PHYSICAL REVIEW D VOLUME 20, NUMBER 4 15 AUGUST 1979

Note on cosmic censorship

Pong Soo Jang
Theoretical Physics Institute, The University of Alberta, Edmonton, Alberta T6G 2J1, Canada
(Received 29 March 1979)

For initial data sets which represent charged black holes we prove some inequalities which relate the total energy, the total charge, and the size of the black hole. One of them is a necessary condition for the validity of cosmic censorship.

I. INTRODUCTION

Singularity theorems[1] in general relativity imply that a large class of physically reasonable initial states evolve into singular spacetimes. However, there has been a hypothesis, called (weak) cosmic censorship,[2,11] which states that an initial state of an asymptotically flat spacetime with a physically reasonable matter field cannot evolve singularities which can be observed from the asymptotic region. In other words, the occurrence of singularities in general relativity cannot interfere seriously with an observer's ability to predict as long as he remains in the asymptotic region. One formulation of this idea, due to Geroch and Horowitz,[2] is the following.

Conjecture 1. The maximal evolution of every asymptotically flat vacuum initial data set, topologically R^3, is an asymptotically flat (with a complete $\mathscr{I}^+$) spacetime.

The sense in which conjecture 1 captures the idea of the cosmic censorship is this: Given an initial data set on a spacelike slice Σ, one can uniquely and maximally evolve the data to obtain a spacetime in which Σ is a Cauchy surface. We suppose this spacetime is asymptotically flat with complete future null infinity $\mathscr{I}^+$ but it may be geodesically incomplete. Then all the past directed causal curves from $\mathscr{I}^+$ must meet Σ since it is a Cauchy surface. What this means is that the observer in the asymptotic region cannot detect the possible singular behavior of spacetime in the future of Σ.

The conjecture appears to be difficult to resolve. However, there have been some partial results of the following type: Assuming the validity of the conjecture one can deduce some restrictions on the global properties of the initial data. For instance, for a time-symmetric initial data set with an apparent horizon $\mathcal{H}$,[3] the following inequality[4] should hold:

$$A_0 \leqslant 16\pi E^2 , \tag{1}$$

where A_0 is the area of $\mathcal{H}$ and E is the total energy associated with the initial data set. There are some results[5] which indicate that this inequality holds for all such data sets. We shall obtain further confirmatory evidence in this vein. That is, we shall show that the cosmic censorship implies the following more stingent inequality for a time-symmetric initial data with a charged black hole:

$$(A_0/4\pi)^{1/2} \leqslant E + (E^2 - e^2)^{1/2} , \tag{2}$$

where e is the total charge inside $\mathcal{H}$. Then we shall find that an argument similar to Ref. 5 shows not only the validity of (2) but also an additional inequality,

$$E - (E^2 - e^2)^{1/2} \leqslant (A_0/4\pi)^{1/2} . \tag{3}$$

This additional relationship could be useful to estimate the energy which can be radiated away from such a system.

Note that conjecture 1 deals only with vacuum spacetimes. In Sec. III, we shall discuss a possible generalization of it to include matter fields. There, the technique used in Sec. II will play an important role.

II. INEQUALITIES FOR A CHARGED BLACK HOLE

Consider a time-symmetric initial data set which is asymptotically flat.[6] In this case, the constraint equations of Einstein's equation reduce to

$$R = 16\pi \mu , \tag{4}$$

where R is the scalar curvature of the positive-definite metric q_{ab} on the initial surface Σ and μ is the local energy density of the matter field. Let this data set have an apparent horizon $\mathcal{H}$. Let the matter field outside $\mathcal{H}$ be just an electromagnetic field and the total charge inside $\mathcal{H}$ be e. That is,

$$\mu = \frac{1}{8\pi} (E^a E_a + B^a B_a) \tag{5}$$

outside $\mathcal{H}$, and

$$\int_S E_a dS^a = 4\pi e \tag{6}$$

for any two-sphere S which encloses $\mathcal{H}$, where E^a and B^a are electric and magnetic fields on Σ, respectively.

If the cosmic censorship is right, this initial data set should settle down to a stationary final state with a black hole. From the theorems of Israel, Hawking, and Robinson,[7] the only stationary electrovacuum black holes are Kerr-Newman solutions. The formula for the area A_f of a Kerr-Newman black hole is

$$A_f = 4\pi[2M^2 - e^2 + 2M^2(M^2 - a^2 - e^2)^{1/2}] \leq 4\pi[M + (M^2 - e^2)^{1/2}]^2, \tag{7}$$

where M is the total mass and a is the angular momentum parameter. The total charge e of the final state should be the same as the total charge of the initial state because charged matter fields are confined within $\mathcal{H}$. Since the energy carried off to infinity by radiation is non-negative, we have $M \leq E$, where E is the total energy associated with the initial data set. By Hawking's area theorem[8] (which is again based on the cosmic censorship hypothesis) the initial area A_i of the horizon cannot be larger than the final area. For the time-symmetric initial data sets we are considering, the apparent horizon $\mathcal{H}$ is a minimal surface which encloses the horizon. Hence, the area A_0 of $\mathcal{H}$ should not be larger than A_i.

Stringing all these inequalities together, we obtain

$$A_0 \leq 4\pi[E + (E^2 - e^2)^{1/2}]^2. \tag{2}$$

In short, if we can find an initial data set which violates (2), then it will provide us with a counterexample to cosmic censorship. Note that the data set has to be regular only outside $\mathcal{H}$ in order to obtain (2) from cosmic censorship.

When e vanishes, (2) reduces to (1). It has been noticed[5] that a slight modification of Geroch's positive-mass argument[9] produces (1). We shall show that a further modification of the same argument produces not only (2) but also (3).

Following Geroch's argument for positivity of mass, we assume there exists a one-parameter family of smooth two-surfaces, parameterized by s, such that s vanishes on $\mathcal{H}$ and satisfies the differential equation

$$\tilde{p} = (D_a s D_b s q^{ab})^{1/2}, \tag{8}$$

where $\tilde{p}$ is the trace of the extrinsic curvature of the $s = \text{const}$ surface. It follows from (8) that the area $A(s)$ of $s = \text{const}$ surface is given by

$$A(s) = A_0 \exp(s). \tag{9}$$

For each value of s, let $f(s)$ denote the following integral over an $s = \text{const}$ surface:

$$f(s) = \frac{r_0}{32\pi} \int e^{s/2}(2\tilde{R} - \tilde{p}^2)dA, \tag{10}$$

where r_0 is $(A_0/4\pi)^{1/2}$ and $\tilde{R}$ is the intrinsic scalar curvature of the two-surface. Then the rate of change of $f(s)$ with respect to s is given by

$$\frac{d}{ds} f(s) \geq \frac{r_0}{2} \int \mu\, e^{s/2} dA. \tag{11}$$

Hence,

$$f(\infty) - f(0) \geq \frac{r_0}{2} \int_0^\infty ds \int e^{s/2} \mu\, dA. \tag{12}$$

The value of $f(0)$ can be obtained using the Gauss-Bonnet theorem and the fact that $\mathcal{H}$ is a minimal two-sphere. The value of $f(\infty)$ is the total energy associated with the data set.[9,10] Therefore, (12) becomes

$$E \geq \frac{r_0}{2} + \frac{r_0}{2} \int_0^\infty ds\, e^{s/2} \int \mu\, dA. \tag{13}$$

Recalling that the matter field outside $\mathcal{H}$ consists of an electromagnetic field, we have

$$\int \mu\, dA = \frac{1}{8\pi} \int (E^a E_a + B^a B_a) dA \geq \frac{1}{8\pi} \int (E^a \hat{r}_a)^2 dA, \tag{14}$$

where $\hat{r}_a$ is the unit vector field normal to the $s = \text{const}$ surfaces. Next we observe that

$$\int (E^a \hat{r}_a)^2 dA \geq \frac{(\int E^a \hat{r}_a dA)^2}{\int dA} = \frac{(4\pi e)^2}{4\pi r_0^2 \exp(s)}, \tag{15}$$

where the inequality is obtained using the Cauchy-Schwarz inequality, i.e.,

$$\left(\int |kh| dA\right)^2 \leq \left(\int k^2 dA\right)\left(\int h^2 dA\right). \tag{16}$$

Therefore, Eq. (13) becomes

$$E \geq \frac{r_0}{2} + \frac{4e^2}{r_0} \int_0^\infty ds \exp(-s/2) = \frac{r_0}{2} + \frac{e^2}{2r_0}. \tag{17}$$

This, in turn, implies

$$E - (E^2 - e^2)^{1/2} \leq r_0 \leq E + (E^2 - e^2)^{1/2}, \tag{18}$$

which is just (2) and (3). Note that $(E^2 - e^2) \geq 0$ is already implied by Eq. (17).

We also note that if we turn the argument around, that is, if we assume cosmic censorship, then (18) imposes the relationship

$$E - (E^2 - e^2)^{1/2} \leq M + (M^2 - e^2)^{1/2} \tag{19}$$

between the final energy M and the initial energy E of such a system. This inequality places an upper limit on the amount of energy which can be radiated off to infinity.

III. A GENERALIZATION OF CONJECTURE 1

As discussed by various authors,[2,11] the inclusion of matter fields into the formulation of the cosmic censorship raises rather difficult problems. Specifically, there are initial data of a gravitating perfect fluid such that when evolved using some equations of state, the resulting spacetimes appear to contradict the idea of cosmic censorship.[12] These spacetimes are often said to have "shell-crossing singularities."

One point of view against these being counterexamples to cosmic censorship has been the following: The model for a matter field in those examples allows the development of an infinite energy density even when treated as a test field on Minkowski space. The singular spacetime, in those examples, appears to be just a reflection of this singular nature of the matter model itself, and the idea of cosmic censorship is not that the inclusion of gravitational interaction via general relativity will smooth out such singular behavior of the matter model or hide it from the observer at infinity.

It is conceivable that the description of matter fields by fluid models with such equations of state will turn out to be essentially right in the limit of high density. So such singular spacetimes in those examples might actually occur in nature whether we consider it as consequences of general relativity or the matter field. However, the issue of what is the right phenomenological description of a matter field at such high densities is itself a difficult one to settle.

Under this situation, what one would like to settle first is whether or not there exists a counterexample to cosmic censorship which will be considered as a consequence of some essential features of general relativity. For instance, one would like to know whether there exists a counterargument to cosmic censorship which assumes as little about properties of the matter field as the usual singularity theorems do. We formulate cosmic censorship as the absence of such a clear-cut counterexample.

Conjecture 2. Given any asymptotically flat initial geometry (that is, the positive-definite metric q_{ab}, the extrinsic curvature p^{ab}, and all the components of the stress-energy tensor) on a R^3-manifold Σ which satisfies the dominant and strong energy conditions,[13] there exists an asymptotically flat spacetime M (possibly geodesically incomplete, but with a complete $\mathscr{I}^+$), such that (i) there is a slice Σ' in M so that the geometry of Σ' is the original geometry, (ii) Σ' is a Cauchy surface of M, and (iii) the local energy conditions are satisfied in the future of Σ'.

If one can find initial geometries which violate conjecture 2, then they will be as strong an argument for the occurrence of naked singularities as the singularity theorems are for the occurrence of singularities. Since there are only some inequalities to be satisfied, one might think that it will be easy to show the validity of conjecture 2. However, note that for a vacuum initial geometry the dominant energy condition prohibits creation of matter in the future of Σ. Hence, testing conjecture 2 for this case would involve constructing a nonstationary, nonspherically symmetric, asymptotically flat solution of the vacuum Einstein's equation. In fact, even for the nonvacuum spherically symmetric case, conjecture 2 seems to be surprisingly difficult to resolve.

Even if conjecture 2 turns out to be true its validity would far from establish cosmic censorship in our universe. To illustrate this we consider initial data sets for the Einstein-Maxwell equations. Suppose that for a certain data set the maximal evolution via the Einstein-Maxwell equations is not asymptotically flat with a complete $\mathscr{I}^+$. Nonetheless, conjecture 2 could still be true since it does not require the spacetime M to satisfy the Einstein-Maxwell equations. In such respects, our formulation is not totally satisfactory. However, we shall see that there are some difficulties even when one tries to include the electromagnetic interaction of the matter field explicitly into conjecture 2. The following spherically symmetric example illustrates this point.

Here, for the sake of simplicity, we consider time-symmetric, spherically symmetric, asymptotically flat initial data with an apparent $\mathscr{H}$. As in Sec. II, let the charge integral over $\mathscr{H}$ be $4\pi e$. However, we now assume that outside $\mathscr{H}$ there are some matter fields in addition to the electromagnetic field. As for the properties of matter fields, we assume the following: (i) The total stress-energy tensor of matter fields including electromagnetic field satisfies the strong and dominant energy conditions and (ii) matter is electrically neutral so that even if it is radiated off to infinity charge is still conserved.

Following the conventional picture of gravitational collapse, assume the final state evolving from such an initial state is the Reissner-Nordström solution with charge e. Hence, assuming cosmic censorship as in Sec. II, we obtain

$$(A_0/4\pi)^{1/2} \leq E + (E^2 - e^2)^{1/2} . \qquad (20)$$

So if we can find, among the above-described initial states, one which violates this inequality (20), then it will provide us with a counterexample to cosmic censorship.

We examine whether there exists such a counterexample. Because of the spherical symmetry, the total energy E can be written as follows:

$$E=\frac{r_0}{2}+\frac{r_0}{2}\int ds\,\exp(s/2)\int dA\,\mu\,, \tag{21}$$

where s is a spherically symmetric function which is related to the area of a metric two-sphere A by $s=2\ln(A/A_0)$. Let $\mathcal{E}^a$ denote a spherically symmetric vector field outside $\mathcal{H}$ such that $D_a\mathcal{E}^a=0$ and the flux integral of it is $4\pi e$. Then Eq. (21) can be cast in the form

$$E=\frac{r_0}{2}+\frac{e^2}{2r_0}+\frac{r_0}{2}\int ds\,\exp(s/2)\int dA\left(\mu-\frac{1}{8\pi}\mathcal{E}^a\mathcal{E}_a\right)\,. \tag{22}$$

Therefore, the question whether (20) is satisfied is reduced to whether the integral of (22) is nonnegative. Do the conditions (i) and (ii) on matter field imply non-negativity of this integral? The first, i.e., the local energy conditions for the *total* stress-energy tensor, implies only μ is nonnegative. The second condition is that the matter field is electrically neutral so that if it is radiated off to infinity charge is still conserved. Even this, combined with (i), does not rule out the possibility that the integral in (22) is negative. Hence, there are initial states which violate the inequality (20), thus which could be considered as counterexamples to cosmic censorship.[14]

However, note that the situation here is somewhat similar to that of the "shell-crossing singularities" we mentioned earlier. Our counterexample depends crucially on the existence of matter for which the above integral is negative, and the availability of such matter in our universe is questionable. Thus, our example is too open to the same criticism as the shell-crossing singularities to be a genuine counterargument against cosmic censorship.

Nonetheless, this additional example makes it clear that the inclusion of general matter fields into any true statement of cosmic censorship has to involve some detailed properties of matter fields. Under the situation where a complete description of all matter fields is not available, the best one can do toward establishment of cosmic censorship seems to be to prove a statement such as conjecture 2.

IV. CONCLUSION

We showed that certain initial data sets which would be counterexamples to the cosmic censorship do not occur in general relativity. More generally, we asked what initial states could be considered as clear-cut counterexamples to cosmic censorship. We then formulated cosmic censorship as conjecture 2 which states that such clear-cut counterexamples do not occur in general relativity. Although this formulation seems to convey only partial aspects of the idea of cosmic censorship, we observed that there are some limitations to enlarging the scope of this conjecture.

ACKNOWLEDGMENTS

I am grateful to Professor W. Israel for enlightening discussions and many suggestions. In addition, thanks are due to Professor R. Geroch for correspondence on this subject. This work was supported by Natural Sciences and Engineering Research Council of Canada.

[1]R. Penrose, Phys. Rev. Lett. 14, 57 (1965); S. W. Hawking, Proc. R. Soc. London A300, 187 (1967); S. W. Hawking, and R. Penrose, *ibid*. A314, 529 (1970).

[2]R. Geroch and G. Horowitz, in *Einstein Centenary Volume*, edited by S. Hawking and W. Israel (Plenum, New York, 1979).

[3]Apparent horizon $\mathcal{H}$ is defined as the outer boundary of the region of Σ which contains trapped or marginally trapped surfaces. See, for example, S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973).

[4]R. Penrose, Ann. N. Y. Acad. Sci. 224, 125 (1973); G. Gibbons, Commun. Math. Phys. 27, 87 (1972).

[5]P. S. Jang and R. M. Wald, J. Math. Phys. 18, 41 (1977).

[6]R. Geroch, in *Asymptotic Structure of Space-Time*, edited by F. P. Esposito and L. Witten (Plenum, New York, 1976).

[7]W. Israel, Phys. Rev. 164, 1776 (1967); Commun. Math. Phys. 8, 245 (1968); S. Hawking, *ibid*. 25, 152 (1972); D. Robinson, Phys. Rev. Lett. 34, 905 (1975); B. Carter, *ibid*. 26, 331 (1971).

[8]S. Hawking, Phys. Rev. Lett. 26, 1344 (1971).

[9]R. Geroch, Ann. N. Y. Acad. Sci. 224, 108 (1973).

[10]J. Goldberg, in *Einstein Centenary Volume*, edited by A. Held (Plenum, New York, 1979).

[11]G. F. R. Ellis *et al.*, in *Einstein Centenary Volume*, edited by A. Held (Plenum, New York, 1979).

[12]H. Muller zum Hagen *et al.*, Commun. Math. Phys. 37, 29 (1974); M. Demianski and J. P. Lasota, As-

trophys. Lett. 1, 205 (1968).

[13]The stress-energy tensor T^{ab} is said to satisfy the strong energy condition if $(T^{ab}-\frac{1}{2}Tg^{ab})t_a t_b \geq 0$ for every timelike vector t^a, and said to satisfy the dominant energy condition if $T^{ab}t_a t_b \geq 0$ and $T^{ab}t_a$ is a nonspacelike vector for every timelike vector t^a. See, for example, Ref. 3.

[14]One would like to find some stronger but still reasonable condition on matter fields to rule out the proposed counterexamples. For instance, when there is no interaction between the electromagnetic field and other matter fields the total stress-energy tensor can be decomposed into the electromagnetic part and the rest. For such cases, it will be reasonable to impose that each part of the stress-energy tensor satisfies the energy condition separately. With this stronger condition the counterexamples are ruled out. However, when there is some interaction, e.g., the polarization of the matter field, there is no natural decomposition of the total stress-energy tensor, as is well known through the Abraham-Minkowski controversy [see, for example, the article by W. Israel and J. M. Steward, in *Einstein Centenary Volume*, edited by A. Held (Plenum, New York, 1979)].