find a unitary transformation which will cause a secondary coupling function $h(\omega,\hat{\epsilon})$, depending on noncommuting quantum variables, to be replaced (in the sense of the above) by $\hat{h}(\omega,\xi)$ which depends only on a commuting set of quantum variables? We are unable to answer this question in general at the present time. One can also ask whether or not a $U(\omega,\xi)$ exists which will cause the original coupling $h(\omega,\hat{\epsilon})$, whatever its dependence on quantum variables, to be transformed to zero. That is, can we have

$$\hat{h}(\omega,\xi)=0 \text{ identically?}$$

This question cannot be answered in general. However, within the context of the application discussed in Sec. III and II, we can supply an answer: It can be done.

It is straightforward to derive the form of the unitary transformation making use of Eq. (5.6). We refer the interested reader to Appendix C where the construction is explicitly shown. The calculation is done for a particular choice of magnetic field $\vec{B}(q)$.

The interesting property of the solution described in Appendix C is that it leaves the spin operator S_3 unchanged, i.e.,

$$S_3\to S'_3=US_3U^{-1}=S_3\ . \tag{5.8}$$

Thus the transformed Hamiltonian is simply

$$\mathcal{H}'=\frac{1}{m}\vec{p}\cdot\vec{\pi}^q-\gamma\frac{\partial}{\partial q_j}B(q)S_3\pi^p_j\ , \tag{5.9}$$

i.e., the Hamiltonian of model 2 after the integrity criteria for model 2 have been satisfied.

As we noted in Sec. III, it was not essential to introduce the secondary coupling function in our model of the Stern-Gerlach experiment. A secondary coupling function is not needed to satisfy conditions (i) and (ii) of Sec. V, as the $X(\eta)$ in this case vanishes. The result (5.9) emphasizes the fact that the secondary coupling function does not directly affect the time development of the apparatus observables. It is the primary coupling functions which have physical significance.

VI. DISCUSSION AND CONCLUDING REMARKS

In this paper we have endeavored to further develop an alternative approach to the quantum measurement process. This approach, originally suggested by Sudarshan,[8] was recently restated in I. The basic feature of the approach is to treat the apparatus as a purely classical system. In I we set up the formalism which would allow such a purely classical system to interact directly with a quantum system. In this paper we set out to examine what constraints should be placed on such interactions, and further we also wished to illustrate the approach by means of a simple example.

In Sec. II we derived, from the strong form of the principle of integrity, a set of constraints which must be imposed on any interaction which is to allow the observables of the apparatus system to retain their classical integrity. We termed these the integrity criteria, as they determine whether or not the principle of integrity is satisfied.

It is not obvious that any interactions exist which satisfy the principle. For this reason we next turned to a simple example, to which we could apply the integrity criteria and look for solutions. The models of Sec. III consist of a quantum spin system, and a freely moving classical particle as the apparatus, and are loosely based on the Stern-Gerlach experiment.[5] We considered two different interactions, which led to model 1 and model 2.

In Sec. IV we applied the integrity criteria to these two models. For model 1, with the more complex interaction, the criteria were satisfied if, and only if, the external magnetic field, which was used to induce the interaction, was undirectional. On the other hand, for model 2, with the simpler interaction, the criteria enforced the less restrictive requirement that the *gradients* of the external magnetic field be unidirectional.

These results are very interesting. They show, to begin with that interactions which preserve the integrity of the classical observables may exist, if magnetic fields of the required form can be constructed. However, in neither case does the required form correspond to the conventional Stern-Gerlach experiment,[5] where the magnetic field is supplied by two magnetic shoes. Thus the simple model we have suggested does not correspond to the conventional Stern-Gerlach experiment, if we require the strong form of the principle of integrity to be satisfied. It remains to be seen whether or not the weak form of the principle of integrity would allow such a simple description of the Stern-Gerlach experiment.

Despite this result, we have shown that a magnetic field of the required form could be realized by means of a nonzero current distribution. The resultant experiment might not be easy to carry out, but it at least establishes the principle that interactions exist which preserve the integrity of the apparatus observables.

In Sec. V we examined the role of the secondary coupling function in our approach. We saw that its occurrence guaranteed the form invariance of the Hamiltonian under a certain class of unitary gauge transformations. We used this fact to demonstrate, for a particularly simple external magnetic field, that the secondary coupling function of model 1, after the integrity criteria are satisfied, could be transformed away. Thus, after carrying out the unitary transformation on the Hamiltonian of model 1, we end up with the Hamiltonian of model 2. This interesting result demonstrates that, for the example we considered, the two models are equivalent.

So far our attention has been focused on setting up an interaction, between the classical apparatus and a quantum system, which would allow the apparatus to retain certain of its classical properties after the interaction has ceased to occur. We have not yet really come to grips with the measurement aspect of the interaction. How does our model mimic the measurement process? How is the information stored in the observables of the (classical) apparatus? Furthermore, we must address the problem of how the observer is to "read" the apparatus after the interaction has occurred. Does the principle of integrity guarantee that the apparatus will be classical from the point of view of an outside observer? Will the observer interact with the quantum-enlarged apparatus system, or the conventional classical apparatus? Such questions remain to be answered before our approach attains the status of a theory, and it is to these questions which we shall direct our attention in our subsequent paper.

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APPENDIX A

In this appendix we supply a derivation of the criteria (B) and (C) quoted in Sec. II. In Sec. II we introduced the set $\{\rho\}$ as the largest algebraically independent subset of the set $\{\phi^\mu\}$. Then $[\rho_i, \rho_j]=0$ and each ϕ^μ can be formed by algebraic combination of the ρ_i. In combinations we allow the coefficients to depend on the ω variables. We consider extensions of this set to an algebraically independent commuting subset of the algebra of dynamical variables. We denote such an extension by $\{\rho\}\cup\{\rho'\}$. We will write for the algebra of operators generated by these two sets $\mathfrak{A}_\omega(\{\rho\})$ and

$\mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\})$, respectively. These are both commutative algebras. The sets $\{\rho\}$ and $\{\rho'\}$ will in general be finite sets, though the algebras generated by them will be infinite dimensional (in the linear sense).

From Eq. (2.7b) we see that

$$\ddot{\omega}^\mu(t) = iF^\nu[F^\mu, \pi^\nu] + i[F^\mu, F^\nu]\pi^\nu + i[F^\mu, G]$$
$$= iF^\nu[\phi^\mu, \pi^\nu] + i[\phi^\mu, G] + iF^\nu\left[\frac{\partial H(\omega)}{\partial\omega^\rho}\epsilon^{\mu\rho}, \pi^\nu\right] .$$

The requirement that $\ddot{\omega}^\mu(t)$ belongs to the commutative algebra tells us that for each m we must have

$$F^\nu[\rho_m, \pi^\nu] + [\rho_m, G] \in \mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\}) . \tag{A1}$$

Thus if $[\rho_m, \pi^\nu]$ either vanishes or already belongs to the algebra, we see that $[\rho_m, G]$ must also belong to the algebra, whereas if $[\rho_m, \pi_\mu]$ lies outside the algebra the secondary coupling function $h(\omega, \hat{\epsilon})$ must be so chosen that the combination (A1) belongs to the algebra.

As a result, for some choice of $\{\rho'\}$ we have $[F^\mu, \mathcal{H}] \in \mathfrak{A}_\omega(\{\rho'\})$ and we can write

$$[F^\mu, \mathcal{H}] = \sum_m a_m(\omega)\zeta_m + \sum_l b_l(\omega)\zeta_l' . \tag{A2}$$

Here we use a linear basis for the algebra $\mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\})$, where $\{\zeta\}$ is generated by the set $\{\rho\}$, and we group the remaining elements of the basis into $\{\zeta'\}$. From Eq. (2.7c) we see that $[[F^\mu, \mathcal{H}], F^\nu\pi^\mu + G]$ must also belong to the algebra. We write this commutator as

$$\sum_m \{a_m(\omega)([\zeta_m, F^\nu]\pi^\nu + F^\nu[\zeta_m, \pi^\nu] + [\zeta_m, G]) + F^\nu[a_m(\omega), \pi^\nu]\zeta_m\}$$
$$+ \sum_l \{b_l(\omega)([\zeta_l', F^\nu]\pi^\nu + F^\nu[\zeta_l', \pi^\nu] + [\zeta_l', G]) + F^\nu[b_l(\omega), \pi^\nu]\zeta_l'\} . \tag{A3}$$

Now, we know already that $[\zeta_m, F^\nu] = [\zeta_l', F^\nu] = 0$ while $F^\nu[\zeta_m, \pi^\nu] + [\zeta_m, G]$, $F^\nu\zeta_m$, and $F^\nu\zeta_l' \in \mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\})$, and so (A3) tells us that we must have

$$F^\nu[\zeta_l', \pi^\nu] + [\zeta_l', G] \in \mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\}) \tag{A4}$$

for all those ζ_l' occurring in the expansion (A2). It is clear that we need only insist upon the condition

$$F^\nu[\rho_l', \pi^\nu] + [\rho_l', G] \in \mathfrak{A}_\omega(\{\rho\}\cup\{\rho'\}) \tag{A5}$$

for those ρ_l' which contribute in the expansion (A2).

The result (A1) is just criterion (B) of Sec. II. It is clear that criterion (C) is just (A5) together with the generalizations to the higher derivatives. In any problem the set $\{\rho\}\cup\{\rho'\}$ will be a finite set so that after a few derivatives the criteria will be exhausted. The example treated in Sec. IV is a case in point.

APPENDIX B

The purpose of this appendix is to derive the result (4.5) for model 1 of Sec. III, and the result (4.16) for model 2 of Sec. III.

We first turn to model 1, and begin with the vanishing of the commutator (4.4), namely

$$[[\phi_i^p, \mathcal{H}], \phi_j^p] = 0 \quad \text{for all } i \text{ and } j . \tag{B1}$$

Using the Hamiltonian operator (3.9) we have

$$[\phi_i^p, \mathcal{H}] = \frac{i}{m}\left(\vec{p}\cdot\frac{\partial}{\partial\vec{q}}\right)(\vec{f}_i\cdot\vec{S}) + i\gamma(\vec{B}\times\vec{f}_i)\cdot\vec{S}$$
$$- i\gamma(\vec{f}_i\times\vec{f}_j)\cdot\vec{S}\pi_j^p$$
$$= i\vec{D}(\vec{f}_i)\cdot\vec{S} , \tag{B2}$$

where

$$\vec{D}(\vec{f}_i) = \frac{1}{m}\left(\vec{p}\cdot\frac{\partial}{\partial\vec{q}}\right)\vec{f}_i + \gamma(\vec{B}\times\vec{f}_i) \tag{B3}$$

and we have used the result (4.3) that $\vec{f}_i$ and $\vec{f}_j$ are parallel. Thus we have

$$[[\phi_i^p, \mathcal{H}], \phi_j^p] = -(\vec{D}(\vec{f}_i)\times\vec{f}_j)\cdot\vec{S} ,$$

and Eq. (B1) is satisfied if and only if

$$\vec{D}(\vec{f}_i)\times\vec{f}_j = 0 . \tag{B4}$$

However, both $\vec{B}$ and $\vec{f}_i$ are independent of $\vec{p}$, so this equation is satisfied if and only if we have both

$$\frac{1}{m}p_l\left(\frac{\partial}{\partial q_l}\vec{f}_i\right)\times\vec{f}_j = 0 \tag{B5}$$

and

$$(\vec{B}\times\vec{f}_i)\times\vec{f}_j = 0 . \tag{B6}$$

Equation (B6) requires that $\vec{B}\times\vec{f}_i$ be parallel to $\vec{f}_j$. However, $\vec{B}\times\vec{f}_i$ is perpendicular to $\vec{f}_i$. We have already seen that $\vec{f}_i$ is parallel to $\vec{f}_j$ in Eq. (4.3). Thus $\vec{B}\times\vec{f}_i$ is perpendicular to $\vec{f}_j$. We can satisfy these conflicting requirements if and only if

$$\vec{B}\times\vec{f}_i = 0 . \tag{B7}$$

Consider the equation $\vec{B}\times\vec{f}_1 = 0$. This is satisfied if and only if there exists a function $\psi_1(q)$ such that

$$\vec{f}_1(q) \equiv -\gamma\frac{\partial}{\partial q_1}\vec{B}(q) = \psi_1(q)\vec{B}(q) ,$$

except at points where $\vec{B}(q)=0$. The solution of this equation is

$$\vec{B}(q)=B(q)\vec{n}\ , \tag{B8}$$

where

$$B(q)=\exp\left[-\frac{1}{\gamma}\int_0^{q_1}\psi_1(q)dq_1'\right]\ , \tag{B9}$$

and $\vec{n}$ is a constant unit vector, in each disjoint region of support.

Since the $\vec{f}_i$ are parallel to $\vec{B}$, we may simply write

$$\frac{1}{m}\left(\vec{p}\cdot\frac{\partial}{\partial\vec{q}}\right)\vec{B}(q)=\Phi(q,p)\vec{B}(q)$$

and

$$\frac{1}{m}\left(\vec{p}\cdot\frac{\partial}{\partial\vec{q}}\right)\vec{f}_i=\Phi_i'(q,p)\vec{B}(q)\ ,$$

so that Eq. (B5) is now automatically satisfied.

Equations (B8) and (B9) give us the required result.

Let us now turn our attention to model 2, using the Hamiltonian operator (3.6). The resulting analysis is altered because the $\vec{B}\times\vec{f}_i$ term is absent from the definition of $\vec{D}(\vec{f}_i)$ in Eq. (B3). Thus Eq. (B1) is satisfied if and only if we have

$$p_l\left(\frac{\partial}{\partial q_l}\vec{f}_i(q)\right)\times\vec{f}_j(q)=0$$

or (B10)

$$\left(\frac{\partial}{\partial q_l}\vec{f}_i(q)\right)\times\vec{f}_j(q)=0\ .$$

In particular, we focus our attention on the equation

$$\left(\frac{\partial}{\partial q_1}\vec{f}_i(q)\right)\times\vec{f}_i(q)=0\ ,$$

which is solved in the same way as Eq. (B7). This equation is satisfied if and only if

$$\vec{f}_i(q)=f_i(q)\vec{n}\ ,$$

where $\vec{n}$ is a constant unit vector. But this is just the result (4.16), which we set out to derive for model 2.

APPENDIX C

In this appendix we shall construct the unitary "gauge" transformation $U(\omega,\xi)$ which yields

$$\hat{h}(\omega,\xi)=0\ , \tag{C1}$$

where $\hat{h}(\omega,\xi)$ is given by Eq. (5.6), for the example considered in Sec. III and IV of this paper. In that example, the Hamiltonian takes the explicit form

$$\mathcal{H}=\frac{1}{m}\vec{p}\cdot\vec{\pi}^{\,q}-\gamma B(q)S_3-\gamma\frac{\partial}{\partial q_i}B(q)S_3\pi_i^p \tag{C2}$$

after the integrity criteria are satisfied. We notice that $X(S)=0$ in this example. Using the form for $\hat{h}(\omega,\xi)$ given in (5.6), Eq. (C1) gives us the following equation for $U(\omega,\xi)$:

$$-\frac{i}{m}p_jU\left(\frac{\partial}{\partial q_j}U^{-1}\right)+i\gamma\frac{\partial B(q)}{\partial q_j}US_3\left(\frac{\partial}{\partial p_j}U^{-1}\right)=\gamma B(q)US_3U^{-1}\ . \tag{C3}$$

To find a solution to this equation we specialize to a magnetic field

$$\gamma B(q)=Aq_1+B\ , \tag{C4}$$

a form which is allowed by the arguments of Sec. IV, and in particular Eq. (4.14). Also, we assume that S_3 is in an irreducible spin multiplet. To begin with we assume the spin of the quantum system to be half-odd-integral. For such spins S_3 can be inverted. We further work in a representation in which S_3 is diagonal.

With these restrictions, Eq. (C3) can be solved using the methods for first-order partial differential equations.[9] As we are not interested in finding the most general solution of Eq. (C3), we find a particular solution of the form

$$U=\exp\left\{i\left[\frac{(p_1)^3}{3mAS_3}+\left(q_1+\frac{B}{A}\right)p_1\right]\right\}. \tag{C5}$$

Since S_3 is diagonal, we have

$$(S_3^{-1})_{ii}=\frac{1}{(S_3)_{ii}}\ .$$

We also treat the case of integer spin. The result in this case follows quite closely the above form. For the case of integer spin s we denote the matrix elements of S_3 by

$$(S_3)_{jk}=\delta_{jk}(s-k+1)\ ,$$

where j and k range over the values 1 to $2s+1$. Then for the entries on the diagonal $(S_3)_{kk}\neq 0$ if and only if $k\neq s+1$. We define the transformation U by means of its matrix elements as follows: for $k\neq s+1$,

$$U_{kk}=\exp\left\{+i\left[\frac{(p_1)^3}{3mA(S_3)_{kk}}-\left(q_1+\frac{B}{A}\right)p_1\right]\right\}\ , \tag{C6a}$$

while for $k=s+1$, we define

$$(U)_{s+1,s+1}=1\ . \tag{C6b}$$

Equations (C5) and (C6) define the required unitary transformation. We note that by construction it is diagonal in all cases. Thus, in particular it commutes with the third spin projection S_3,

$$[S_3,U]=0\ .$$

This tells us that S_3 is unaltered by the transformation. The effect of the unitary transformation (C6) on the Hamiltonian (C2), which for the magnetic field (C4) is

$$\mathcal{H} = \frac{1}{m}\vec{p}\cdot\vec{\pi}^{q} - AS_3\pi_1^{q} - (Aq_1 + B)S_3 , \qquad (C7)$$

can now be seen to be

$$\mathcal{H} \to \mathcal{H}' = U\mathcal{H}U^{-1}$$
$$= \frac{1}{m}\vec{p}\cdot\vec{\pi}^{q} - AS_3\pi_1^{p} . \qquad (C8)$$

However, (C8) is just the Hamiltonian for model 2 using a magnetic field whose only nonvanishing gradient is

$$\frac{\partial}{\partial q_1}B(q) = (0, 0, A) .$$

A further interesting comment is that Eq. (C3) actually is the zero-eigenvalue equation for the Hamiltonian (C2), namely,

$$\mathcal{H}U^{-1} = 0 ,$$

and thus the matrix elements of U^{-1} are the zero-eigenvalue eigenfunctions of $\mathcal{H}$ corresponding to the allowed values of S_3.

The inverse problem can be solved in a similar manner. That is, we can write down a unitary transformation which transforms the Hamiltonian (C8) into the form of a Hamiltonian for model 1 [i.e., like (C7)]. We denote the transformation by V and a solution is

$$V_{kk} = \begin{cases} \exp\left\{-i\left[\frac{(p_1)^3}{3mA(S_3)_{kk}} + \left(q_1 + \frac{B}{A}\right)p_1\right]\right\} & \text{if } k \neq s+1 , \\ 1 & \text{if } k = s+1 , \end{cases}$$

where B is an arbitrary constant.

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[1]T. N. Sherry and E. C. G. Sudarshan, Phys. Rev. D 18, 4580 (1978). A rigorous treatment of the general theory is contained in the work of B. Misra and E. C. G. Sudarshan (unpublished).

[2]G. C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. 88, 101 (1952).

[3]In fact, in the integrity criteria (b) and (c), it suffices to require that $F^{\nu}[\rho_m, \pi^{\nu}] + [\rho_m, G]$ and $F^{\nu}[\rho'_m, \pi^{\nu}] + [\rho'_m, G]$ commute with ϕ^{μ} for all the relevant ρ_m and ρ'_m. The Jacobi identities then ensure that the other commutators vanish as required.

[4]T. N. Sherry, E. C. G. Sudarshan, and S. R. Gautam, CPT report (unpublished).

[5]See, for example, N. Ramsey, *Molecular Beams* (Oxford Univ. Press, London, 1955) and D. Bohm, *Quantum Theory* (Prentice-Hall, New York, 1951) for discussions of the Stern-Gerlach experiment.

[6] We are grateful to Dr. F. J. Belinfante for bringing this result to our attention.

[7]In such a case, in place of (4.5) we find that the allowed form of the magnetic field is $\sum_m B^m(q)\vec{n}_m$ where $\vec{n}_m$ are constant unit vectors and the support regions of $B^m(q)$ and $B^{m'}(q)$ are disjoint for $m \neq m'$. However, as discussed in Sec. II, in this case there are extra integrity criteria. Those used so far are necessary but not sufficient. In particular, we must check whether or not $[p^i(0), \phi^j(t)] = 0$ for all times t. Let us consider the case $B^1(q)\vec{n}_1 + B^2(q)\vec{n}_2$. Then the coupling function is

$$\phi^j = -\gamma\frac{\partial B^1}{\partial q^j}\vec{S}\cdot\vec{n}_1 - \gamma\frac{\partial B^2}{\partial q_j}\vec{S}\cdot\vec{n}_2 .$$

Now let us suppose the particle passes through the region 1 first. While in this region $B^2(q) \equiv 0$, and so S_{n_1} remains fixed, but $S_{n_2}(t)$ becomes dependent on $\pi(t)$, since the coefficient of π in $\mathcal{H}$ does not commute with $S_{n_2}(t)$. Thus, when the particle arrives in region 2, $p^j(t)$ will depend upon $S_{n_2}(t_2)$ which is, as we have seen, a function of the unobservables π. Thus the final criterion cannot be satisfied, when the magnetic field has the form $B^1(q)\vec{n}_1 + B^2(q)\vec{n}_2$. It is clear then that the integrity criteria can only be satisfied when the magnetic field is of the more restrictive form (4.5).

[8]E. C. G. Sudarshan, Pramāna 6, 117 (1976).

[9]See, for example, E. T. Copson, *Partial Differential Equations* (Cambridge Univ. Press, Cambridge, England, 1975).